Instabilities Driven Equilibration of the Quark-Gluon Plasma

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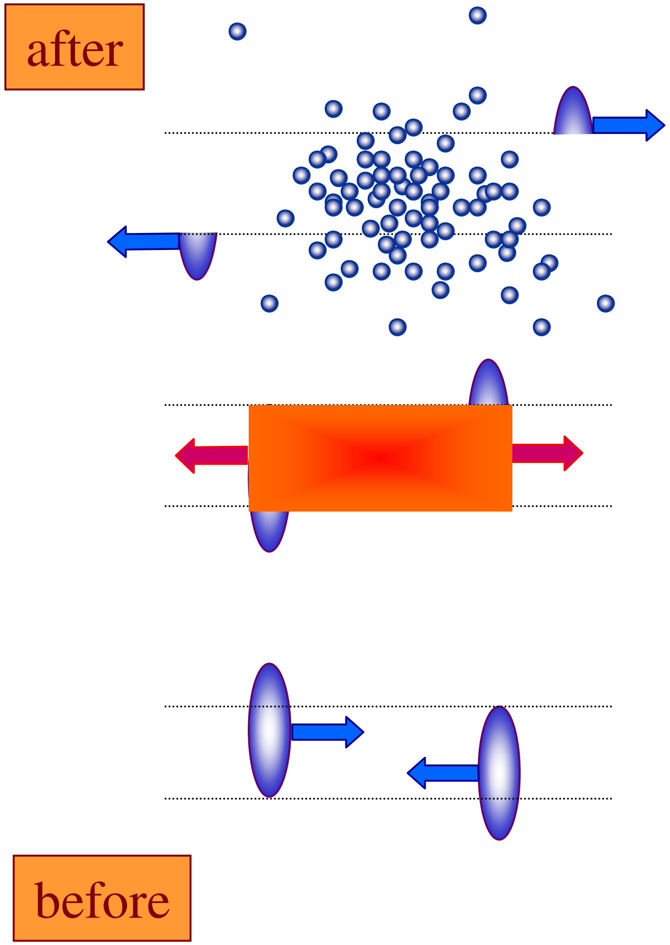
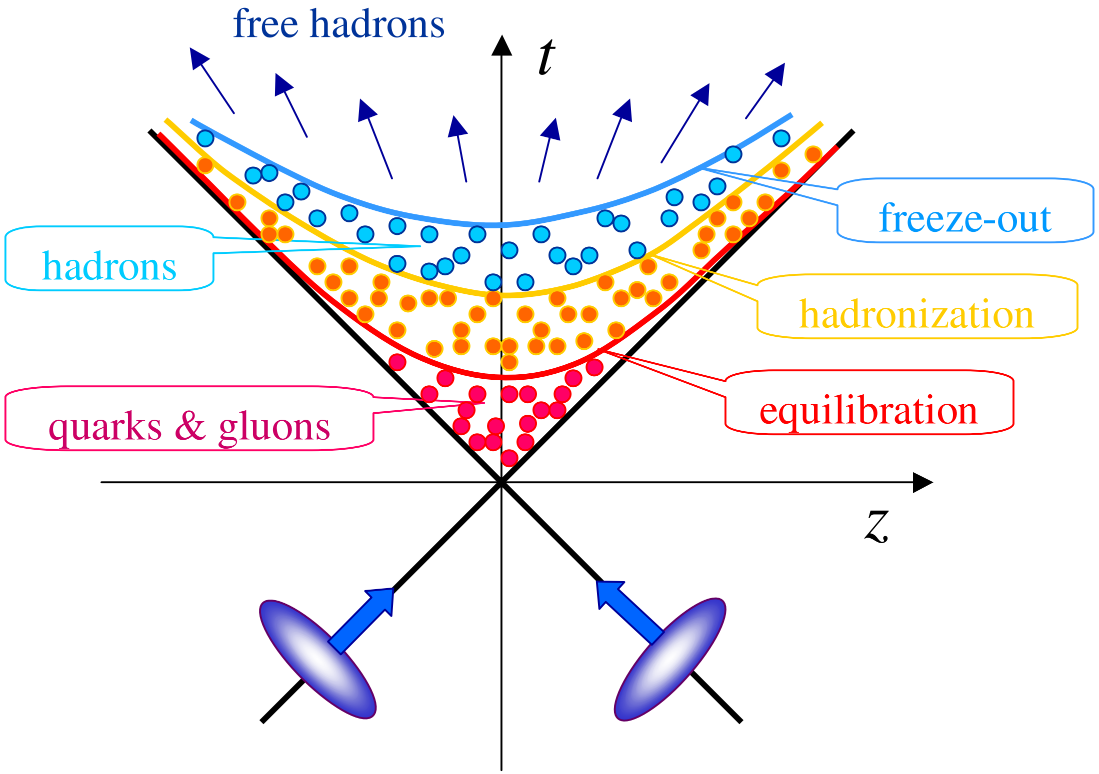
Course of relativistic heavy-ion collisions



after

before



free hadrons

hadrons

quarks & gluons

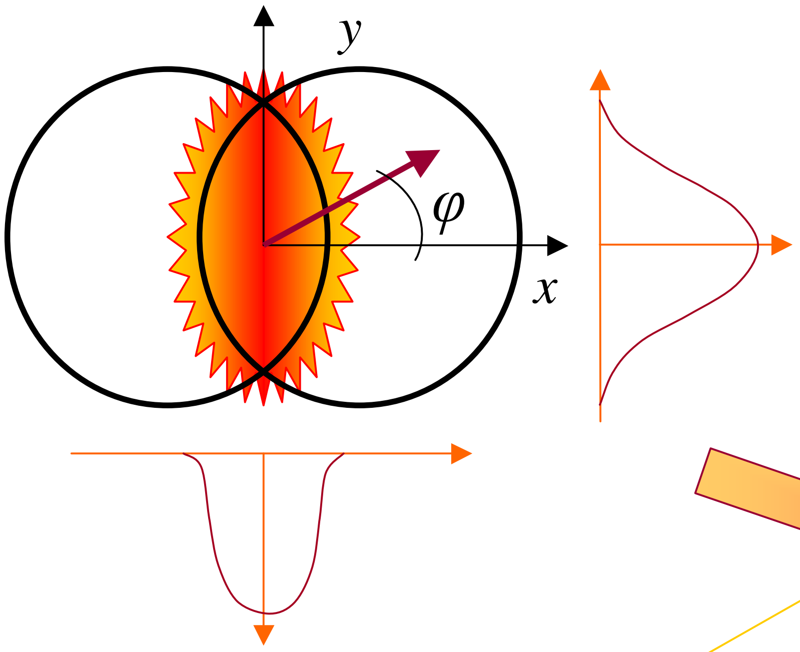
freeze-out

hadronization

equilibration

Evidence of the early stage equilibration

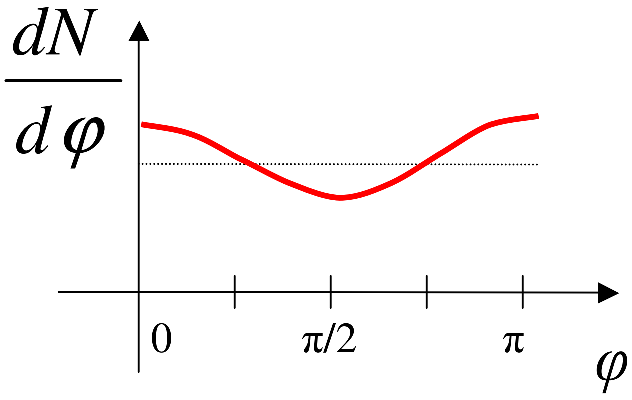
Success of hydrodynamic models in describing elliptic flow



Hydrodynamics

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)\mathbf{v}=-\frac{\nabla p}{\rho}$$

Hydrodynamics requires local thermodynamical equilibrium!



Equilibration is fast

$$v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle$$

Eccentricity decays due to the free streaming!

$$\varepsilon \searrow \;\Rightarrow\; v_2 \searrow \quad \rightarrow \quad t_{eq} \le 1 \;\; \mathrm{fm}/c$$

time of equilibration

U. Heinz, AIP Conf. Proc.**739,** 163 (2004)

Decay of Eccentricity

free-streaming model

Y – parton’s rapidity

$$f(\mathbf{p},\mathbf{r},t) \sim \exp\left[-\frac{(x-v_x t)^2}{2\sigma_x^2}-\frac{(y-v_y t)^2}{2\sigma_y^2}-\frac{(z-v_z t)^2}{2\sigma_z^2}-\frac{Y^2}{2\Delta Y^2}\right]\frac{P(p_T)}{p_T^2\,\mathrm{ch}\,Y}$$

$$\left\{\begin{aligned} \left\langle x^2\right\rangle &= \int\frac{d^3p}{(2\pi)^3}\int d^3r\, x^2\, f(\mathbf{p},\mathbf{r},t) = \sigma_x^2+\alpha t^2 \\ \left\langle y^2\right\rangle &= \int\frac{d^3p}{(2\pi)^3}\int d^3r\, y^2\, f(\mathbf{p},\mathbf{r},t) = \sigma_y^2+\alpha t^2 \end{aligned}\right.$$

$m=0$

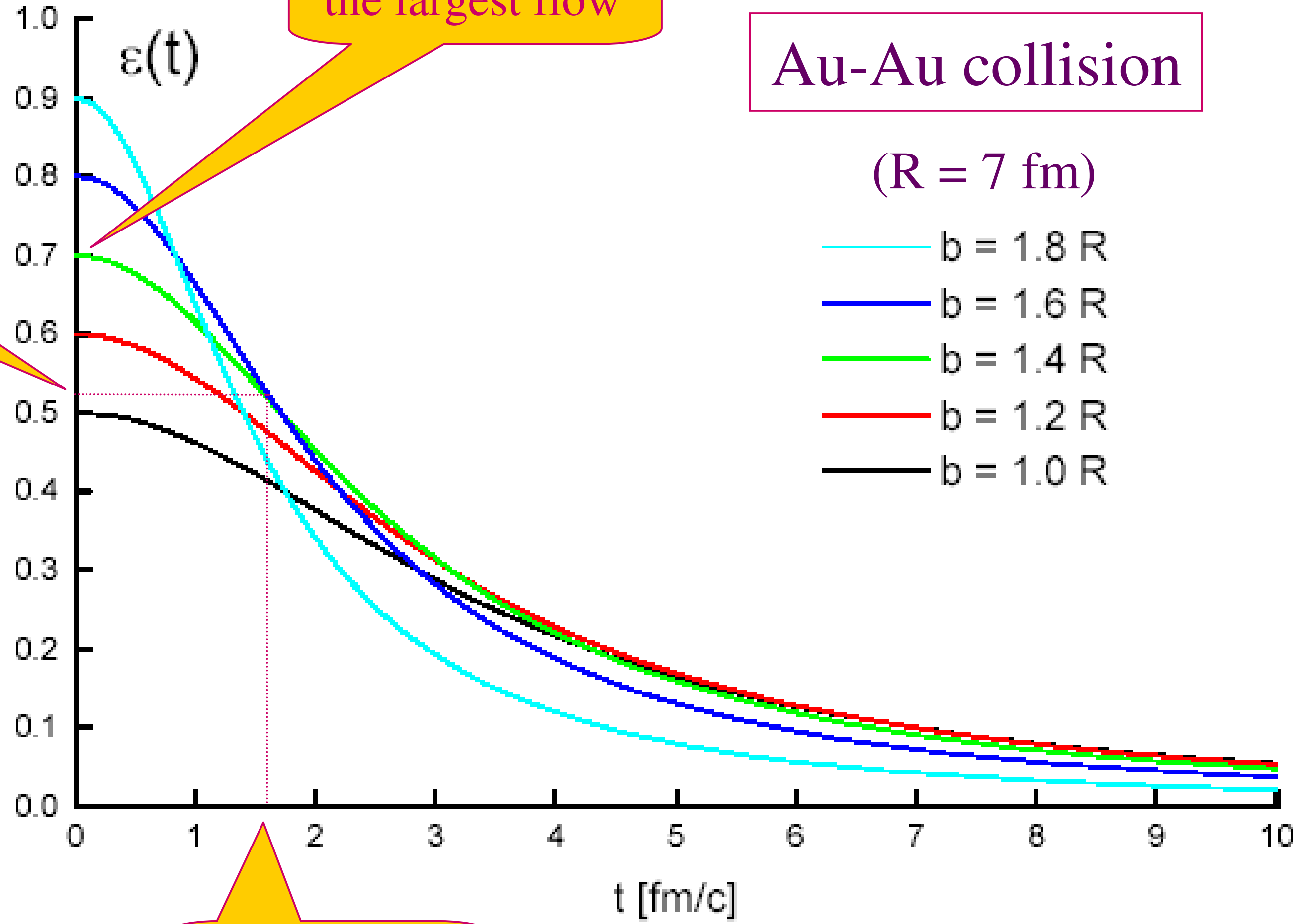
$$\varepsilon(t) = \frac{\left\langle y^2\right\rangle-\left\langle x^2\right\rangle}{\left\langle y^2\right\rangle+\left\langle x^2\right\rangle} = \frac{\varepsilon(0)}{1+\dfrac{\alpha}{R_T^2}t^2}$$

$$\varepsilon(0)=\frac{\sigma_y^2-\sigma_x^2}{\sigma_y^2+\sigma_x^2}\,, \qquad R_T^2=\frac{\sigma_y^2+\sigma_x^2}{2}$$

$$\alpha = 1/2 \;\; at \;\; Y \approx 0$$

W. Jas and St. Mrówczyński, Phys. Rev. **C76** (2007) 044905.

Decay of Eccentricity cont.



the largest flow

Au-Au collision

(R = 7 fm)

$0.75\ \varepsilon_0$

$t_{eq} = 1.5$ fm/c

Collisions in weakly coupled QGP

Assumption: **QGP is weakly coupled !** $\alpha_s \equiv \frac{g^2}{4\pi} << 1$ – QCD coupling constant

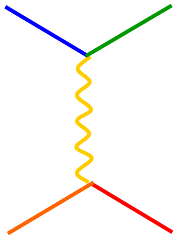
Time scale of equilibration driven by hard parton-parton scatterings

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

hard scattering ~ momentum transfer of order of T

either single hard scattering or multiple soft scatterings

dominated by



Collisions are too slow !

Instabilities

stationary state

$$\boxed{A(t) = A_0 + \delta A(t)}$$

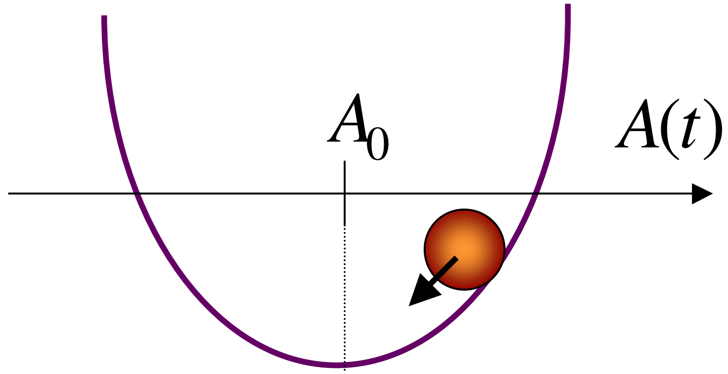
fluctuation

Instability

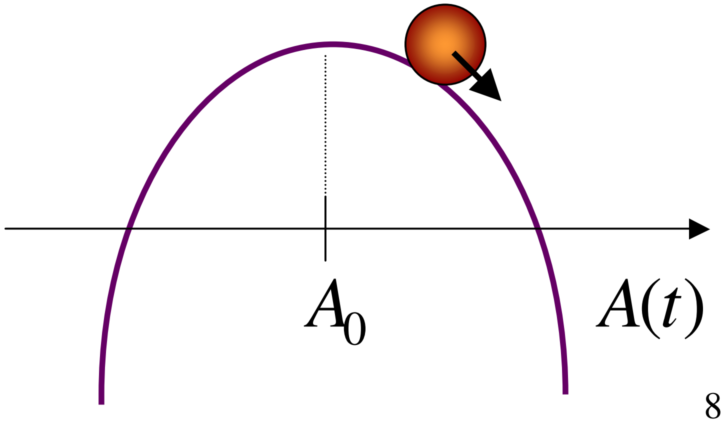
$$\boxed{\delta A(t) \propto \mathrm{e}^{\gamma t}}$$

$$\gamma > 0$$

stable configuration



unstable configuration



Terminology

Plasma instabilities – interplay of particles and classical fields

Quantum Field Theory – no particles, no classical fields

- particles – $\overbrace{\text{hard excitations, hard modes}}^{p_{\text{hard}} \sim T}$
- classical fields – $\underbrace{\text{highly populated}}_{\sim 1/g^2}$ $\underbrace{\text{soft excitations, soft modes}}_{p_{\text{soft}} \sim gT}$

Plasma instabilities

- **instabilities in configuration space – hydrodynamic instabilities**

- **instabilities in momentum space – kinetic instabilities**

instabilities due to non-equilibrium momentum distribution

$$f(\mathbf{p}) \;\text{ is }\; \text{not} \;\sim \exp\left(-\frac{E}{T}\right)$$

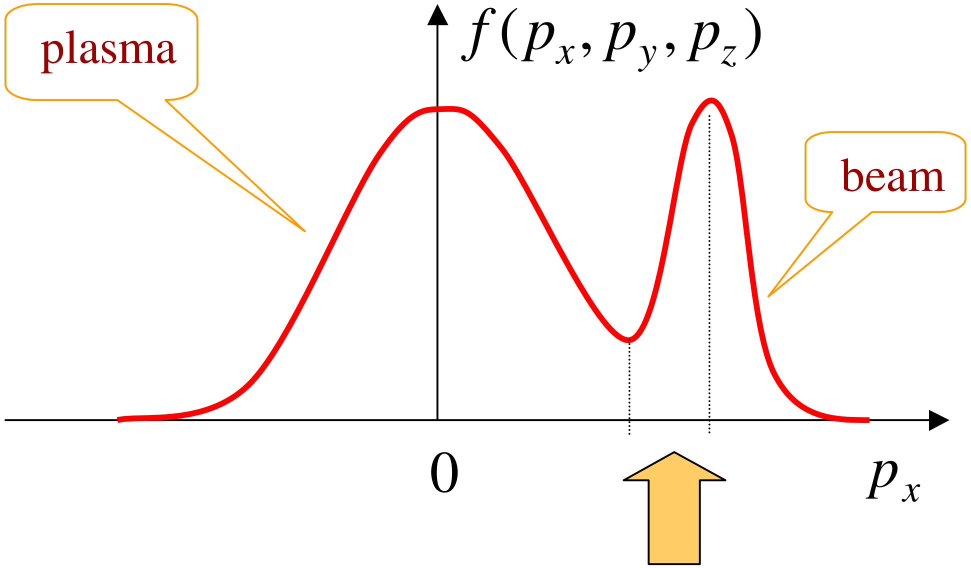
Kinetic instabilities

- ▶ **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}, \; \delta\rho \sim e^{-i(\omega t - \mathbf{kr})}$
- ▶ **transverse modes** – $\mathbf{k} \perp \mathbf{E}, \; \delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{kr})}$

E – electric field, **k** – wave vector, ρ – charge density, **j** - current

Logitudinal modes

unstable configuration

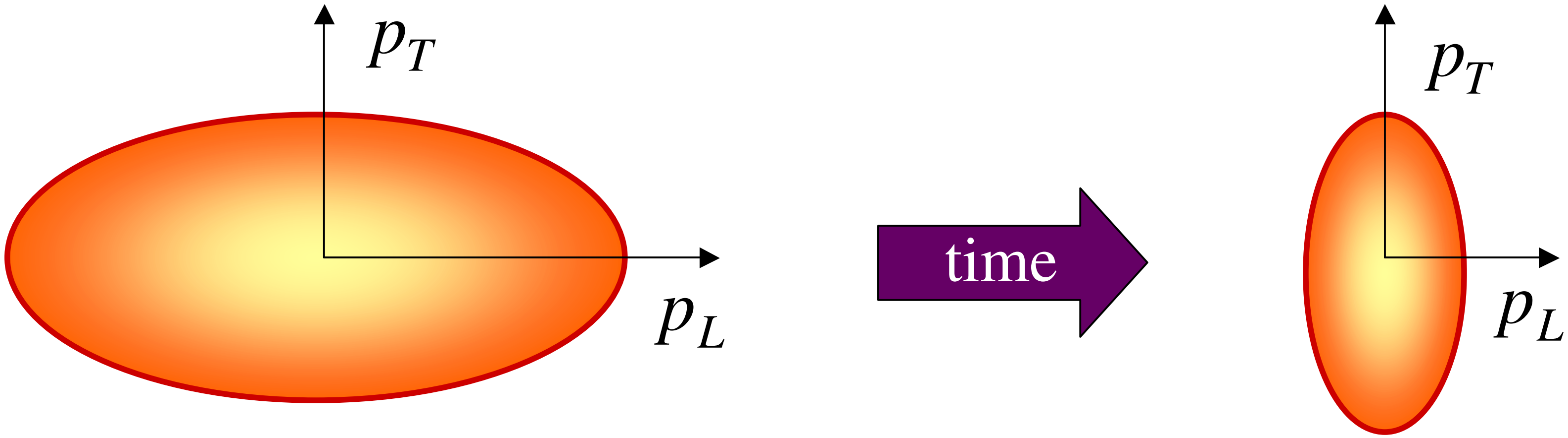


Energy is transferred from particles to fields

Transverse modes

Unstable modes occur due to anisotropy of the momentum distribution

Parton momentum distribution is initially strongly anisotropic

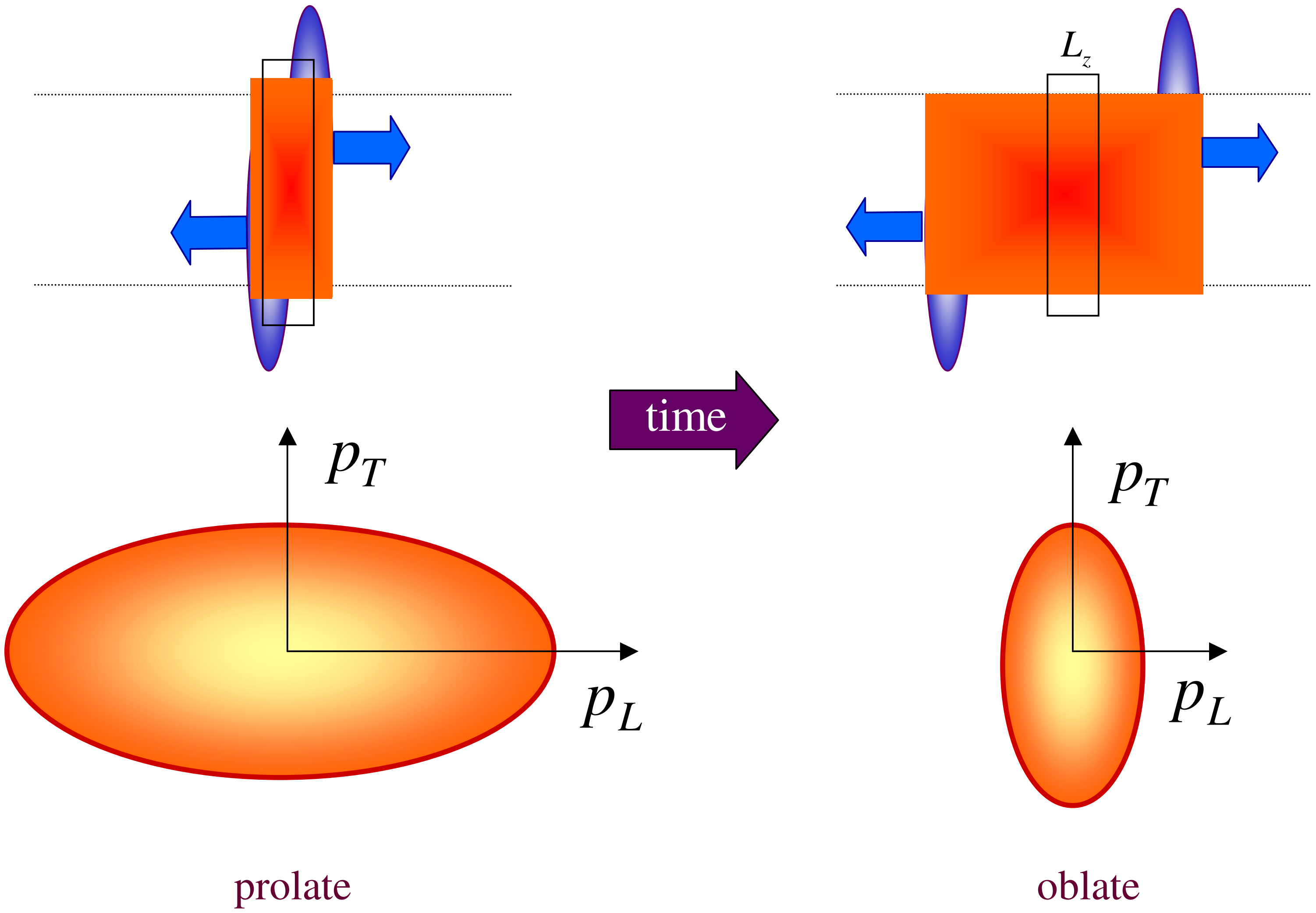


CM after 1-st collisions

local rest frame

Transverse modes are relevant for relativistic nuclear collisions!

Evolution of Parton Momentum Distribution

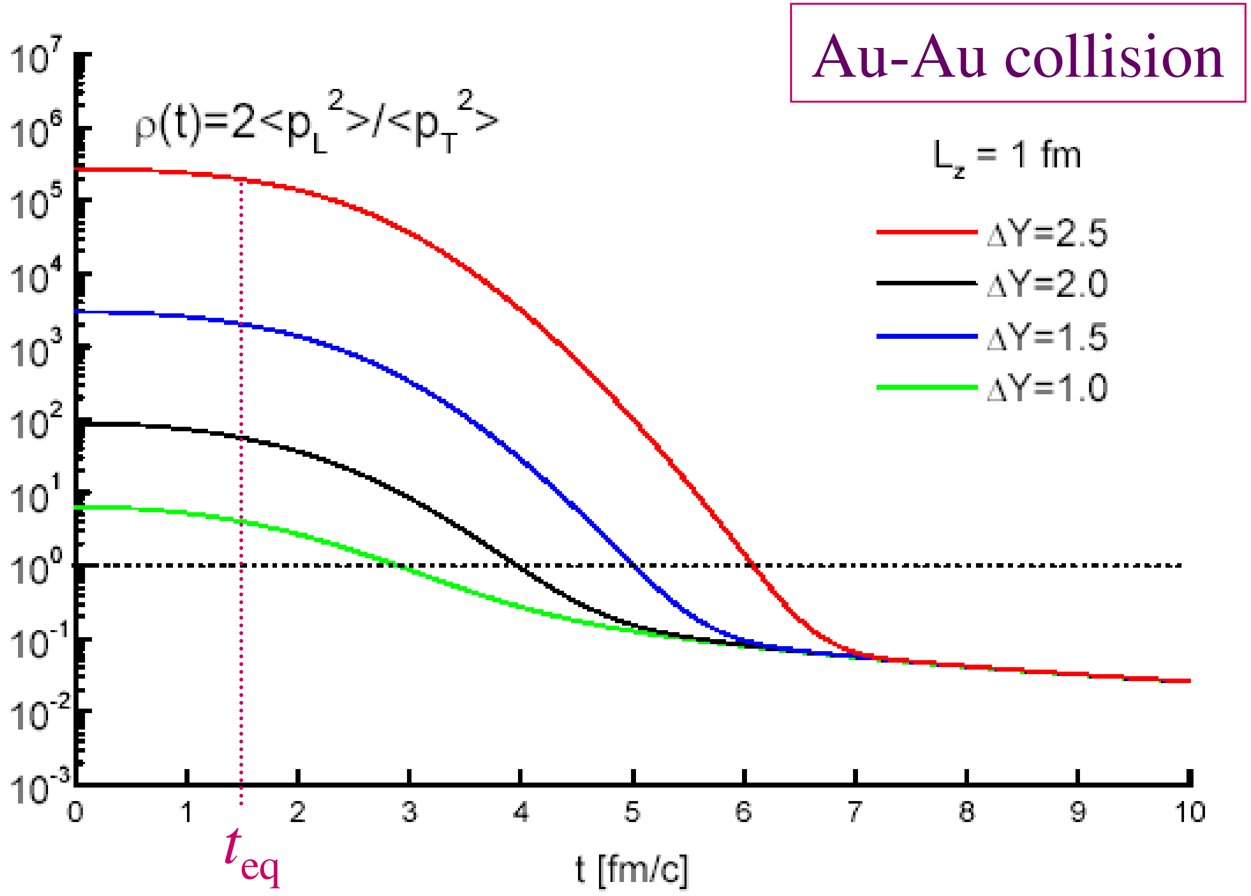


Evolution of Momentum Distribution cont.

free-streaming model

Au-Au collision

$$\rho(0) = e^{2\Delta Y^2} - 1$$



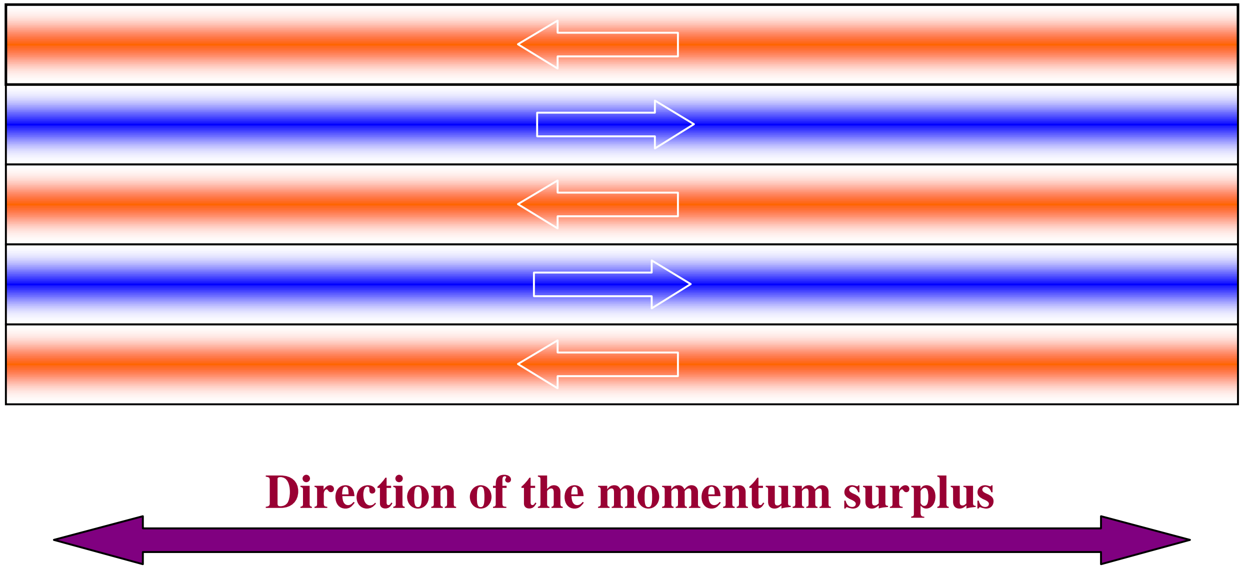
$$\sigma_x = \sigma_y = L_x = L_y = 3\ \text{fm}, \quad \sigma_z = L_z = 1\ \text{fm}$$

Seeds of instability

$\left\langle j_a^{\mu}(x) \right\rangle = 0$ **but current fluctuations are finite**

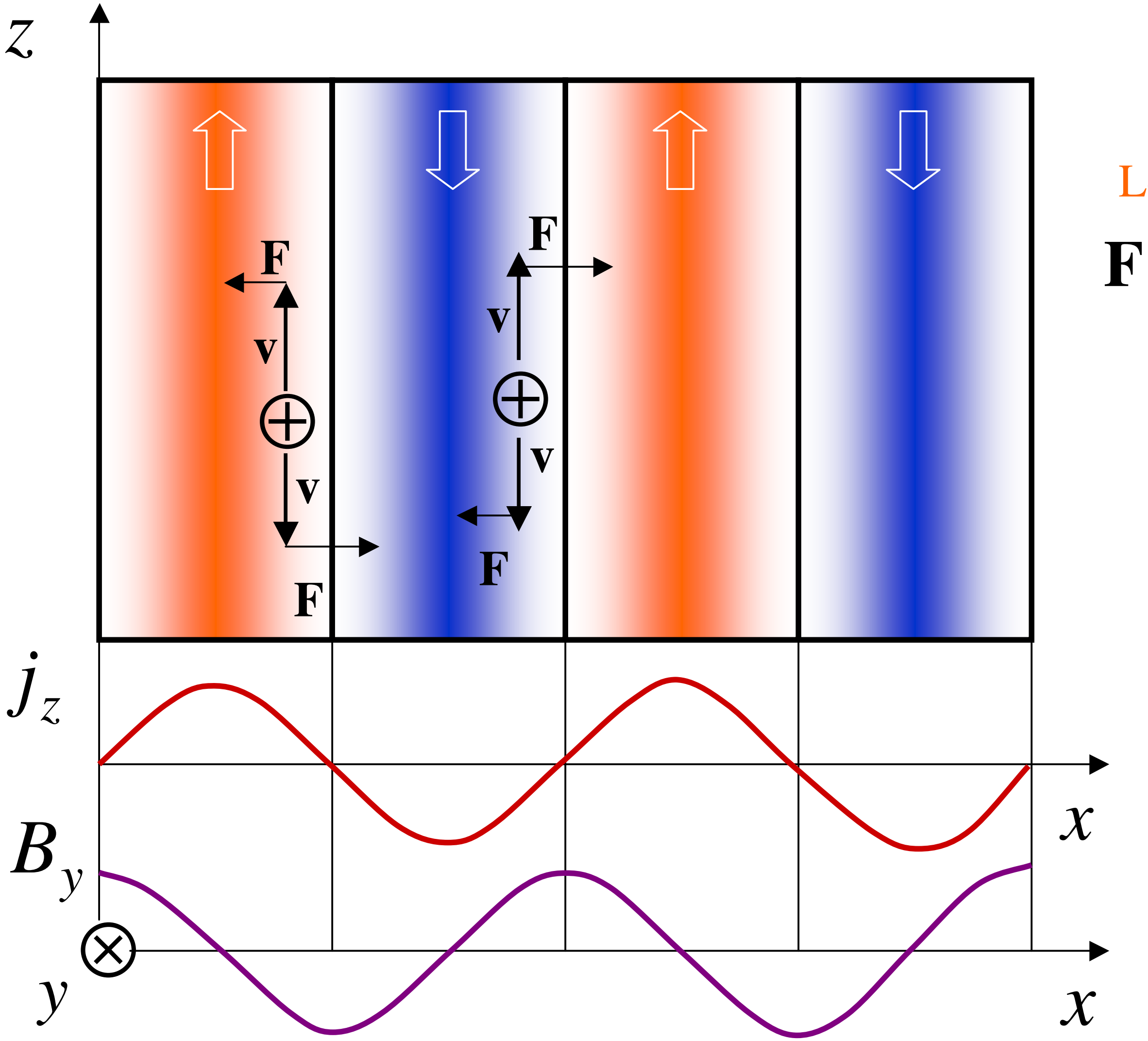
$$\boxed{\left\langle j_a^{\mu}(x_1)\, j_b^{\nu}(x_2) \right\rangle = \frac{1}{2}\delta^{ab} \int \frac{d^3p}{(2\pi)^3} \frac{p^{\mu} p^{\nu}}{E_p^2} f(\mathbf{p}) \delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0}$$

$$x_1 = (t_1, \mathbf{x}_1)\,, \quad x_2 = (t_2, \mathbf{x}_2)\,, \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus

Mechanism of filamentation



Lorentz force

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

Ampere’s law

$$\nabla \times \mathbf{B} = \mathbf{j}$$

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with Imω > 0 $\Rightarrow \quad A^\mu(x) \sim e^{\mathrm{Im}\,\omega t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. How to get it?

Transport theory

fundamental

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial^\nu_p Q\} = C$$

quarks

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial^\nu_p \overline{Q}\} = \overline{C}$$

antiquarks

adjoint

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{\mathcal{F}_{\mu\nu}, (x)\partial^\nu_p G\} = C_g$$

gluons

free streaming

mean-field force

collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, ...], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$\boxed{D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]}$$

mean-field generation

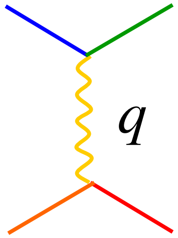
collisionless limit: $C = \overline{C} = C_g = 0$

Time scale of collisional processes

Time scale of processes driven by parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

$$t_{\text{soft}} \sim \frac{1}{g^2 \ln(1/g)\, T}$$



hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Time scale of collective phenomena

$$t_{\text{collec}} \sim \frac{1}{g\,T}$$

$$g^2 << 1 \quad \Rightarrow \quad t_{\text{hard}} >> t_{\text{soft}} >> t_{\text{collec}}$$

The instabilities are fast!

Transport theory - linearization

fluctuation

$$Q(p,x) = Q_0(p) + \delta Q(p,x)$$

stationary colorless state $Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| >> |\delta Q(p,x)|, \quad |\partial_p^\mu Q_0(p)| >> |\partial_p^\mu \delta Q(p,x)|$$

Linearized transport equations

$$\boxed{\begin{aligned} &p_\mu D^\mu \delta Q(p,x) - g p^\mu F_{\mu\nu}(x) \partial_p^\nu Q_0(p) = 0 \\ &p_\mu D^\mu \delta \overline{Q}(p,x) + g p^\mu F_{\mu\nu}(x) \partial_p^\nu \overline{Q}_0(p) = 0 \\ &p_\mu \mathcal{D}^\mu \delta G(p,x) - g p^\mu \mathcal{F}_{\mu\nu}(x) \partial_p^\nu G_0(p) = 0 \end{aligned}}$$

Transport theory – polarization tensor

$$\delta Q(p,x) = g\int d^4x'\,\Delta_p(x-x')\,p^\mu F_{\mu\nu}(x)\partial_p^\nu Q_0(p)$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$j^\mu[\delta Q, \delta\overline{Q}, \delta G]$$

$$j^\mu(k) = \Pi^{\mu\nu}(k)A_\nu(k)$$

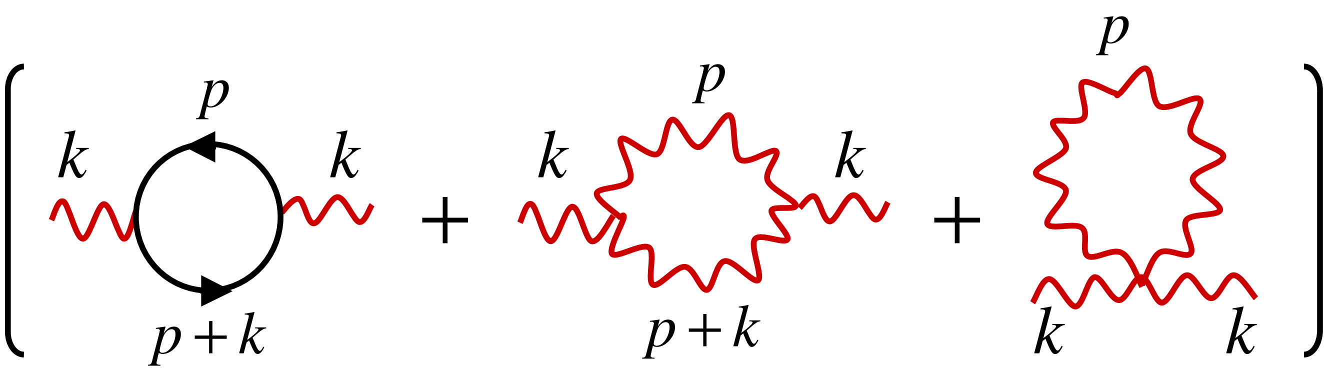
$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \overline{n}(\mathbf{p}) + 2n_g(\mathbf{p})$$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\Big[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\Big]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad \boxed{k_\mu \Pi^{\mu\nu}(k) = 0}$$

Diagrammatic Hard Loop approach

$\Pi^{\mu\nu}(k) =$



Hard loop approximation: $k^\mu << p^\mu$

$$\boxed{\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\left[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\right]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

St. M. & M. Thoma, Phys. Rev. C **62**, 036011 (2000)

Dispersion equation

Dispersion equation

$$\boxed{\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0}$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$ **chromodielectric tensor**

$$k^\mu \equiv (\omega, \mathbf{k})$$

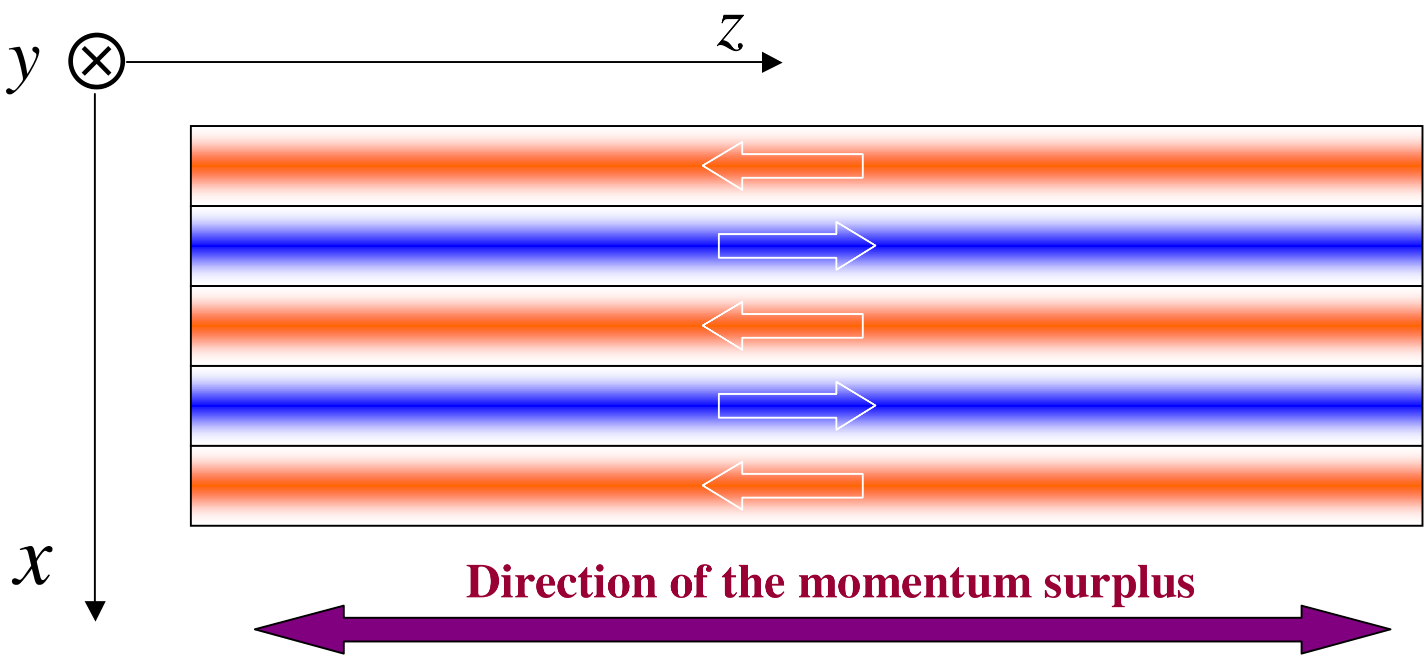
Dispersion equation

$$\boxed{\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0}$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int \frac{d^3p}{(2\pi)^3} \frac{v^i}{\omega - \mathbf{kv} + i0^+} \frac{\partial f(\mathbf{p})}{\partial p^l}\Big[\Big(1 - \frac{\mathbf{kv}}{\omega}\Big)\delta^{lj} + \frac{k^l v^j}{\omega}\Big]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

Dispersion equation – configuration of interest



Direction of the momentum surplus

$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$

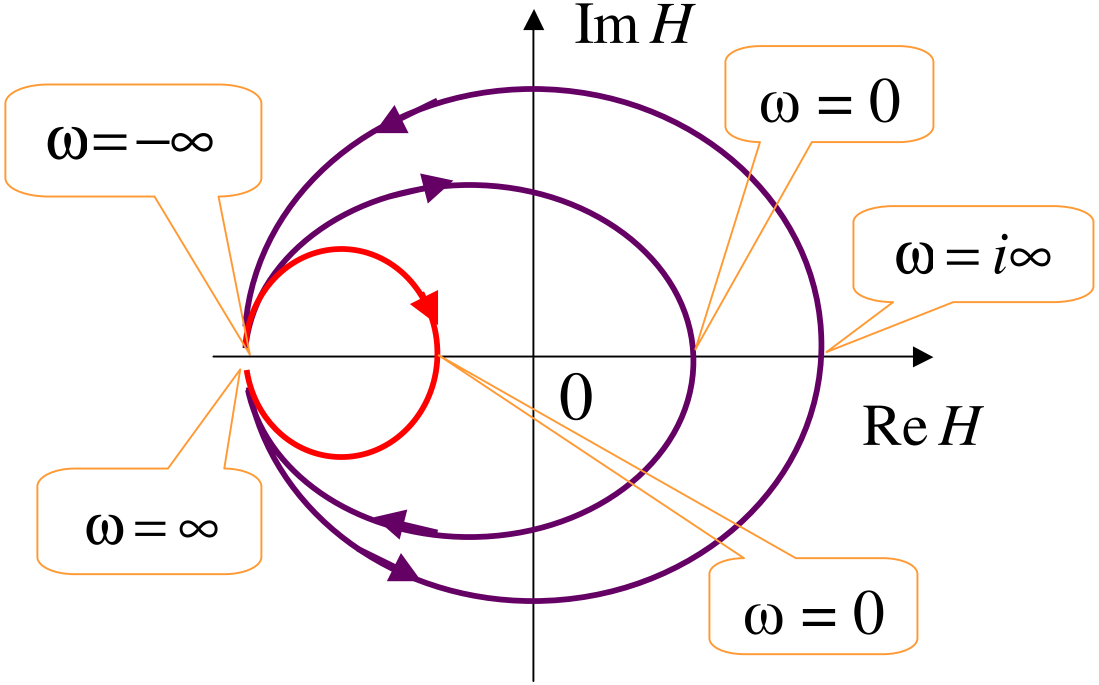
Dispersion equation

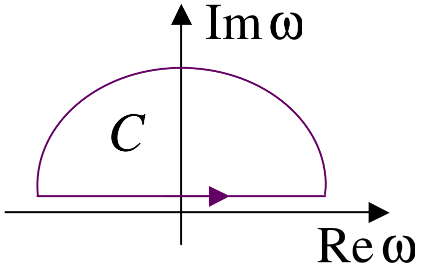
$$\boxed{k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0}$$

Existence of unstable modes – Penrose criterion

$$\boxed{H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)}$$

$$\oint_C \frac{d\omega}{2\pi i}\frac{1}{H(\omega)}\frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i}\frac{d\ln H(\omega)}{d\omega} = \ln H(\omega)\Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$





There are unstable modes if

$$\boxed{H(\omega = 0) < 0}$$

Anisotropy!

Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho\sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2+\sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

$$\rho = 6\ \mathrm{fm}^{-3}$$

$$\alpha_s = g^2/4\pi = 0.3$$

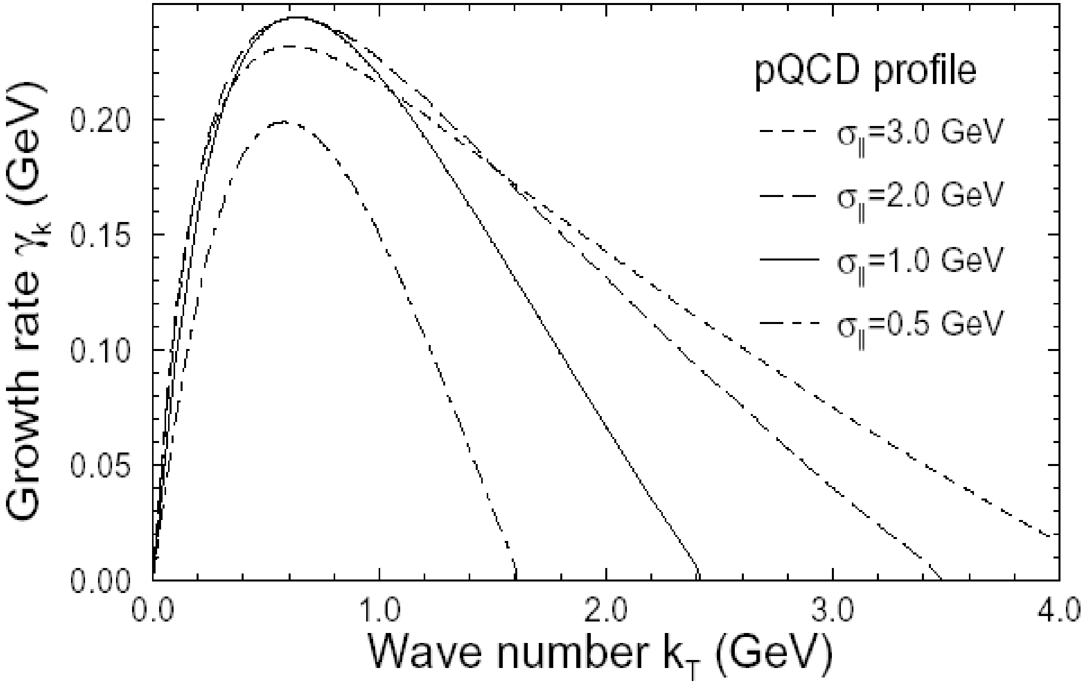
$$\sigma_\perp = 0.3\ \mathrm{GeV}$$

$$k^2 - \omega^2\varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i\,\gamma_k$$

$$0 < \gamma_k \in \Re$$



J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

Hard-Loop dynamics

Soft fields in the passive background of hard particles

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\text{eff}} = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\Big[f(\mathbf{p}) F^a_{\mu\nu}(x) \Big(\frac{p^\nu p^\rho}{(p\cdot D)^2}\Big)_{ab} F^{b\mu}_{\rho}(x) + i\frac{C_F}{3}\tilde{f}(\mathbf{p})\psi(x)\frac{p\cdot\gamma}{p\cdot D}\psi(x)\Big]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\mu \Lambda^\mu(p,q,k) = \Sigma(p) + \Sigma(q)$$

St. M., A. Rebhan & M. Strickland, Phys. Rev. D **74**, 025004 (2004)

Growth of instabilities – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

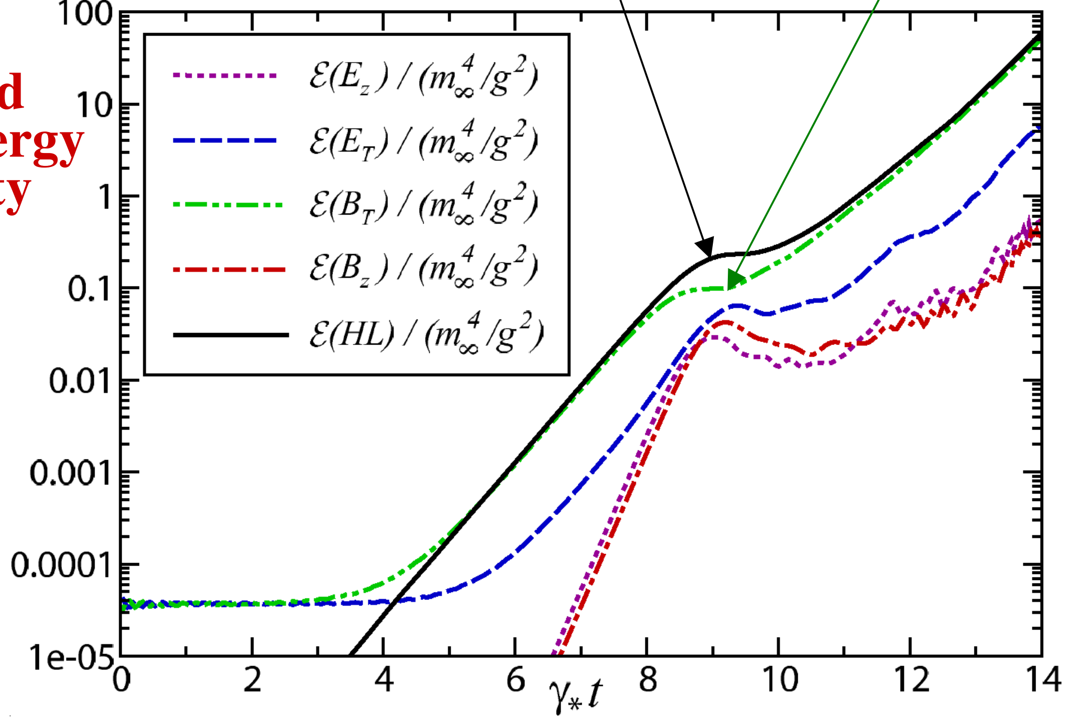
1+1 dimensions

$$A_a^\mu = A_a^\mu(t,z)$$

Scaled field energy density

total

transverse magnetic



Anisotropic particle's momentum distribution

$$f(\mathbf{p}) = f_{\text{iso}}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi}\int_0^\infty dp\, p^2 \frac{df_{\text{iso}}(p)}{dp}$$

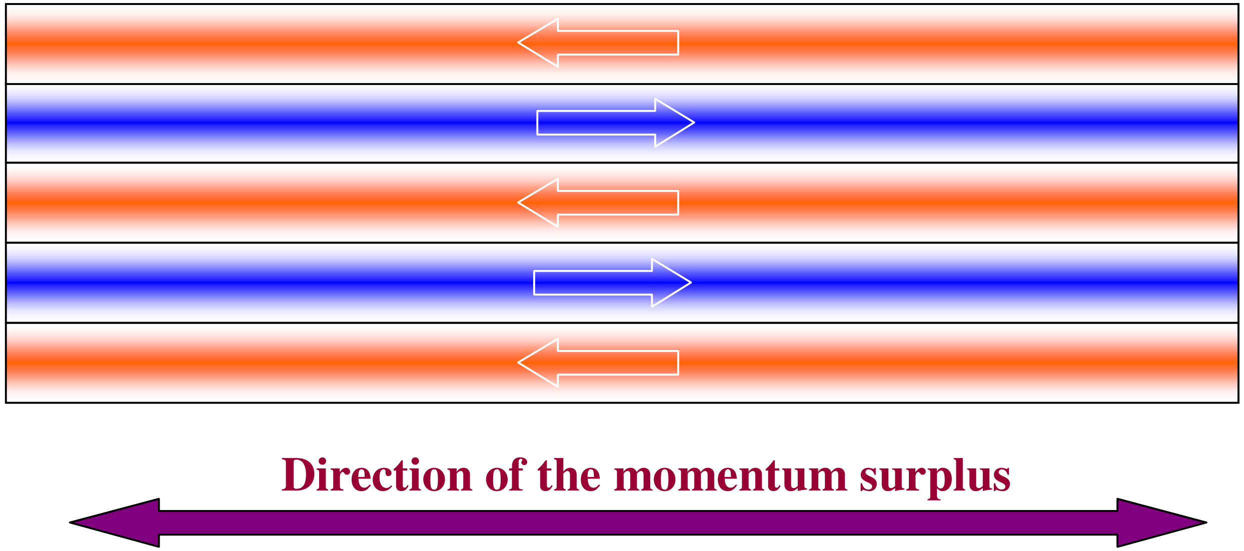
$$(m_D, \zeta)$$

Strong anisotropy $\zeta = 10$

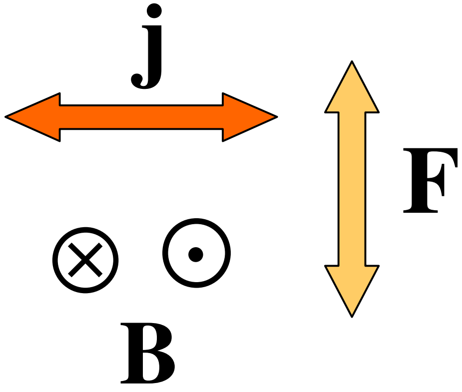
γ_* - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

Isotropization - particles

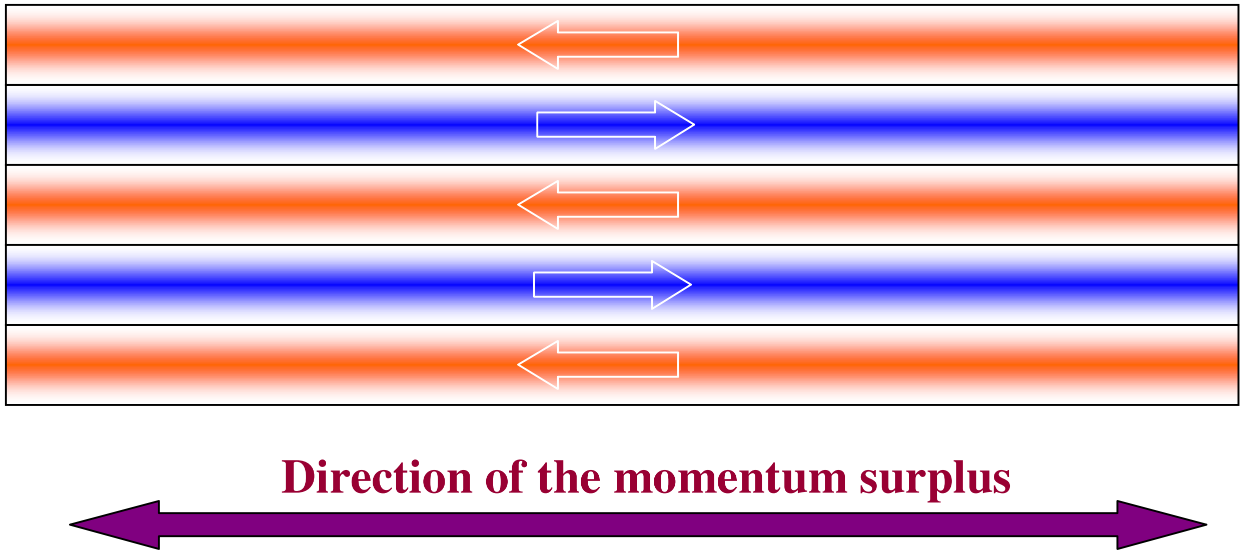


Direction of the momentum surplus

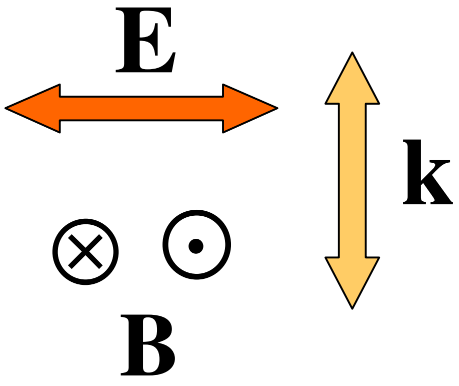


$$\boxed{\Delta\mathbf{p} = \int dt\, \mathbf{F}}$$

Isotropization - fields



Direction of the momentum surplus



$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^{a} \times \mathbf{E}^{a} \sim \mathbf{k}$$

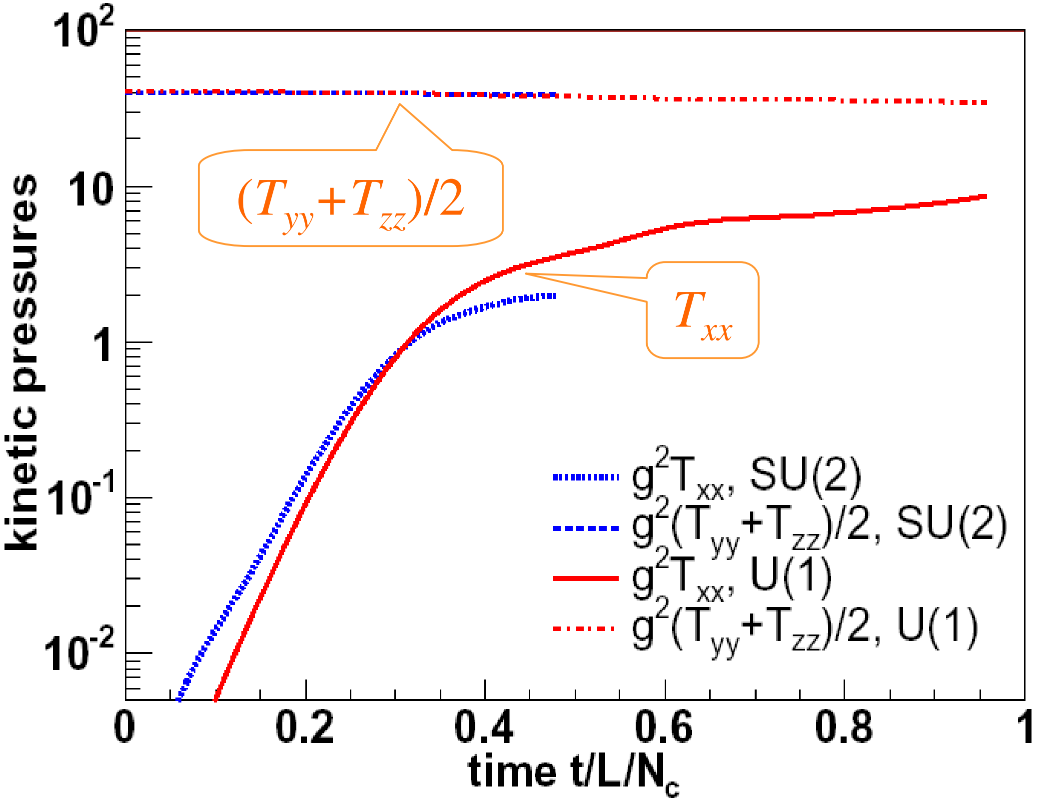
Isotropization – numerical simulation

Classical system of colored particles & fields

$$T_{ij} = \int \frac{d^3p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

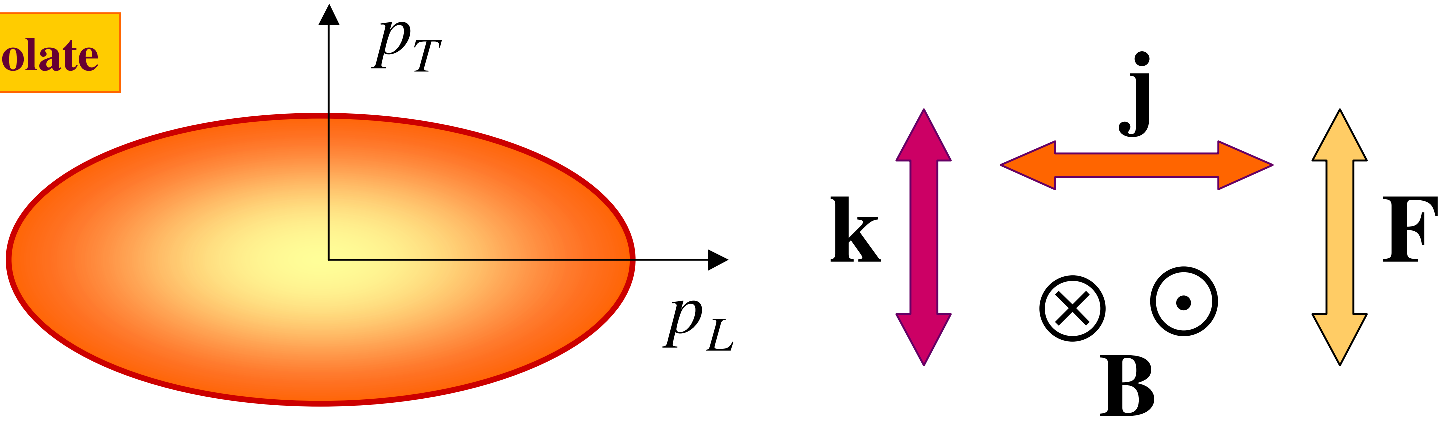
$$T_{xx} = (T_{yy} + T_{zz})/2$$



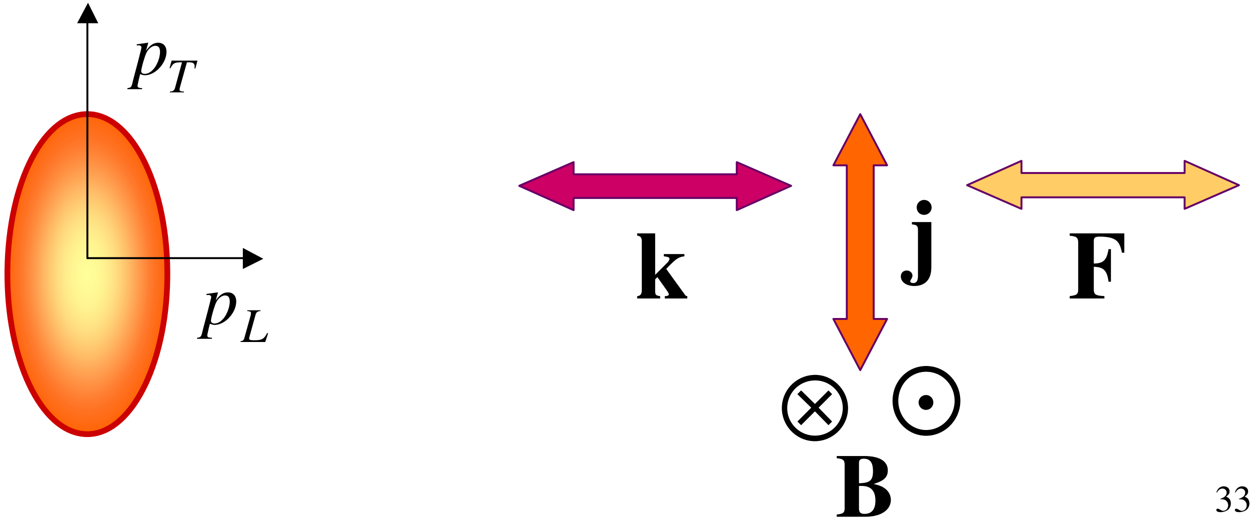
A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

Prolate vs. oblate

prolate



oblate



Conclusion

The scenario of instabilities driven equilibration provides a plausible solution to the fast equilibration problem of weakly coupled plasma