Transport theory of massless fields

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Using the Schwinger-Keldysh technique we discuss how to derive the transport equations for the system of massless quantum fields. We analyze the scalar field models with quartic and cubic interaction terms. In the ϕ^4 model the massive quasiparticles appear due to the self-interaction of massless bare fields. Therefore, the derivation of the transport equations strongly resembles one of the massive fields, but the subset of diagrams which provides the quasiparticle mass has to be resummed. The kinetic equation for the finite width quasiparticles is found, where, except for the mean-field and collision terms, there are terms which are absent in the standard Boltzmann equation. The structure of these terms is discussed. In the massless ϕ^3 model the massive quasiparticles do not emerge and presumably there is no transport theory corresponding to this model. It is not surprising since the ϕ^3 model is, in any case, ill defined. [S0556-2821(97)07016-1]

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I. INTRODUCTION

Transport theory is a very convenient tool to study many-body nonequilibrium systems, nonrelativistic as well as relativistic. The kinetic equations which play a central role in the transport approach can usually be derived by means of simple heuristic arguments similar to those which were used by Boltzmann over a hundred years ago when he introduced his famous equation. However, such arguments are insufficient when one studies a system of very complicated dynamics as the quark-gluon plasma governed by QCD. Then, one has to refer to a formal scheme which allows one to derive the transport equation directly from the underlying quantum field theory. The formal scheme is also needed to specify the limits of the kinetic approach. Indeed, the derivation shows the assumptions and approximations which lead to the transport theory, and hence the domain of its applicability can be established.

Until now the transport equations of the QCD plasma have been successfully derived in the mean-field or collisionless limit [1,2] and the structure of these equations is well understood [1–4]. In particular, it has been shown that in quasiequilibrium these equations provide [2,4] the so-called hard thermal loops [5]. The collisionless transport equations can be applied to a variety of problems. However, one needs the collision terms to discuss dissipative phenomena. In spite of some efforts [6–8], the general form of these terms in the transport equations of the quark-gluon plasma remains unknown.

The so-called Schwinger-Keldysh [9] formulation of quantum field theory provides a very promising basis to derive the transport equation beyond the mean-field limit. Kadanoff and Baym [10] developed the technique for nonrelativistic quantum systems, which has been further generalized to relativistic ones [11–19]. We mention here only the papers which provide a more or less systematic analysis of the collision terms.

The treatment of the massless fields, which are crucial for the gauge theories as QED or QCD, is particularly difficult when the transport equations are derived. Except the well-known infrared divergences which plague the perturbative expansion, there is a specific problem of nonequilibrium massless fields. The inhomogeneities in the system cause the off-mass-shell propagation of particles and then the perturbative analysis of the collision terms appears hardly tractable. More specifically, it appears very difficult, if possible at all, to express the field self-energy as the transition-matrix element squared and consequently we lose the probabilistic character of the kinetic theory. The problem is absent for the massive fields when the system is assumed homogeneous at the inverse mass or Compton scale. This is a natural assumption within the transport theory which, in any case, deals with the quantities averaged over a certain scale which can be identified with the Compton one.

The problem of the massless *nonequilibrium* fields has not been fully recognized in the existing literature. One has usually assumed, explicitly or implicitly, the on-mass-shell propagation. Such an assumption is indeed reasonable when the quasihomogeneous system near global equilibrium is considered [2]. However, the condition should be imposed that the inhomogeneity length is much larger than the inverse quasiparticle mass. It has also been shown on the phenomenological level [20] that the off-mass-shell propagation plays a very important role in the parton system which is far from equilibrium. Thus, we intend to develop a systematic approach to the transport of massless fields, which allows one to treat these fields in a very similar manner to the massive ones. The basic idea is rather obvious.

The fields which are massless in vacuum gain an effective mass in a medium due to the interaction. Therefore, the minimal scale at which the transport theory works is not an inverse bare mass, which is infinite for massless fields, but the inverse effective one. The staring point of the perturbative computation should no longer be free fields but the interacting ones. In physical terms, we postulate the existence of the massive quasiparticles and look for their transport equation.

At the technical level, we begin with the Lagrangian of

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the massless fields and make a formal trick which is well known in the quantum field theory at finite temperature, see, e.g., [21–29]. Namely, the auxiliary mass term is added to the free Lagrangian and then is subtracted due to a redefinition of the interaction term. As a result the subset of diagrams which contributes to the mass, which is determined in a self-consistent way, is effectively resummed in the perturbative expansion. A somewhat similar technique was applied to the kinetic theory in [18].

In this paper we show how the suggested method works for the self-interacting scalar fields. We discuss in detail the ϕ^3 and ϕ^4 models which appear to be qualitatively different. We successfully derive the transport equations for the ϕ^4 model and show why the method does not work for the ϕ^3 case. Our discussion closely follows the scheme of derivation which was earlier developed for the massive fields: self-interacting scalar fields [15] and the spinor fields interacting with the scalar and vector ones [16].

The main steps of the derivation are the following. We define the contour Green's function with the time arguments on the contour in a complex time plane. This function is a key element of the Schwinger-Keldysh approach. After discussing its properties and relevance for nonequilibrium systems, we write down the exact equations of motion, i.e., the Dyson-Schwinger equations. Assuming the macroscopic quasihomogeneity of the system, we perform the gradient expansion and the Wigner transformation. Then, the pair of Dyson-Schwinger equations are converted into the transport and mass-shell equations both satisfied by the Wigner function. The latter equation allows one to identify the initially introduced fictitious mass with the effective one generated by the interaction. We further perform the perturbative analysis showing how the Vlasov terms and the collisional ones emerge. Finally we define the distribution functions of standard probabilistic interpretation and find the transport equations satisfied by these functions.

Throughout this work we use natural units where $\hbar = c = 1$. The signature of the metric tensor is $(+,-,-,-)$. As long as possible, we keep the convention of Bjorken and Drell [30].

II. PRELIMINARIES

We consider the system of massless scalar fields with the Lagrangian density of the form

$$\mathcal{L}(x) = \frac{1}{2}\,\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{g}{n!}\,\phi^n(x), \tag{1}$$

where n equals 3 or 4. The renormalization counterterms are omitted in the Lagrangian. We introduce an auxiliary position-dependent mass $m_*(x)$ which can be treated as an external field. Specifically, we redefine the Lagrangian as

$$\mathcal{L}_m(x) = \frac{1}{2}\,\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}\,m_*^2(x)\phi^2(x) + \mathcal{L}_I(x) \tag{2}$$

with the interaction term

$$\mathcal{L}_I(x) = +\frac{1}{2}\,m_*^2(x)\phi^2(x) - \frac{g}{n!}\,\phi^n(x).$$

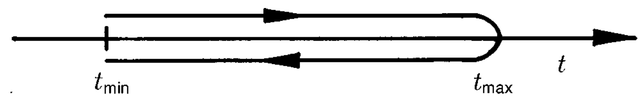

FIG. 1. The contour along the time axis for an evaluation of the operator expectation values.

The fields which satisfy the equation of motion

$$[\partial^2 + m_*^2(x)]\phi(x) = 0, \tag{3}$$

represent *free quasiparticles* with mass m_*. We observe that it is not *a priori* clear whether massive quasiparticles emerge due to the field self-interaction. It is even less clear whether the limit of free quasiparticles exist. As will be shown it is indeed the case for the ϕ^4 model, but not for the ϕ^3 one.

We write down the energy-momentum tensor defined as

$$T^{\mu\nu}(x) = \partial^\mu\phi(x)\partial^\nu\phi(x) - g^{\mu\nu}\mathcal{L}(x).$$

Subtracting the total derivative

$$\frac{1}{4}\,\partial^\mu\partial^\nu\phi^2(x) - g^{\mu\nu}\,\frac{1}{4}\,\partial^\sigma\partial_\sigma\phi^2(x),$$

we get the energy-momentum tensor which, for the free fields, is of a form convenient for our purposes: i.e.,

$$T_0^{\mu\nu}(x) = -\frac{1}{4}\,\phi(x)\overleftrightarrow{\partial^\mu}\overleftrightarrow{\partial^\nu}\phi(x). \tag{4}$$

The fields are assumed here to satisfy the equation of motion (3).

III. GREEN'S FUNCTIONS

The central role in our considerations plays the contour Green's function defined as

$$i\Delta(x,y) \stackrel{\rm def}{=} \langle \tilde{T}\phi(x)\phi(y)\rangle,$$

where the angular brackets denote the ensemble average at time t_0 (usually identified with $-\infty$); $\tilde{T}$ is the time-ordering operation along the directed contour shown in Fig. 1. The parameter $t_{\rm max}$ is shifted to $+\infty$ in the calculations. The time arguments are complex with an infinitesimal positive or negative imaginary part, which locates them on the upper or on the lower branch of the contour. The ordering operation is defined as

$$\tilde{T}\phi(x)\phi(y) \stackrel{\rm def}{=} \Theta(x_0,y_0)\phi(x)\phi(y) + \Theta(y_0,x_0)\phi(y)\phi(x),$$

where $\Theta(x_0,y_0)$ equals 1 if x_0 succeeds y_0 on the contour, and equals 0 when x_0 precedes y_0.

If the field is expected to develop a finite expectation value, as it happens when the symmetry is spontaneously broken, the contribution $\langle\phi(x)\rangle\langle\phi(y)\rangle$ is subtracted from the right-hand side of the equation defining the Green's function, see, e.g., [15,16]. Then, one concentrates on the field fluctuations around the expectation values. Since $\langle\phi(x)\rangle$ is expected to vanish in the models defined by the Lagrangians (1) we neglect this contribution in the Green's function definition.

We also use four other Green's functions with real-time arguments:

$$i\Delta^{>}(x,y)\overset{\text{def}}{=}\langle\phi(x)\phi(y)\rangle,$$

$$i\Delta^{<}(x,y)\overset{\text{def}}{=}\langle\phi(y)\phi(x)\rangle,$$

$$i\Delta^{c}(x,y)\overset{\text{def}}{=}\langle T^{c}\phi(x)\phi(y)\rangle,$$

$$i\Delta^{a}(x,y)\overset{\text{def}}{=}\langle T^{a}\phi(x)\phi(y)\rangle,$$

where $T^{c}(T^{a})$ prescribes (anti)chronological time ordering:

$$T^{c}\phi(x)\phi(y)\overset{\text{def}}{=}\Theta(x_0-y_0)\phi(x)\phi(y)+\Theta(y_0-x_0)\phi(y)\phi c(x),$$

$$T^{a}\phi(x)\phi(y)\overset{\text{def}}{=}\Theta(y_0-x_0)\phi(x)\phi(y)+\Theta(x_0-y_0)\phi(y)\phi(x).$$

These functions are related to the contour Green's functions in the following manner:

$$\Delta^{c}(x,y)\equiv\Delta(x,y)\quad\text{for } x_0,y_0 \text{ from the upper branch},$$

$$\Delta^{a}(x,y)\equiv\Delta(x,y)\quad\text{for } x_0,y_0 \text{ from the lower branch},$$

$$\Delta^{>}(x,y)\equiv\Delta(x,y)\quad\text{for } x_0 \text{ from the upper branch and } y_0 \text{ from the lower one},$$

$$\Delta^{<}(x,y)\equiv\Delta(x,y)\quad\text{for } x_0 \text{ from the lower branch and } y_0 \text{ from the upper one}.$$

One easily finds the identities which directly follow from the definitions

$$\Delta^{c}(x,y)=\Theta(x_0-y_0)\Delta^{>}(x,y)+\Theta(y_0-x_0)\Delta^{<}(x,y),$$

$$\Delta^{a}(x,y)=\Theta(y_0-x_0)\Delta^{>}(x,y)+\Theta(x_0-y_0)\Delta^{<}(x,y). \tag{5}$$

One also observes that

$$[i\Delta^{\gtrless}(x,y)]^{\dagger}=i\Delta^{\gtrless}(x,y),$$

$$[i\Delta^{a}(x,y)]^{\dagger}=i\Delta^{c}(x,y),$$

where † denotes Hermitian conjugation, i.e., complex conjugation with an exchange of the Green's-function arguments. Because the fields are real, the functions $i\Delta^{\gtrless}(x,y)$ satisfy the relation

$$\Delta^{>}(x,y)=\Delta^{<}(y,x). \tag{6}$$

It appears convenient to introduce the retarded $(+)$ and advanced $(-)$ Green's functions

$$\Delta^{\pm}(x,y)\overset{\text{def}}{=}\pm[\Delta^{>}(x,y)-\Delta^{<}(x,y)]\Theta(\pm x_0\mp y_0). \tag{7}$$

One immediately finds the identity

$$\Delta^{+}(x,y)-\Delta^{-}(x,y)=\Delta^{>}(x,y)-\Delta^{<}(x,y). \tag{8}$$

Let us now briefly discuss the physical interpretation of the Green's functions. The function $\Delta^{c}(x,y)$ describes the propagation of disturbance in which a single particle is added to the many-particle system in space-time point y and then is removed from it in a space-time point x. An antiparticle disturbance is propagated backward in time. The meaning of $\Delta^{a}(x,y)$ is analogous but particles are propagated backward in time and antiparticles forward. In the zero density limit $\Delta^{c}(x,y)$ coincides with the Feynman propagator.

The physical meaning of functions $\Delta^{>}(x,y)$ and $\Delta^{<}(x,y)$ is more transparent when one considers the Wigner transform defined as

$$\Delta^{\gtrless}(X,p)\overset{\text{def}}{=}\int d^4u\, e^{ipu}\Delta^{\gtrless}\left(X+\frac{1}{2}u, X-\frac{1}{2}u\right). \tag{9}$$

Then, the free-field energy-momentum tensor (4) averaged over an ensemble can be expressed as

$$\langle T_0^{\mu\nu}(X)\rangle=\int\frac{d^4p}{(2\pi)^4}\,p^{\mu}p^{\nu}i\Delta^{<}(X,p). \tag{10}$$

One recognizes the standard form of the energy-momentum tensor in the kinetic theory with the function $i\Delta^{<}(X,p)$ giving the density of particles with four-momentum p in a space-time point X. Therefore, $i\Delta^{<}(X,p)$ can be treated as a quantum analogue of the classical distribution function. Indeed, the function $i\Delta^{<}(X,p)$ is Hermitian. However, it is not positively definite and the probabilistic interpretation is only approximately valid. One should also observe that, in contrast to the classical distribution functions, $i\Delta^{<}(X,p)$ can be nonzero for the off-mass-shell four-momenta.

IV. GREEN'S-FUNCTION EQUATIONS OF MOTION

The Dyson-Schwinger equations satisfied by the contour Green's function are

$$[\partial_x^2+m_*^2(x)]\Delta(x,y) =-\delta^{(4)}(x,y)+\int_C d^4x'\,\Pi(x,x')\Delta(x',y), \tag{11}$$

$$[\partial_y^2+m_*^2(y)]\Delta(x,y) =-\delta^{(4)}(x,y)+\int_C d^4x'\,\Delta(x,x')\Pi(x',y), \tag{12}$$

where $\Pi(x,y)$ is the self-energy; the integration over x_0' is performed on the contour and the function $\delta^{(4)}(x,y)$ is defined on the contour as

$$\delta^{(4)}(x,y)=\begin{cases}\delta^{(4)}(x-y) & \text{for } x_0,y_0 \text{ from the upper branch,}\\ 0 & \text{for } x_0,y_0 \text{ from the different branches,}\\ -\delta^{(4)}(x-y) & \text{for } x_0,y_0 \text{ from the lower branch.}\end{cases}$$

Let us split the self-energy into three parts as

$$\Pi(x,y)=\Pi_\delta(x)\delta^{(4)}(x,y) +\Pi^>(x,y)\Theta(x_0,y_0)+\Pi^<(x,y)\Theta(y_0,x_0).$$

As we shall see later, Π_δ provides a dominant contribution to the mean-field while $\Pi^{\gtrless}$ determines the collision terms in the transport equations. Therefore, we call Π_δ the mean-field self-energy and $\Pi^{\gtrless}$ the collisional self-energy.

With the help of the retarded and advanced Green's functions (7) and the retarded and advanced self-energies defined in an analogous way, Eq. (11) and (12) can be rewritten as

$$[\partial_x^2+m_*^2(x)-\Pi_\delta(x)]\Delta^{\gtrless}(x,y) =\int d^4x'[\Pi^{\gtrless}(x,x')\Delta^-(x',y)+\Pi^+(x,x')\Delta^{\gtrless}(x',y)], \tag{13}$$

$$[\partial_y^2+m_*^2(y)-\Pi_\delta(y)]\Delta^{\gtrless}(x,y) =\int d^4x'[\Delta^{\gtrless}(x,x')\Pi^-(x',y)+\Delta^+(x,x')\Pi^{\gtrless}(x',y)], \tag{14}$$

where all time integrations run from $-\infty$ to $+\infty$.

Let us also write down the equations satisfied by the functions $\Delta^\pm$:

$$[\partial_x^2+m_*^2(x)-\Pi_\delta(x)]\Delta^\pm(x,y) =-\delta^{(4)}(x-y)+\int d^4x'\,\Pi^\pm(x,x')\Delta^\pm(x',y), \tag{15}$$

$$[\partial_y^2+m_*^2(y)-\Pi_\delta(y)]\Delta^\pm(x,y) =-\delta^{(4)}(x-y)+\int d^4x'\,\Delta^\pm(x,x')\Pi^\pm(x',y). \tag{16}$$

V. TOWARDS TRANSPORT EQUATIONS

The transport equations are derived under the assumption that the Green's functions and the self-energies depend weakly on the sum of their arguments, and that they are significantly different from zero only when the difference of their arguments is close to zero. To express these properties it is convenient to define a new set of variables as

$$\Delta(X,u)\equiv\Delta\left(X+\frac{1}{2}u,X-\frac{1}{2}u\right).$$

For homogeneous systems, the dependence on $X=(x+y)/2$ drops out entirely due to the translational invariance and $\Delta(x,y)$ depends only on $u=x-y$. For weakly inhomogeneous, or quasihomogeneous systems, the Green's functions and self-energies are assumed to vary slowly with X. We additionally assume that the Green's functions and self-energies are strongly *peaked* near $u=0$. The effective mass $m_*(x)$ is simply assumed to be weakly dependent on x.

We will now convert Eqs. (13) and (14) into transport equations by implementing the above approximation and performing the Wigner transformation (9) for all Green's functions and self-energies. This is done using the following set of translation rules which can be easily derived:

$$\int d^4x'\,f(x,x')g(x',y) \rightarrow f(X,p)g(X,p)+\frac{i}{2}\left[\frac{\partial f(X,p)}{\partial p_\mu}\frac{\partial g(X,p)}{\partial X^\mu} -\frac{\partial f(X,p)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu}\right],$$

$$h(x)g(x,y)\rightarrow h(X)g(X,p)-\frac{i}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu},$$

$$h(y)g(x,y)\rightarrow h(X)g(X,p)+\frac{i}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu},$$

$$\partial_x^\mu f(x,y) \rightarrow \left(-ip^\mu + \frac{1}{2}\,\partial^\mu \right) f(X,p),$$

$$\partial_y^\mu f(x,y) \rightarrow \left(ip^\mu + \frac{1}{2}\,\partial^\mu \right) f(X,p).$$

Here $X \equiv (x+y)/2$, $\partial^\mu \equiv \partial/\partial X_\mu$ and the functions $f(x,y)$ and $g(x,y)$ satisfy the assumptions discussed above. The function $h(x)$ is assumed to be weakly dependent on x.

Applying these translation rules to Eqs. (13) and (14), we obtain

$$\left(\frac{1}{4}\,\partial^2 - ip^\mu \partial_\mu - p^2 + m_*^2(X) - \Pi_\delta(X) - \frac{i}{2}\,\partial_\mu (m_*^2(X) - \Pi_\delta(X)) \partial_p^\mu \right) \Delta^{\gtrless}(X,p)$$

$$= \Pi^{\gtrless}(X,p)\Delta^-(X,p) + \Pi^+(X,p)\Delta^{\gtrless}(X,p) + \frac{i}{2}\,\{\Pi^{\gtrless}(X,p),\Delta^-(X,p)\} + \frac{i}{2}\,\{\Pi^+(X,p),\Delta^{\gtrless}(X,p)\}, \tag{17}$$

$$\left(\frac{1}{4}\,\partial^2 + ip^\mu \partial_\mu - p^2 + m_*^2(X) - \Pi_\delta(X) + \frac{i}{2}\,\partial_\mu (m_*^2(X) - \Pi_\delta(X)) \partial_p^\mu \right) \Delta^{\gtrless}(X,p)$$

$$= \Delta^{\gtrless}(X,p)\Pi^-(X,p) + \Delta^+(X,p)\Pi^{\gtrless}(X,p) + \frac{i}{2}\,\{\Delta^{\gtrless}(X,p),\Pi^-(X,p)\} + \frac{i}{2}\,\{\Delta^+(X,p),\Pi^{\gtrless}(X,p)\}, \tag{18}$$

where we have introduced the Poisson-like brackets defined as

$$\{C(X,p),D(X,p)\} \equiv \frac{\partial C(X,p)}{\partial p_\mu}\,\frac{\partial D(X,p)}{\partial X^\mu} - \frac{\partial C(X,p)}{\partial X^\mu}\,\frac{\partial D(X,p)}{\partial p_\mu}.$$

The kinetic theory deals only with averaged system characteristics. Thus, one usually assumes that the system is homogeneous on a scale of the Compton wavelength of the quasiparticles. In other words, the characteristic length of inhomogeneities is assumed to be much larger than the inverse mass of quasiparticles. Therefore, we impose the condition

$$|\Delta^{\gtrless}(X,p)| \gg \left| \frac{1}{m_*^2}\,\partial^2 \Delta^{\gtrless}(X,p) \right|, \tag{19}$$

which leads to *the quasiparticle approximation*. As discussed in the next section and in the Appendix, the requirement (19) renders the off-shell contributions to the Green's functions $\Delta^{\gtrless}$ negligible. Thus, we deal with the quasiparticles having on-mass-shell momenta. Unfortunately, the assumption (19) cannot be applied to massless particles and for this reason we have introduced the effective mass m_*.

Let us now take the difference and the sum of Eqs. (17) and (18), where the ∂^2 terms have been neglected due to the quasiparticle approximation (19). Then, one gets

$$\left[p^\mu \partial_\mu + \frac{1}{2}\,\partial_\mu (m_*^2(X) - \Pi_\delta(X)) \partial_p^\mu \right] \Delta^{\gtrless}(X,p)$$

$$= \frac{i}{2}\,[\Pi^>(X,p)\Delta^<(X,p) - \Pi^<(X,p)\Delta^>(X,p)]$$

$$- \frac{1}{4}\,\{\Pi^{\gtrless}(X,p),\Delta^+(X,p) \quad + \Delta^-(X,p)\}$$

$$- \frac{1}{4}\,\{\Pi^+(X,p) + \Pi^-(X,p),\Delta^{\gtrless}(X,p)\}, \tag{20}$$

$$[-p^2 + m_*^2(X) - \Pi_\delta(X)]\Delta^{\gtrless}(X,p) = \frac{1}{2}\,\{\Pi^{\gtrless}(X,p)[\Delta^+(X,p) + \Delta^-(X,p)] + [\Pi^+(X,p) + \Pi^-(X,p)]\Delta^{\gtrless}(X,p)\}$$

$$+ \frac{i}{4}\,\{\Pi^>(X,p),\Delta^<(X,p)\} - \frac{i}{4}\,\{\Pi^<(X,p),\Delta^>(X,p)\}, \tag{21}$$

where we have used the identity (8) applied to the Green's functions and self-energies.

One recognizes Eq. (20) as a transport equation while Eq. (21) as a so-called mass-shell equation. We will write down these equation in a more compact way. From the definition (7) one finds that

$$\Delta^\pm(X,p) = \pm\frac{1}{2}\,[\Delta^>(X,p) - \Delta^<(X,p)] + \frac{1}{2\pi i}\,P\int d\omega'\,\frac{\Delta^>(X,\omega',\mathbf{p}) - \Delta^<(X,\omega',\mathbf{p})}{\omega - \omega'}. \tag{22}$$

The first term on the right-hand side (RHS) is anti-Hermitian while the second one is Hermitian. Thus, we introduce

$$\mathrm{Im}\ \Delta^{\pm}(X,p) \equiv \pm\frac{1}{2i}\,[\Delta^{>}(X,p)-\Delta^{<}(X,p)], \tag{23}$$

$$\mathrm{Re}\ \Delta^{\pm}(X,p) \equiv \frac{1}{2\pi i}\,P\int d\omega'\,\frac{\Delta^{>}(X,\omega',\mathbf{p})-\Delta^{<}(X,\omega',\mathbf{p})}{\omega-\omega'}. \tag{24}$$

With the help of Eq. (24) and analogous formulas for $\Pi^{\pm}$, Eqs. (20) and (21) can be rewritten as

$$\{p^2-m_*^2(X)+\Pi_\delta(X)+\mathrm{Re}\ \Pi^{+}(X,p),\Delta^{\gtrless}(X,p)\} = i[\Pi^{>}(X,p)\Delta^{<}(X,p)-\Pi^{<}(X,p)\Delta^{>}(X,p)]-\{\Pi^{\gtrless}(X,p),\mathrm{Re}\ \Delta^{+}(X,p)\}, \tag{25}$$

$$[p^2-m_*^2(X)+\Pi_\delta(X)+\mathrm{Re}\Pi^{+}(X,p)]\Delta^{\gtrless}(X,p) = -\Pi^{\gtrless}(X,p)\mathrm{Re}\Delta^{+}(,p)-\frac{i}{4}\{\Pi^{>}(X,p),\Delta^{<}(X,p)\}+\frac{i}{4}\{\Pi^{<}(X,p),\Delta^{>}(X,p)\}. \tag{26}$$

In the case of fields with finite bare mass, the gradient terms on the right-hand sides of Eqs. (25) and (26) are *small* [15,16] and are usually neglected. When the bare fields are massless, as those studied here, there is no reason to neglect the gradient terms. The equation analogous to Eq. (25) was derived earlier in [10,11].

It appears useful to write down the transport and mass-shell equations satisfied by the retarded and advanced Green's functions. Starting with Eqs. (15) and (16) one finds

$$\{p^2-m_*^2(X)+\Pi_\delta(X)+\Pi^{\pm}(X,p),\Delta^{\pm}(X,p)\}=0, \tag{27}$$

$$[p^2-m_*^2(X)+\Pi_\delta(X)+\Pi^{\pm}(X,p)]\Delta^{\pm}(X,p)=1. \tag{28}$$

We observe that the gradient terms drop out entirely in Eq. (28). Nevertheless, the equation holds within the first order of gradient expansion. Because of the absence of the gradients, Eq. (28) can be immediately solved as

$$\Delta^{\pm}(X,p)=\frac{1}{p^2-m_*^2(X)+\Pi_\delta(X)+\Pi^{\pm}(X,p)}. \tag{29}$$

One notices that $\Delta^{\pm}$ of the form (29) solves not only Eq. (28) but Eq. (27) as well. Indeed, any function f of K satisfies the equation $\{K,f(K)\}=0$.

The real and imaginary parts of $\Delta^{\pm}(X,p)$, which are needed in our further considerations, are

$$\mathrm{Re}\ \Delta^{\pm}(X,p)=\frac{p^2-m_*^2(X)+\Pi_\delta(X)+\mathrm{Re}\ \Pi^{+}(X,p)}{[p^2-m_*^2(X)+\Pi_\delta(X)+\mathrm{Re}\ \Pi^{+}(X,p)]^2+[\mathrm{Im}\ \Pi^{+}(X,p)]^2}, \tag{30}$$

$$\mathrm{Im}\ \Delta^{\pm}(X,p)=\frac{\pm\mathrm{Im}\ \Pi^{+}(X,p)}{[p^2-m_*^2(X)+\Pi_\delta(X)+\mathrm{Re}\ \Pi^{+}(X,p)]^2+[\mathrm{Im}\ \Pi^{+}(X,p)]^2}. \tag{31}$$

VI. FREE QUASIPARTICLES

Before further analysis the equations obtained in the previous section we consider here a very important limit which corresponds to the free quasiparticles. Specifically, we assume that $\Pi_\delta=\Pi^{\gtrless}=0$. Then, Eqs. (25) and (26) read

$$\left(p^\mu\partial_\mu+\frac{1}{2}\,\partial_\mu m_*^2(X)\partial_p^\mu\right)\Delta_0^{\gtrless}(X,p)=0, \tag{32}$$

$$[p^2-m_*^2(X)]\Delta_0^{\gtrless}(X,p)=0. \tag{33}$$

Although the quasiparticles are assumed to be free, the transport equation is of the Vlasov, not of the free form. This is a simple consequence of the X dependence of the effective mass.

Because of Eq. (33), $\Delta_0^{\gtrless}(X,p)$ is proportional to $\delta(p^2-m_*^2)$, and consequently *free quasiparticles are always on mass shell*. If the quasiparticle approximation (19) is *not* applied, the mass-shell equation gets the form

$$\left(\frac{1}{4}\,\partial^2-p^2+m_*^2(X)\right)\Delta_0^{\gtrless}(X,p)=0,$$

and the off-shell contribution to the Green's function $\Delta_0^{\gtrless}$ is nonzero. A detailed discussion of the quasiparticle approximation is given in the Appendix.

We also discuss the (anti)chronological Green's functions $\Delta_0^{c(a)}$ in the limit of free quasiparticles. One easily finds their equations of motion as

$$\left(p^{\mu}\partial_{\mu}+\frac{1}{2}\,\partial_{\mu}m_{*}^{2}(X)\partial_{p}^{\mu}\right)\Delta_{0}^{c}(X,p)=0,$$

$$[p^{2}-m_{*}^{2}]\Delta_{0}^{c}(X,p)=1.$$

For the antichronological function Δ^{a}, the right-hand side of the mass-shell equation equals -1 instead of $+1$. The solution of these equation can be written as

$$\Delta_{0}^{c}(X,p)=\frac{1}{p^{2}-m_{*}^{2}+i0^{+}}+\Theta(-p_{0})\Delta_{0}^{>}(X,p)$$
$$+\Theta(p_{0})\Delta_{0}^{<}(X,p),$$

where $\Delta_{0}^{\gtrless}(X,p)$ is assumed to satisfy Eqs. (32) and (33). It is worth mentioning that any function which depends on (X,p) through $(p^{2}-m_{*}^{2})$ solves the Vlasov equation (32). $\Delta_{0}^{c}(X,p)$ obeys the initial condition of the standard Feynman propagator. It also satisfies relation (5).

The antichronological Green's function is

$$\Delta_{0}^{a}(X,p)=\frac{-1}{p^{2}-m_{*}^{2}-i0^{+}}+\Theta(p_{0})\Delta_{0}^{>}(X,p)$$
$$+\Theta(-p_{0})\Delta_{0}^{<}(X,p).$$

Knowing Δ_{0}^{c} and Δ_{0}^{a} one immediately gets the retarded and advanced functions

$$\Delta_{0}^{\pm}(X,p)=\frac{1}{p^{2}-m_{*}^{2}\pm ip_{0}0^{+}}, \tag{34}$$

which obey the respective initial conditions. Confronting the expressions of $\Delta^{\pm}$ for free (34) and interacting quasiparticles (29), one finds that

$$\mathrm{Im}\;\Pi^{+}(X,p)=-\mathrm{Im}\;\Pi^{-}(X,p)=\begin{cases}0^{+} & \text{for } p_{0}>0,\\ 0^{-} & \text{for } p_{0}<0,\end{cases} \tag{35}$$

in the limit of free quasiparticles.

It appears useful to express the Green's functions $\Delta_{0}^{\gtrless}$ through the distribution function f_{0} as

$$\Theta(p_{0})i\Delta_{0}^{<}(X,p)=\Theta(p_{0})2\pi\delta(p^{2}-m_{*}^{2})f_{0}(X,\mathbf{p})$$
$$=\frac{\pi}{E_{p}}\,\delta(E_{p}-p_{0})f_{0}(X,\mathbf{p}), \tag{36}$$

where $E_{p}\equiv\sqrt{\mathbf{p}^{2}+m_{*}^{2}}$. This equation should be treated as a definition of f_{0}.

Because of relation (6) we have

$$\Delta^{<}(X,p)=\Delta^{>}(X,-p),$$

and consequently

$$\Theta(p_{0})i\Delta_{0}^{>}(X,-p)=\Theta(p_{0})2\pi\delta(p^{2}-m_{*}^{2})f_{0}(X,\mathbf{p})$$
$$=\frac{\pi}{E_{p}}\,\delta(E_{p}-p_{0})f_{0}(X,\mathbf{p}). \tag{37}$$

In that way we express the positive-energy part of $\Delta_{0}^{<}(X,p)$ and the negative-energy part of $\Delta_{0}^{>}(X,p)$ through $f_{0}(X,p)$. We extend these expressions to the whole energy domain using identity (8). With the help of the explicit form of the retarded and advanced functions (34) we get the formula

$$i\Delta_{0}^{>}(X,p)-i\Delta_{0}^{<}(X,p)$$
$$=2\pi\delta(p^{2}-m_{*}^{2})[\Theta(p_{0})-\Theta(-p_{0})], \tag{38}$$

which is discussed in detail in the next section.

Combining Eqs. (36) and (37) with Eq. (38), one finds the desired expression of the Green's functions $\Delta_{0}^{\gtrless}$ in terms of the distribution function f_{0}. Namely,

$$i\Delta_{0}^{<}(X,p)=\frac{\pi}{E_{p}}\,\delta(E_{p}-p_{0})f_{0}(X,\mathbf{p})$$
$$+\frac{\pi}{E_{p}}\,\delta(E_{p}+p_{0})[f_{0}(X,-\mathbf{p})+1], \tag{39}$$

$$i\Delta_{0}^{>}(X,p)=\frac{\pi}{E_{p}}\,\delta(E_{p}-p_{0})[f_{0}(X,\mathbf{p})+1]$$
$$+\frac{\pi}{E_{p}}\,\delta(E_{p}+p_{0})f_{0}(X,-\mathbf{p}). \tag{40}$$

When the system is in thermodynamical equilibrium the distribution functions reads

$$f_{0}^{\mathrm{eq}}(\mathbf{p})=\frac{1}{e^{\beta^{\mu}p_{\mu}}-1}, \tag{41}$$

where $\beta^{\mu}\equiv u^{\mu}/T$ with u^{μ} being the hydrodynamical velocity and T the temperature. In the *local* equilibrium the two parameters are X dependent.

VII. SPECTRAL FUNCTION

In this section we introduce one more function which appears useful in the analysis of the interacting systems. The spectral function A is defined as

$$A(x,y)\overset{\mathrm{def}}{=}i\Delta^{>}(x,y)-i\Delta^{<}(x,y).$$

Thus,

$$A(x,y)=\langle[\phi(x),\phi(y)]\rangle,$$

where $[\phi(x),\phi(y)]$ denotes the field commutator.

Because of the equal-time commutation relations

$$[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})]=0,\quad [\dot{\phi}(t,\mathbf{x}),\phi(t,\mathbf{y})]=-i\,\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

with the dot denoting the time derivative, the Wigner transformed spectral function satisfies the two identities

$$\int\frac{dp_{0}}{2\pi}\,A(X,p)=0,\qquad\int\frac{dp_{0}}{2\pi}\,p_{0}A(X,p)=1. \tag{42}$$

One also sees that [cf. Eq. (23)]

$$A(X,p) = \mp 2 \mathrm{Im} \Delta^{\pm}(X,p). \quad (43)$$

Finally, we observe that the identity (6), which holds for the real fields, provides the relation

$$A(X,p) = -A(X,-p).$$

From the transport and mass-shell equations (25) and (26) one immediately finds the equations for $A(X,p)$ as

$$\{p^2 - m_*^2(X) + \Pi_\delta(X) + \mathrm{Re}\ \Pi^+(X,p), A(X,p)\}$$
$$= 2\{\mathrm{Im}\ \Pi^+(X,p), \mathrm{Re}\ \Delta^+(X,p)\}, \quad (44)$$

$$[p^2 - m_*^2(X) + \Pi_\delta(X) + \mathrm{Re}\ \Pi^+(X,p)] A(X,p)$$
$$= 2\ \mathrm{Im}\ \Pi^+(X,p) \mathrm{Re}\ \Delta^+(X,p). \quad (45)$$

Substituting Re $\Delta^+(X,p)$ from Eq. (30) into the algebraic equation (45) we find its solution as

$$A(X,p) = \frac{2\ \mathrm{Im}\ \Pi^+(X,p)}{[p^2 - m_*^2(X) + \Pi_\delta(X) + \mathrm{Re}\ \Pi^+(X,p)]^2 + [\mathrm{Im}\ \Pi^+(X,p)]^2}. \quad (46)$$

Then, one easily shows that the function of the form (46) solves Eq. (44) as well. In fact, the spectral function (46) can be found directly from Eq. (30) due to the relation (43).

The spectral function of the free quasiparticles can be, obviously, obtained from Eq. (46) but the limit should be taken with care. We first write the spectral function (46) as

$$A(X,p) = \frac{i}{p^2 - m_*^2(X) + \Pi_\delta(X) + \mathrm{Re}\ \Pi^+(X,p) + i\ \mathrm{Im}\ \Pi^+(X,p)} - \frac{i}{p^2 - m_*^2(X) + \Pi_\delta(X) + \mathrm{Re}\ \Pi^+(X,p) - i\ \mathrm{Im}\ \Pi^+(X,p)}.$$

Then we take the limit $\Pi \to 0$ keeping in mind condition (35). Using the well-known identity

$$\frac{1}{x \pm i0^+} = P\ \frac{1}{x} \mp i\pi\delta(x),$$

we get the spectral function of noninteracting quasiparticles as

$$A_0(X,p) = 2\pi\delta(p^2 - m_*^2)[\Theta(p_0 - \Theta(-p_0)], \quad (47)$$

which, of course, coincides with Eq. (38).

Let us also consider a specific approximate form of the spectral function. If the condition

$$\mathbf{p}^2 + m_*^2(X) - \Pi_\delta(X) - \mathrm{Re}\ \Pi^+(X,p) \gg |\mathrm{Im}\ \Pi^+(X,p)|$$

is satisfied, A as a function of p_0 is close to zero everywhere except two narrow regions around $p_0 = \pm E_\pm (E_\pm > 0)$ which solve the equations

$$E_\pm^2(X,\mathbf{p}) = \mathbf{p}^2 + m_*^2(X) - \Pi_\delta(X) - \mathrm{Re}\ \Pi^+(X, p_0 = \pm E_\pm, \mathbf{p}).$$

Then, the function (46) can be approximated as

$$A(X,p) = \frac{1}{E_+(X,\mathbf{p})} \frac{\Gamma_+(X,\mathbf{p})}{[E_+(X,\mathbf{p}) - p_0]^2 + \Gamma_+^2(X,\mathbf{p})} - \frac{1}{E_-(X,\mathbf{p})} \frac{\Gamma_-(X,\mathbf{p})}{[E_-(X,\mathbf{p}) + p_0]^2 + \Gamma_-^2(X,\mathbf{p})}, \quad (48)$$

with

$$\Gamma_\pm(X,\mathbf{p}) \equiv \pm \frac{\mathrm{Im}\ \Pi^+(X, p_0 = \pm E_\pm, \mathbf{p})}{2E_\pm(X,\mathbf{p})}. \quad (49)$$

One easily checks that the spectral function of the form (48) satisfies the sum rules (42).

VIII. PERTURBATIVE EXPANSION

As discussed in, e.g., [13,14,31] the contour Green's functions admit a perturbative expansion very similar to that known from vacuum field theory with essentially the same Feynman rules. However, the time integrations do not run from $-\infty$ to $+\infty$, but along the contour shown in Fig. 1. The right turning point of the contour (t_{max}) must be above the largest time argument of the evaluated Green's function. In practice, t_0 is shifted to $-\infty$ and t_{max} to $+\infty$. The second difference is the appearance of tadpoles, i.e., loops formed by single lines, which give zero contribution in the vacuum case. A tadpole corresponds to a Green's function with two equal space-time arguments. Since the Green's function $\Delta(x,y)$ is not well defined for $x=y$ we ascribe the function $-i\Delta^<(x,x)$ to each tadpole. The rest of Feynman rules can be taken from the textbook of Bjorken and Drell [30].

In this section we consider the lowest-order contributions to the self-energies. It should be stressed that the Green's functions, which are represented by the lines of the Feynman diagrams, correspond to those of free quasiparticles not of noninteracting fields.

A. ϕ^4 model

The lowest-order contribution to the self-energy which is associated with the graphs from Fig. 2 is

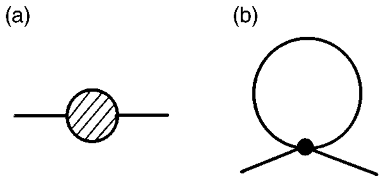

FIG. 2. The lowest-order diagrams of the self-energy in the ϕ^4 model. The bubble in (a) denotes the additional interaction due to the effective mass.

$$\Pi(x,y)=\delta^{(4)}(x,y)\left(m_*^2(x)-\frac{ig}{2}\,\Delta_0^<(x,x)\right),$$

giving

$$\Pi_\delta(x)=m_*^2(x)-\frac{ig}{2}\,\Delta_0^<(x,x), \tag{50}$$

and

$$\Pi^>(x,y)=\Pi^<(x,y)=0.$$

Substituting $\Delta_0^<$ given by Eq. (39), where $\Delta_0^<$ is expressed through the distribution function, into Eq. (50) one finds Π_δ as

$$\Pi_\delta(x)=m_*^2(x)-\frac{g}{2}\int\frac{d^3p}{(2\pi)^3 2E_p}\,[2f_0(x,\mathbf{p})+1].$$

As seen the integral is quadratically divergent even in the vacuum limit when $f_0\to 0$. This type of divergence, which appears due to the zero-mode fluctuations, is well known in the field theory. We remove it by subtracting the vacuum value from Π_δ. Thus one gets

$$\Pi_\delta(x)=m_*^2(x)-\frac{g}{2}\int\frac{d^3p}{(2\pi)^3E_p}\,f_0(x,\mathbf{p}). \tag{51}$$

We compute Π_δ for the equilibrium system when the distribution function is given by Eq. (41) with $u^\mu=(1,0,0,0)$. For $T\gg m_*$ we get after elementary integration the well-known result, see e.g., [26],

$$\Pi_\delta(x)=m_*^2(x)-\frac{gT^2(x)}{24}, \tag{52}$$

where the temperature is x dependent in the case of *local* equilibrium.

B. ϕ^3 model

The lowest-order contribution to the self-energy corresponding to the graphs from Figs. 3(a) and 3(b) is

$$\Pi_a(x,y)=\delta^{(4)}(x,y)\left(m_*^2-\frac{i}{2}\,g^2\int_C d^4x'\,\Delta_0(x',x)\Delta_0^<(x',x')\right).$$

Locating the argument x on the upper branch of the contour, one finds

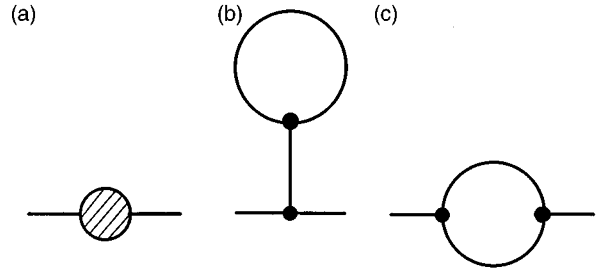

FIG. 3. The lowest-order diagrams of the self-energy in the ϕ^3 model. The bubble in (a) denotes the additional interaction due to the effective mass.

$$\Pi_\delta(x)=m_*^2-\frac{i}{2}\,g^2\int d^4x'[\Delta_0^c(x',x)-\Delta_0^<(x',x)]\Delta_0^<(x',x'),$$

where the time integration runs from $-\infty$ to $+\infty$. Observing that $\Delta^c-\Delta^<=\Delta^+$ we get

$$\Pi_\delta(x)=m_*^2-\frac{i}{2}\,g^2\int d^4x'\,\Delta_0^+(x',x)\Delta_0^<(x',x').$$

Using the explicit form of Δ_0^+ (34) and expressing $\Delta_0^<$ through the distribution function, one finds

$$\Pi_\delta(x)=m_*^2(x)-\frac{g^2}{2}\int\frac{d^4x'\,d^4p}{(2\pi)^4}\,\frac{d^3k}{(2\pi)^3E_k}\times\frac{e^{-ip(x'-x)}}{p^2-m_*^2+ip_0 0^+}\,f_0(x',\mathbf{k}), \tag{53}$$

where as previously we have subtracted the vacuum contribution.

In the case of global equilibrium, when $f_0(x,\mathbf{k})$ is independent of x, the integral from Eq. (53) can be computed as

$$\Pi_\delta=m_*^2+\frac{g^2}{2m_*^2}\int\frac{d^3k}{(2\pi)^3E_k}\,f_0(\mathbf{k}). \tag{54}$$

In the limit $T\gg m_*$ one finally finds

$$\Pi_\delta=m_*^2+\frac{g^2}{24}\,\frac{T^2}{m_*^2}. \tag{55}$$

The graph from Fig. 3(c) corresponds to

$$\Pi_b(x,y)=-\frac{i}{2}\,g^2\Delta_0(x,y)\Delta_0(y,x),$$

and it gives

$$\Pi^{\gtrless}(x,y)=-\frac{i}{2}\,g^2\Delta_0^{\gtrless}(x,y)\Delta_0^{\lessgtr}(y,x). \tag{56}$$

Substituting $\Delta_0^{\gtrless}$ expressed through the distribution function to Eq. (56) one finds

$$\Pi^{>}(X,p)=\frac{i}{2}g^2\int\frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})\{\delta(p_0-E_k-E_q)f_0(X,\mathbf{k})f_0(X,-\mathbf{q})$$
$$+\delta(p_0-E_k+E_q)f_0(X,\mathbf{k})[f_0(X,\mathbf{q})+1]+\delta(p_0+E_k-E_q)[f_0(X,-\mathbf{k})+1]f_0(X,-\mathbf{q})$$
$$+\delta(p_0+E_k+E_q)[f_0(X,-\mathbf{k})+1][f_0(X,\mathbf{q})+1]\}, \tag{57}$$

$$\Pi^{<}(X,p)=\frac{i}{2}g^2\int\frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k})\{\delta(p_0-E_k-E_q)[f_0(X,\mathbf{k})+1][f_0(X,-\mathbf{q})+1]$$
$$+\delta(p_0-E_k+E_q)[f_0(X,\mathbf{k})+1]f_0(X,\mathbf{q})+\delta(p_0+E_k-E_q)f_0(X,-\mathbf{k})[f_0(X,-\mathbf{q})+1]$$
$$+\delta(p_0+E_k+E_q)f_0(X,-\mathbf{k})f_0(X,\mathbf{q})\}. \tag{58}$$

Let us observe here that due to the trivial kinematical reasons $\Pi^{<}(X,p)=\Pi^{<}(X,p)=0$ for on-mass-shell four-momenta p, i.e., when $p^2=m_*^2$.

We further compute $\Pi^{<}(X,p)$ for $\mathbf{p}=0$. In the case of equilibrium when the distribution function is of the form (41) and, as previously, $u^\mu=(1,0,0,0)$, one finds $\Pi^{\gtrless}(X,p)$ for $p=(\pm\omega,\mathbf{0})$ with $\omega>0$ as

$$\Pi^{>}(X,\omega,\mathbf{0})=\Pi^{<}(X,-\omega,\mathbf{0})=-\frac{ig^2}{16\pi}\Theta(\omega-2m_*)\sqrt{1-4m_*^2/\omega^2}\,\frac{1}{[\exp(\beta\omega/2)-1]^2}, \tag{59}$$

$$\Pi^{<}(X,\omega,\mathbf{0})=\Pi^{>}(X,-\omega,\mathbf{0})=-\frac{ig^2}{16\pi}\Theta(\omega-2m_*)\sqrt{1-4m_*^2/\omega^2}\,\frac{\exp(\beta\omega)}{[\exp(\beta\omega/2)-1]^2}. \tag{60}$$

We also compute

$$\mathrm{Im}\,\Pi^{+}(X,\pm\omega,\mathbf{0})=\frac{1}{2i}[\Pi^{>}(X,\pm\omega,\mathbf{0})-\Pi^{<}(X,\pm\omega,\mathbf{0})]=\pm\frac{g^2}{16\pi}\Theta(\omega-2m_*)\sqrt{1-4m_*^2/\omega^2}\,\frac{1}{\exp(\beta\omega/2)-1}.$$

One sees that the condition (35) is indeed satisfied by the perturbative self-energy when $g\to 0$ and m_* is kept fixed.

IX. QUASIPARTICLE MASS

The structure of Eqs. (25) and (26) motivates the definition of the quasiparticle mass as a solution of the equation

$$\Pi_\delta(X)+\mathrm{Re}\,\Pi^{+}(X,p_0=m_*,\mathbf{p}=0)=0. \tag{61}$$

The definition is not Lorentz invariant but statistical systems usually break such an invariance. In the case of global equilibrium there is, for example, a preferential reference frame related to the thermostat.

Let us now look for the explicit expression of m_* in the lowest nontrivial order in g within the two models under consideration.

A. ϕ^4 model

The only nonvanishing contribution to the self-energy in the lowest order of perturbative expansion comes from Π_δ, which is given by Eq. (51). Therefore, the effective mass is a solution of the equation

$$m_*^2(x)=\frac{g}{2}\int\frac{d^3p}{(2\pi)^3E_p}f_0(x,\mathbf{p}). \tag{62}$$

One should keep in mind that E_p and f_0 depend on m_*.

For the equilibrium system with $T\gg m_*$ we immediately get the well-known result

$$m_*^2(x)=\frac{gT^2(x)}{24}. \tag{63}$$

One sees that the condition $T\gg m_*$ is automatically satisfied in the perturbative limit where $g^{-1}\gg 1$.

B. ϕ^3 model

Now we have the nonzero contributions to the self-energy not only from Π_δ (53) but from $\Pi^{\gtrless}$ (57) and (58) as well. The real part of Π^{+} which enters Eq. (61) is given by the equation analogous to Eq. (24). Thus,

$$\mathrm{Re}\,\Pi^{+}(X,m_*,\mathbf{0})=\frac{1}{2\pi i}P\times\int d\omega\,\frac{\Pi^{>}(X,\omega,\mathbf{0})-\Pi^{<}(X,\omega,\mathbf{0})}{m_*-\omega}.$$

Using Eqs. (59) and (60) one finds

$$\mathrm{Re}\,\Pi^{+}(X,m_*,\mathbf{0})=-\frac{g^2}{16\pi^2}m_*\int_{2m_*}^{\infty}\frac{d\omega}{\omega^2-m_*^2}\times\sqrt{1-4m_*^2/\omega^2}\,\frac{1}{\exp(\beta\omega/2)-1}$$
$$=-\frac{g^2}{32\pi^2}I(\beta m_*), \tag{64}$$

where

$$I(a)\equiv\int_1^{\infty}\frac{dx}{x^2-1/4}\,\frac{\sqrt{1-1/x^2}}{e^{ax}-1}.$$

Thus, in the case of *global* equilibrium with $T\gg m_*$, Eq. (61) reads

$$m_*^2+\frac{g^2}{24}\,\frac{T^2}{m_*^2}-\frac{g^2}{32\pi^2}\,I(\beta m_*)=0. \tag{65}$$

One observes that

$$I(a)<\frac{1}{a}\int_1^{\infty}\frac{dx}{x^2}=\frac{1}{a}.$$

Thus, $I(\beta m_*)<T/m_*$ and consequently the absolute value of the third term on the LHS of Eq. (65) is much smaller then that of the second one in the limit of $T\gg m_*$. Therefore, there is *no* real positive m_* which satisfies Eq. (65). It means that the massive quasiparticles do not emerge in the massless ϕ^3 model. It is not surprising since the potential $V(\phi)=(g/3!)\phi^3$ from the Lagrangian (1) has no minimum, even a local one. So, we do not consider the ϕ^3 model any more and concentrate on the ϕ^4 model.

Having the mass of quasiparticles we can determine their dispersion relation. Since all self-energies except $\Pi_\delta(X)$ vanish in the lowest order of the perturbative expansion, the mass-shell equation (26) coincides with that one of the free quasiparticles (33). Thus, we have explicitly shown that in the first order of g, the ϕ^4 model provides the system of free quasiparticles described by the transport equation of the Vlasov form, i.e.,

$$\left(p^\mu\partial_\mu+\frac{1}{2}\,\partial_\mu m_*^2(X)\partial_p^\mu\right)f_0(X,\mathbf{p})=0, \tag{66}$$

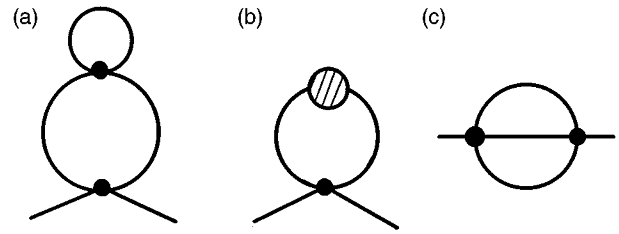

FIG. 4. The second-order diagrams of the self-energy in the ϕ^4 model.

with the effective mass given by Eq. (62).

X. HIGHER-ORDER SELF-ENERGY

We discuss here the g^2 contributions to the self-energy in the ϕ^4 model, which are represented by graphs shown in Fig. 4. The bubble is again related to the effective mass. The contributions corresponding to Figs. 4(a) and 4(b) can be easily computed. However, they are pure real and the only effect of these contributions is a higher-order modification of the effective mass. Thus, we do not explicitly calculate these diagrams but instead we analyze Fig. 4(c) which provides a qualitatively new effect.

The graph from Fig. 4(c) gives the contour self-energy as

$$\Pi_c(x,y)=\frac{g^2}{6}\,\Delta_0(x,y)\Delta_0(y,x)\Delta_0(x,y),$$

and consequently

$$\Pi^{\gtrless}(x,y)=\frac{g^2}{6}\,\Delta_0^{\gtrless}(x,y)\Delta_0^{\lessgtr}(y,x)\Delta_0^{\gtrless}(x,y).$$

Substituting $\Delta_0^{\gtrless}$ expressed through the distribution function f_0 as in Eqs. (39) and (40) one finds the self-energy $\Pi^{\gtrless}$ in the form

$$\begin{aligned}\Pi^<(X,p)=i\,\frac{g^2}{6}\int\frac{d^3k}{(2\pi)^3 2E_k}\,\frac{d^3q}{(2\pi)^3 2E_q}\,\frac{d^3r}{(2\pi)^3 2E_r}\,(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k}-\mathbf{r})[\delta(p_0-E_k-E_q-E_r)f_0^k f_0^{-q} f_0^r\\ +\delta(p_0-E_k-E_q+E_r)f_0^k f_0^{-q}(f_0^{-r}+1)\\ +\delta(p_0-E_k+E_q-E_r)f_0^k(f_0^q+1)f_0^r+\delta(p_0-E_k+E_q+E_r)f_0^k(f_0^q+1)(f_0^{-r}+1)\\ +\delta(p_0+E_k-E_q-E_r)(f_0^{-k}+1)f_0^{-q}f_0^r+\delta(p_0+E_k-E_q+E_r)(f_0^{-k}+1)f_0^{-q}(f_0^{-r}+1)\\ +\delta(p_0+E_k+E_q-E_r)(f_0^{-k}+1)(f_0^q+1)f_0^r+\delta(p_0+E_k+E_q+E_r)(f_0^{-k}+1)(f_0^q+1)(f_0^{-r}+1)],\end{aligned} \tag{67}$$

$$\begin{aligned}\Pi^>(X,p)=i\,\frac{g^2}{6}\int\frac{d^3k}{(2\pi)^3 2E_k}\,\frac{d^3q}{(2\pi)^3 2E_q}\,\frac{d^3r}{(2\pi)^3 2E_r}\,(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k}-\mathbf{r})[\delta(p_0+E_k+E_q+E_r)f_0^{-k}f_0^q f_0^{-r}\\ +\delta(p_0+E_k+E_q-E_r)f_0^{-k}f_0^q(f_0^r+1)+\delta(p_0+E_k-E_q+E_r)f_0^{-k}(f_0^{-q}+1)f_0^{-r}\\ +\delta(p_0+E_k-E_q-E_r)f_0^{-k}(f_0^{-q}+1)(f_0^r+1)\\ +\delta(p_0-E_k+E_q+E_r)(f_0^k+1)f_0^q f_0^{-r}+\delta(p_0-E_k+E_q-E_r)(f_0^k+1)f_0^q(f_0^r+1)\\ +\delta(p_0-E_k-E_q+E_r)(f_0^k+1)(f_0^{-q}+1)f_0^{-r}+\delta(p_0-E_k-E_q-E_r)(f_0^k+1)(f_0^{-q}+1)(f_0^r+1)],\end{aligned} \tag{68}$$

with

$$f_0^k \equiv f_0(X,k), \quad f_0^{-k} \equiv f_0(X,-k).$$

It is important to notice that in contrast to the similar expressions of the ϕ^3 model, i.e., Eqs. (57) and (58), the self-energies (68) and (67) are nonzero not only for the off-shell but for on-shell momentum p as well. However, the number of terms from Eq. (68) or (67) which contribute to $\Pi^{\gtrless}$ is reduced when $p^2 = m_*^2$. Indeed, Eqs. (67) and (68) simplify in this case as

$$\begin{aligned}\Theta(p_0)\Pi^<(X,p) &= i\,\frac{g^2}{6}\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k}-\mathbf{r})[\delta(p_0-E_k-E_q+E_r)f_0^k f_0^{-q}(f_0^{-r}+1)\\ &\quad + \delta(p_0-E_k+E_q-E_r)f_0^k(f_0^q+1)f_0^r + \delta(p_0+E_k-E_q-E_r)(f_0^{-k}+1)f_0^{-q}f_0^r]\\ &= i\,\frac{g^2}{2}\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(4)}(p+q-k-r)(f_0^q+1)f_0^k f_0^r,\end{aligned} \tag{69}$$

$$\begin{aligned}\Theta(p_0)\Pi^>(X,p) &= i\,\frac{g^2}{6}\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(3)}(\mathbf{p}+\mathbf{q}-\mathbf{k}-\mathbf{r})\\ &\quad\times[\delta(p_0-E_k-E_q+E_r)(f_0^k+1)(f_0^{-q}+1)f_0^{-r} + \delta(p_0-E_k+E_q-E_r)(f_0^k+1)f_0^q(f_0^r+1)\\ &\quad + \delta(p_0+E_k-E_q-E_r)f_0^{-k}(f_0^{-q}+1)(f_0^r+1)]\\ &= i\,\frac{g^2}{2}\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(4)}(p+q-k-r)f_0^q(f_0^k+1)(f_0^r+1),\end{aligned} \tag{70}$$

where $p_0 = \sqrt{m_*^2 + \mathbf{p}^2}$.

The self-energies $\Pi^{\gtrless}$ provide, through the equations analogous to Eqs. (23) and (24), the Re Π^+ and Im Π^+ which enter the transport (25) and mass-shell (26) equations. The imaginary part of Π^+ is of particular interest. Because of the finite value of Im Π^+, the spectral function (46) is no longer δ-like but it is of the Breit-Wigner shape. Thus, the quasiparticles are of finite lifetime. For the on-mass-shell momenta with $p_0 > 0$ the imaginary part of Π^+ equals

$$\begin{aligned}\mathrm{Im}\,\Pi^+(X,p) &= \frac{1}{2i}[\Pi^>(X,p) - \Pi^<(X,p)]\\ &= \frac{g^2}{4}\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(4)}(p+q-k-r)[f_0^q(f_0^k+1)(f_0^r+1) - (f_0^q+1)f_0^k f_0^r].\end{aligned} \tag{71}$$

This function was computed for the equilibrium distribution in [26,28].

XI. INTERACTING QUASIPARTICLES

In this section we discuss the dispersion relation of the interacting quasiparticles and then define the respective distribution function.

Having the self-energies calculated in g^2 order we can determine the quasiparticle dispersion relation in this order. As previously, the quasiparticle mass is found as a solution of Eq. (61). Therefore, the singular self-energy $\Pi_\delta(X)$ as well as Re $\Pi^+(X,m_*,0)$ are included in $m_*^2(X)$. To avoid the double counting, the expression

$$p^2 - m_*^2(X) + \Pi_\delta(X) + \mathrm{Re}\,\Pi^+(X,p)$$

is replaced everywhere by

$$p^2 - m_*^2(X) + \mathrm{Re}\,\tilde{\Pi}^+(X,p),$$

with

$$\begin{aligned}\mathrm{Re}\,\tilde{\Pi}^+(X,p) &\equiv +\mathrm{Re}\,\Pi^+(X,p) - \Theta(p_0)\mathrm{Re}\,\Pi^+(X,m_*,0)\\ &\quad - \Theta(-p_0)\mathrm{Re}\,\Pi^+(X,-m_*,0).\end{aligned}$$

The dispersion relation is given by the equation

$$p^2 - m_*^2(X) + \mathrm{Re}\,\tilde{\Pi}^+(X,p) = 0, \tag{72}$$

but according to Eq. (46), which determines the spectral function, the quasiparticles are of the finite width and the relation (72) gives only the most probable quasiparticle four-momentum. We call the four-momenta, which satisfies the relation (72), as "on-mass shell," however, one should keep in mind that the meaning of this term differs for the finite and zero width quasiparticles.

Equation (72) can be easily solved if Re $\tilde{\Pi}^+$ provides only a small correction to the free quasiparticle dispersion relation. Then, one finds the on-mass-shell momentum as $p^{\pm} = (\pm E_p^{\pm}, \mathbf{p})$ with

$$E_p^{\pm} = \sqrt{m_*^2(X) + \mathbf{p}^2 + \mathrm{Re}\,\tilde{\Pi}^+(X, \pm\sqrt{m_*^2(X)+\mathbf{p}^2}, \mathbf{p})}. \tag{73}$$

The distribution function $f(X,p)$ of the interacting quasiparticles is defined in a way analogous to Eq. (36), i.e.,

$$\Theta(p_0)i\Delta^{<}(X,p)=\Theta(p_0)A(X,p)f(X,p), \tag{74}$$

where $A(X,p)$ is the spectral function (46). In contrast with the case of free quasiparticles, $f(X,p)$ depends not on the three-vector $\mathbf{p}$ but on the four-vector p. Because of the identities

$$\Delta^{<}(X,p)=\Delta^{>}(X,-p), \quad A(X,p)=i\Delta^{>}(X,p)-i\Delta^{<}(X,p),$$

we have

$$i\Delta^{>}(X,p)=\Theta(p_0)A(X,p)[f(X,p)+1] -\Theta(-p_0)A(X,p)f(X,-p), \tag{75}$$

$$i\Delta^{<}(X,p)=\Theta(p_0)A(X,p)f(X,p) -\Theta(-p_0)A(X,p)[f(X,-p)+1]. \tag{76}$$

There is a very important property of $\Delta^{\gtrless}$ expressed in the form of Eqs. (75) and (76). Namely, if the Green's functions $\Delta^{\gtrless}$ satisfy the transport equation (25) and the spectral function solves Eq. (45), the mass-shell equation of $\Delta^{\gtrless}$, i.e., Eq. (26), is satisfied *automatically* in the zeroth order of the gradient expansion. Let us derive this result.

The transport and mass-shell equations (25) and (26) with the gradient terms neglected read

$$0=\Pi^{>}(X,p)\Delta^{<}(X,p)-\Pi^{<}(X,p)\Delta^{>}(X,p),$$

$$[p^2-m_*^2(X)+\mathrm{Re}\,\tilde{\Pi}^{+}(X,p)]\Delta^{\gtrless}(X,p) =-\Pi^{\gtrless}(X,p)\mathrm{Re}\,\Delta^{+}(X,p).$$

Substituting $\Delta^{\gtrless}$ in the form of Eqs. (75) and (76) into the first equation and taking only the terms corresponding to $p_0>0$, the equation is

$$0=A(x,p)\{\Pi^{>}(X,p)f(X,p)-\Pi^{<}(X,p)[f(X,p)+1]\}. \tag{77}$$

Now we substitute $\Delta^{<}$ given by Eq. (76) into the mass-shell equation and get

$$[p^2-m_*^2(X)+\mathrm{Re}\,\tilde{\Pi}^{+}(X,p)]A(x,p)f(X,p) =-\Pi^{<}(X,p)\mathrm{Re}\,\Delta^{+}(X,p), \tag{78}$$

where p_0 is assumed to be positive. Using the spectral function equation (45), Eq. (78) is manipulated to the form

$$\mathrm{Re}\,\Delta^{+}(x,p)[\Pi^{>}(X,p)f(X,p)-\Pi^{<}(X,p)(f(X,p)+1)]=0. \tag{79}$$

One sees that if f solves Eq. (77), it automatically satisfies Eq. (79). Similar considerations can be easily repeated for $p_0<0$ and then for $\Delta^{<}$ with $p_0>0$ and $p_0<0$.

Let us observe that the quasiparticles studied in this paper are *narrow*. Indeed, the effective mass (62) is of order $g^{1/2}$ while the width of the Breit-Wigner distribution given by Eq. (49) is proportional to g in power at least 3/2. Thus, the width of quasiparticles is much smaller than their mass (g is obviously assumed to be small). Due to this property we will often refer to the case of zero-width quasiparticles or on-mass-shell momenta.

XII. TRANSPORT EQUATION

The distribution function f satisfies the transport equation which can be obtained from Eq. (25) for $\Delta^{>}$ or $\Delta^{<}$. After using Eq. (44) one finds

$$\begin{aligned}A(X,p)\{p^2-m_*^2(X)+\mathrm{Re}\,\tilde{\Pi}^{+}(X,p),f(X,p)\}\\ =iA(X,p)\{\Pi^{>}(X,p)f(X,p)-\Pi^{<}(X,p)[f(X,p)+1]\}+if(X,p)\{\Pi^{>}(X,p),\mathrm{Re}\,\Delta^{+}(X,p)\}\\ -i(f(X,p)+1)\{\Pi^{<}(X,p),\mathrm{Re}\,\Delta^{+}(X,p)\},\end{aligned} \tag{80}$$

where $p_0>0$. We have also used here the following property of the Poisson-like brackets:

$$\{A,BC\}=\{A,B\}C+\{A,C\}B.$$

Since Eq. (80) is one of the main results of this paper we discuss it in detail.

The left-hand side of Eq. (80) is a straightforward generalization of the drift term of the standard relativistic transport equation. Computing the Poisson-like bracket and imposing the mass-shell constraint (72) one finds the familiar structure

$$\frac{1}{2}\,\Theta(p_0)\{p^2-m_*^2(X)+\mathrm{Re}\,\tilde{\Pi}^{+}(X,p),f(X,p)\}=E_p^{+}\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)f(X,p)+\nabla V(X)\nabla_p f(X,p),$$

where

$$V(X)\equiv m_*^2(X)-\mathrm{Re}\,\tilde{\Pi}^{+}(X,p),$$

and the velocity $\mathbf{v}$ equals $\partial E_p^{+}/\partial\mathbf{p}$ with the energy E_p^{+} given by Eq. (73).

Let us now analyze the right-hand side of Eq. (80). Since the quasiparticles of interest are narrow, we take into account only those terms contributing to the self-energies $\Pi^{\gtrless}$ which are nonzero for the on-mass-shell momenta. The other terms are negligibly small. Then, $\Pi^{\gtrless}$ from the transport equation (80) are given by the formulasanalogous to Eqs. (69) and (70) with f instead of f_0. Consequently,

$$\Theta(p_0)\Pi^<(X,p)=i\,\frac{g^2}{2}\int\frac{d^4kA_k^+}{(2\pi)^4}\frac{d^4qA_q^+}{(2\pi)^4}\frac{d^4rA_r^+}{(2\pi)^4}\times(2\pi)^4\delta^{(4)}(p+q-k-r)(f^q+1)f^kf^r, \tag{81}$$

$$\Theta(p_0)\Pi^>(X,p)=i\,\frac{g^2}{2}\int\frac{d^4kA_k^+}{(2\pi)^4}\frac{d^4qA_q^+}{(2\pi)^4}\frac{d^4rA_r^+}{(2\pi)^4}\times(2\pi)^4\delta^{(4)}(p+q-k-r)f^q(f^k+1)(f^r+1), \tag{82}$$

where

$$A_k^+\equiv\Theta(k_0)A(X,k).$$

One sees that in the limit of zero-width quasiparticles

$$\frac{d^4kA_k^+}{(2\pi)^4}\to\frac{d^3k}{(2\pi)^3 2E_k}.$$

The first term on the RHS of the transport equation (80) is very similar to the standard collision term of the relativistic transport equation [32]. Indeed,

$$i\{\Pi^>(X,p)f(X,p)-\Pi^<(X,p)[f(X,p)+1]\}=\frac{g^2}{2}\int\frac{d^4kA_k^+}{(2\pi)^4}\frac{d^4qA_q^+}{(2\pi)^4}\frac{d^4rA_r^+}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+q-k-r)[(f^p+1)(f^q+1)f^kf^r-f^pf^q(f^k+1)(f^r+1)]. \tag{83}$$

The last two terms from the RHS of Eq. (80), analogous to those found a long time ago in [10,11], are absent in the usual transport equation. We are going to show that in the local equilibrium, when the collision term (83) vanishes, we reproduce the standard collisionless equation if the four-momentum is on-mass shell.

As is well known [32], the standard collision term, which emerges from Eq. (83) when the quasiparticle width tends to zero [cf. Eq. (47)], vanishes for the local equilibrium distribution function of the form (41). Following [32], one easily shows that the collision term (83) also vanishes for the distribution function (41) with the particle momentum p no longer constrained by the mass-shell condition.

One observes that in the local equilibrium the collisional self-energies can be written as

$$\Pi^>(X,p)=2i\ \mathrm{Im}\ \Pi^+(X,p)[f^{\mathrm{eq}}(X,p)+1],$$

and

$$\Pi^<(X,p)=2i\ \mathrm{Im}\ \Pi^+(X,p)f^{\mathrm{eq}}(X,p).$$

The transport equation (80) then simplifies to

$$A(X,p)\{p^2-m_*^2(X)+\mathrm{Re}\ \tilde{\Pi}^+(X,p),f^{\mathrm{eq}}(X,p)\}=2\ \mathrm{Im}\ \Pi^+(X,p)\{f^{\mathrm{eq}}(X,p),\mathrm{Re}\ \Delta^+(X,p)\}.$$

Using Eqs. (30) and (46) one manipulates this equation to the form

$$\mathrm{Im}\ \Pi^+(X,p)\{p^2-m_*^2(X)+\mathrm{Re}\ \tilde{\Pi}^+(X,p),f^{\mathrm{eq}}(X,p)\}=[p^2-m_*^2(X)+\mathrm{Re}\ \tilde{\Pi}^+(X,p)]\times\{\mathrm{Im}\ \Pi^+(X,p),f^{\mathrm{eq}}(X,p)\}. \tag{84}$$

As seen, the term on the RHS drops down for the on-mass-shell momenta and then we reproduce the usual Vlasov equation, i.e.,

$$\{p^2-m_*^2(X)+\mathrm{Re}\ \tilde{\Pi}^+(X,p),f^{\mathrm{eq}}(X,p)\}=0. \tag{85}$$

The role of the two unusual terms from the RHS of the transport equation (80) beyond the local equilibrium is rather unclear and needs further studies.

XIII. SUMMARY AND CONCLUDING REMARKS

We have discussed in this paper the nonequilibrium features of the massless fields. The derivation of the kinetic equation in such a case faces serious difficulties because there is no natural length scale over which the system inhomogeneities can be integrated over. As known the transport theory deals with the quantities averaged over an elementary phase-space cell of the minimal size given by the particle Compton wavelength.

The fields with the zero bare mass usually gain an effective mass due to the self-interaction. Therefore, we have introduced the auxiliary mass term in the Lagrangian and then, the transport theory has been derived in a way very similar to the earlier studied [15,16] case of massive fields. However, due to the position dependence of the effective mass, the limit of the noninteracting quasiparticles corresponds to the Vlasov rather than the free particle case. The smallness of the effective mass has also forced us to take into account some extra gradient terms which are usually neglected in the transport equation.

We have considered in detail the ϕ^3 and ϕ^4 models which appear to be very different from each other. In the ϕ^4 model the effective mass is generated in the lowest nontrivial order of the perturbative expansion. In contrast, the massive quasiparticles do not emerge in the ϕ^3 model and most probably there is no transport limit of this model which, as is well known, is, in any case, ill defined.

Within the ϕ^4 model we have derived the transport equation for the finite width quasiparticles. The distribution function has been defined in such a way that the mass-shell constraint is automatically satisfied (in the gradient zeroth order). We have found, except for the mean-field and collision terms, the specific ones which are absent in the standard transport equation. However, in the case of local equilibrium we have been able to reproduce the usual collisionless equation if the four-momentum is on the mass shell.

The massless fields play a crucial role in the gauge theories such as QED or QCD. We believe that the methods developed in this study will be useful in the discussion of the transport theory of quarks and gluons. However, the application of our approach to QCD is not straightforward. The QCD effective action, which is analogous to our Eq. (2), is known only for the equilibrium case [24,23,29]. The generalization of this result to an inhomogeneous system is a serious problem which should be solved before the complete QCD transport equations could be derived.

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APPENDIX

We discuss here the quasiparticle approximation for the system of noninteracting fields. To simplify the discussion the bare mass m is assumed to be nonzero or equivalently m_* is treated as a constant. The transport equation and the mass-shell constraint read

$$p_\mu \partial^\mu \Delta_0^{\gtrless}(X,p)=0, \tag{A1}$$

$$\left(\frac{1}{4}\,\partial^2-p^2+m^2\right)\Delta_0^{\gtrless}(X,p)=0. \tag{A2}$$

These equations, which directly follow from the field equation of motion (3) with $m=m_*$, are exact in the case of the massive free fields — the gradient expansion is not needed to derive them.

The mass-shell constraint (A2) shows that the function $\Delta_0^{\gtrless}(X,p)$ is indeed nonzero for the off-shell momenta, i.e., $\Delta_0^{\gtrless}(X,p)\neq 0$ for $p^2\neq m^2$. This result looks surprising if one keeps in mind that the field, which solves the equation of motion (3), is, in a sense, on-mass shell. The field is the sum of the plane waves

$$\phi(x)=\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\,[e^{-ikx}a(\mathbf{k})+e^{ikx}a^*(\mathbf{k})], \tag{A3}$$

where $k\equiv(\omega_k,\mathbf{k})$ with $\omega_k\equiv\sqrt{\mathbf{k}^2+m^2}$. Thus, $k^2=m^2$. Substituting the field (A3) into the $\Delta^{\gtrless}(X,p)$ definition, one finds that the off-shell contribution to $\Delta_0^{\gtrless}(X,p)$ comes from the interference of the positive- and negative-energy parts present in Eq. (A3). Let us consider when such a contribution can be neglected.

One easily shows that the transport equation (A1) is solved by the function which depends on the four-position $X=(t,\mathbf{x})$ only through $\mathbf{x}-\mathbf{v}t$: i.e.,

$$\Delta_0^{\gtrless}(X,p)=F(\mathbf{x}-\mathbf{v}t,p),$$

where $\mathbf{v}\equiv\mathbf{p}/p_0$. The quasiparticle condition (19) applied to the function F reads

$$|F(\mathbf{x}-\mathbf{v}t,p)|\gg\frac{1}{m^2}\left|(v_i v_j-\delta_{ij})\,\frac{\partial^2 F(\mathbf{x}-\mathbf{v}t,p)}{\partial(\mathbf{x}-\mathbf{v}t)_i\partial(\mathbf{x}-\mathbf{v}t)_j}\right|.$$

If this condition is satisfied for every $\mathbf{x}$ at a given moment of time, say t_0, it is satisfied at *any* time. In other words, if the initial condition at t_0 is sufficiently homogeneous that the quasiparticle approximation can be applied, then this approximation is applicable at any time — the system remains homogeneous.

The question arises whether $\Delta_0^{\gtrless}$, which simultaneously solves the transport (A1) and mass-shell (A2) equations, *can* satisfy the quasiparticle condition. We introduce the Fourier-transformed function $\tilde{\Delta}_0^{\gtrless}(Q,p)$ defined as

$$\tilde{\Delta}_0^{\gtrless}(Q,p)\stackrel{\text{def}}{=}\int d^4X e^{iQ\cdot X}\Delta_0^{\gtrless}(X,p). \tag{A4}$$

The equations corresponding to Eqs. (A1) and (A2), respectively, read

$$p_\mu Q^\mu \tilde{\Delta}_0^{\gtrless}(Q,p)=0,$$

$$\left(-\frac{1}{4}\,Q^2-p^2+m^2\right)\tilde{\Delta}_0^{\gtrless}(Q,p)=0.$$

They are both solved by

$$i\tilde{\Delta}_0^{\gtrless}(Q,p)=\delta(p\cdot Q)\,\delta\!\left(-\frac{1}{4}\,Q^2-p^2+m^2\right)A(Q,p), \tag{A5}$$

with $A(Q,p)$ controlled by the initial condition. Since $i\Delta_0^{\gtrless}(X,p)$ is real, $A(Q,p)$ has the property

$$A(Q,p)=A^*(-Q,p). \tag{A6}$$

The solution of the Eqs. (A1) and (A2) satisfies the quasiparticle condition (19) when

$$|A(Q,p)|\gg\left|\frac{Q^2}{m^2}\,A(Q,p)\right|, \tag{A7}$$

or equivalently $A(Q,p)\neq 0$ only for $Q^2\ll m^2$.

It is instructive to consider the explicit solution of Eqs. (A1) and (A2) in 1+1 dimensions. Using Eq. (A5) we get

$$\begin{aligned} i\Delta_0^{\gtrless}(X,p)&=\int \frac{d^2Q}{(2\pi)^2}\,e^{-iQ\cdot X}\delta(p\cdot Q)\\ &\quad\times\delta\!\left(-\frac{1}{4}\,q^2-p^2+m^2\right)A(Q,p),\\ &=[\Theta(-p^2)+\Theta(p^2-m^2)]\,\frac{1}{(2\pi)^2|p^2|}\,\sqrt{p^2/(p^2-m^2)}\\ &\quad\times[e^{-\tilde{Q}X}A(\tilde{Q},p)+e^{i\tilde{Q}X}A(-\tilde{Q},p)], \end{aligned} \tag{A8}$$

where $\tilde{Q}$ denotes the two-vector

$$\widetilde{Q} \equiv 2|p_0|\sqrt{(p^2-m^2)/p^2}\left(\frac{p_1}{p_0},1\right).$$

Keeping in mind the property (A6), the solution (A8) can be rewritten as

$$i\Delta_0^{\gtrless}(X,p) = [\Theta(-p^2)+\Theta(p^2-m^2)] \times [h(p)\sin(\widetilde{Q}X)+g(p)\cos(\widetilde{Q}X)], \quad \text{(A9)}$$

where $h(p)$ and $g(p)$ are the real functions of p determined by the initial condition.

The quasiparticle condition (19) is satisfied by Eq. (A9) if

$$|\widetilde{Q}^2| = 4|p^2-m^2| \ll m^2.$$

One also sees that $\Delta_0^{\gtrless}(X,p) \sim \delta(p^2-m^2)$ only for $\widetilde{Q}=0$. In other words, the function $\Delta_0^{\gtrless}(X,p)$ is strictly zero for the off-mass-shell momenta when the system is exactly homogeneous. If we are interested in the weakly nonhomogeneous systems, the functions are nonzero for $p^2 > m^2$. Equivalently, if $p^2 \cong m^2$ then $p^2 > m^2$ but not $p^2 < m^2$.

The properties of the function $\Delta_0^{\gtrless}$ in $1+1$ dimensions can be trivially generalized to the $3+1$ case showing the limitations of the quasiparticle approximation.

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