

Local equilibrium of the quark-gluon plasma

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Within kinetic theory, we look for the local equilibrium configurations of a quark-gluon plasma by maximizing the local entropy. We use the well-established transport equations in the Vlasov limit, supplemented by the Waldmann-Snyder collision terms. Two different classes of local equilibrium solutions are found. The first one corresponds to the configurations that comply with the so-called collisional invariants. The second one is given by the distribution functions that cancel the collision terms, representing the most probable binary interactions with soft gluon exchange in the t channel. The two sets of solutions agree with each other if we go beyond these dominant processes and take into account subleading quark-antiquark annihilation or creation and gluon number nonconserving processes. The local equilibrium state appears to be colorful, as the color charges are not locally neutralized. The properties of such an equilibrium state are analyzed. In particular, the related hydrodynamic equations of a colorful fluid are derived. Possible neutralization processes are also briefly discussed.

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I. INTRODUCTION

In the course of equilibration a many-body system first reaches local equilibrium and then evolves hydrodynamically, usually at a much slower rate, towards global equilibrium. The distribution function of local equilibrium is typically of the form of global equilibrium, but its parameters—temperature, hydrodynamic velocity, chemical potentials—are space-time dependent. However, the local equilibrium can also qualitatively differ from the global one. For example, the electron-ion plasma, which is homogeneously neutral in global equilibrium, can be locally charged before the global equilibrium is reached, see, e.g., [1]. Thus parameters that are irrelevant for global equilibrium might be needed to describe local equilibrium. While the state of global equilibrium is unique, the local equilibrium evolves and even its qualitative features can change in time. The processes of charge neutralization are, for example, very fast in the electron-ion plasma. Therefore the system is locally neutral after a short time but the electric currents survive for much longer. Thus we deal with various local equilibrium states, depending on the time scale of interest. The form of local equilibrium is an important characteristic of a system. Knowing the respective distribution function, one can formulate a hydrodynamic description of the system. Let us again refer to the case of the electron-ion plasma. The fact mentioned above that the plasma is neutralized fast but the currents flow for a longer time justifies the magnetohydrodynamics with no electric fields.

The aim of this paper is to discuss local equilibrium of the quark-gluon plasma. While the global equilibrium features of

the system have been studied in detail, see, e.g., the review [2], not much is known about its local equilibrium. Although the problem was formulated long ago [3–6], the key questions remain unanswered. In particular, the scenario of equilibration of color degrees of freedom is far from established. It is unclear whether the regime analogous to magnetohydrodynamics in the electron-ion plasma occurs in the quark-gluon plasma. However, Yang-Mills magnetohydrodynamics has been already considered [3,5–10].

We intend to address these issues which are now of particular interest because of the large scale experimental program at the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Laboratory, where high-energy nucleus-nucleus interactions are studied, see, e.g., [11]. At the early stage of such a collision, when the energy density is sufficiently high, the generation of the quark-gluon plasma is expected. The most spectacular experimental result obtained at RHIC is presumably an observation of a large magnitude of the so-called elliptic flow [12]. The phenomenon, which is just sensitive to the collision early stage, is naturally explained within hydrodynamics as a result of large density gradients [13]. Since the hydrodynamic description is applicable for a system in local thermodynamic equilibrium, the large elliptic flow suggests a surprisingly short, below 1 fm/c [14], equilibration time. Other characteristics of relativistic heavy-ion collisions are also consistent with a model assuming equilibrium state of strongly interacting matter produced in the collisions, see, e.g., [15]. Thus understanding of the equilibration mechanism of the quark-gluon plasma is a key problem for RHIC physics.

The question of local equilibrium is related to a serious difficulty of the transport theory of the quark-gluon plasma. The local equilibrium is defined as a state which maximizes the local entropy. However, the entropy production occurs not due to the Vlasov evolution, which is rather well under-

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stood [2,16], but there is a dissipative phenomenon caused by the particle collisions. Thus the collision terms of the transport equations are needed to discuss the local equilibrium. However, a derivation of these terms has been a very complex task and only the special case of quasi-equilibrium plasma has been seriously examined [17–21]. Fortunately, the structure of the collision terms can be guessed referring to the analogies between the spin and color systems, and this is not only a superficial similarity of degrees of freedom governed by the SU(2) and SU(3) groups, respectively. The relationship appears to be much deeper. The covariance of spin dynamics with respect to the rotation of quantization axis strongly resembles the gauge covariance of QCD. Thus it was argued long ago [22] that the QCD collision terms are of the Waldmann-Snyder type [23] known from the studies of spin systems. More recently, guided by the same analogy, the Waldmann-Snyder transport equations have been used to compute color conductivity of the quark-gluon plasma [19], as well as other transport coefficients [24,25].

Once the collision terms of transport equations are known, the problem of finding the state of local equilibrium is well posed, see, e.g., [26]. Namely, one looks for a configuration which maximizes the local entropy. In fact, such a configuration can be also found without a detailed knowledge on the collision terms. One only needs the so-called collisional invariants—the conditions obeyed by the collision terms, coming from the conservation laws. In such an approach, already followed in [4,6], we, however, gain no information about the time scale corresponding to the local equilibrium state. We also do not know whether the local equilibrium configuration dictated by the collisional invariants is the most general maximum entropy state. To answer these questions an explicit form of the collision terms is required. Then, one looks for a configuration that cancels the collision terms.

In this paper we follow both approaches. After introducing the kinetic theory of the quark-gluon plasma in Sec. II, we find in Sec. III the local equilibrium state provided by the collisional invariants. Then, we select the most probable binary interactions and we derive in Sec. IV the local equilibrium functions which cancel the Waldmann-Snyder collision terms corresponding to these dominant processes. The derivation requires solving a whole set of rather complicated matrix equations. To simplify the analysis, we consider particles obeying classical statistics, although we believe that the physical picture emerging from our analysis is not much changed when quantum statistics is incorporated. The local equilibrium states, which come from the approaches of Secs. III and IV, are colorful and their color structure is exactly the same. However, the baryon chemical potential of (anti-)quarks and the scalar chemical potential of gluons remain unconstrained by the dominant processes. The constraints provided by the collisional invariants only appear when the subleading quark-antiquark annihilation or creation and gluon number nonconserving processes are included. To better understand properties of the colorful local equilibrium, we derive in Sec. V the resulting hydrodynamic equations.

Finally, we consider the applicability of our results and briefly discuss possible processes responsible for the color neutralization in the quark-gluon plasma. Some formulas of the SU(N_c) generators are collected in the Appendix.

Throughout the paper [except Eqs. (115)–(117) where c is restored] we use the natural units with $c = \hbar = k_B = 1$ and the metric $(1, -1, -1, -1)$.

II. KINETIC THEORY OF THE QUARK-GLUON PLASMA

In this section we discuss the transport theory of quarks and gluons [16,27]. The SU(N_c) gauge group is left unspecified but we pay particular attention to the cases $N_c = 2$ and $N_c = 3$ for their possible applications to the different high temperature phases of the standard model. Generically speaking, we call gluons the particles associated to the vector bosons of SU(N_c), which carry charge in the adjoint representation, and we call quarks or antiquarks the particles with the charge in the fundamental representation. We will also call any of those particles a parton.

A. Distribution functions and transport equations

The distribution function of quarks $Q(p, x)$ is a Hermitian $N_c \times N_c$ matrix in color space [for a SU(N_c) color group]; x denotes the space-time quark coordinate and p its momentum, which is not constrained by the mass-shell condition. The spin of quarks and gluons is taken into account as an internal degree of freedom. The distribution function transforms under a local gauge transformation U as

$$Q(p, x) \rightarrow U(x)Q(p, x)U^\dagger(x), \quad (1)$$

that is, it transforms covariantly in the fundamental representation. Here and in most cases below, the color indices are suppressed. The distribution function of antiquarks, which we denote by $\bar{Q}(p, x)$, is also a Hermitian $N_c \times N_c$ matrix in color space, which in a natural way should transform covariantly in the conjugate fundamental representation. However, we will express the antiquark distribution function in the same representation as quarks throughout, and then it transforms according to Eq. (1). The distribution function of (hard) gluons is a Hermitian $(N_c^2 - 1) \times (N_c^2 - 1)$ matrix, which transforms as

$$G(p, x) \rightarrow \mathcal{U}(x)G(p, x)\mathcal{U}^\dagger(x), \quad (2)$$

where

$$\mathcal{U}_{ab}(x) = 2 \text{Tr}[\tau^a U(x)\tau^b U^\dagger(x)], \quad (3)$$

with τ^a , $a = 1, \dots, N_c^2 - 1$ being the SU(N_c) group generators in the fundamental representation with $\text{Tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}$. We note that $\mathcal{U}^T = \mathcal{U}^{-1} = \mathcal{U}^\dagger$. Therefore not only G but also G^T transforms covariantly, i.e.,

$$G^T(p, x) \rightarrow \mathcal{U}(x)G^T(p, x)\mathcal{U}^\dagger(x). \quad (4)$$

The color current is expressed in the fundamental representation as

$$j^\mu(x) = -\frac{g}{2} \int dP p^\mu \left[Q(p,x) - \bar{Q}(p,x) - \frac{1}{N_c} \text{Tr}[Q(p,x) - \bar{Q}(p,x)] + 2\tau^a \text{Tr}[T^a G(p,x)] \right], \quad (5)$$

where the momentum measure

$$dP \equiv \frac{d^4 p}{(2\pi)^3} 2\Theta(p_0) \delta(p^2) \quad (6)$$

takes into account the mass-shell condition $p_0 = |\mathbf{p}|$. Throughout the paper, we neglect the quark masses, although those might be easily taken into account by modifying the mass-shell constraint in the momentum measure. A sum over helicities, two per particle, and over quark flavors N_f is understood in Eq. (5), even though it is not explicitly written down. The $SU(N_c)$ generators in the adjoint representation are expressed through the structure constants $T_{bc}^a = -if_{abc}$, and are normalized as $\text{Tr}[T^a T^b] = N_c \delta^{ab}$. The current can be decomposed as $j^\mu(x) = j_a^\mu(x) \tau^a$ with $j_a^\mu(x) = 2 \text{Tr}[\tau_a j^\mu(x)]$.

Gauge invariant quantities are given by the traces of the distribution functions. Thus the baryon current and the energy-momentum tensor read

$$b^\mu(x) = \frac{1}{3} \int dP p^\mu \text{Tr}[Q(p,x) - \bar{Q}(p,x)], \quad (7)$$

$$t^{\mu\nu}(x) = \int dP p^\mu p^\nu \text{Tr}[Q(p,x) + \bar{Q}(p,x) + G(p,x)], \quad (8)$$

where we use the same symbol $\text{Tr}[\dots]$ for the trace in the fundamental and adjoint representations.

The entropy flow is defined as [4]

$$s^\mu(x) = - \int dP p^\mu \text{Tr} [Q \ln Q + (1-Q) \ln(1-Q) + \bar{Q} \ln \bar{Q} + (1-\bar{Q}) \ln(1-\bar{Q}) + G \ln G - (1+G) \ln(1+G)]. \quad (9)$$

If the effects of quantum statistics are neglected, Eq. (9) simplifies to

$$s^\mu(x) = - \int dP p^\mu \text{Tr} [Q(\ln Q - 1) + \bar{Q}(\ln \bar{Q} - 1) + G(\ln G - 1)]. \quad (10)$$

The distribution functions of quarks and gluons satisfy the transport equations:

$$p^\mu D_\mu Q(p,x) + \frac{g}{2} p^\mu \{ F_{\mu\nu}(x), \partial_p^\nu Q(p,x) \} = C[Q, \bar{Q}, G], \quad (11a)$$

$$p^\mu D_\mu \bar{Q}(p,x) - \frac{g}{2} p^\mu \{ F_{\mu\nu}(x), \partial_p^\nu \bar{Q}(p,x) \} = \bar{C}[Q, \bar{Q}, G], \quad (11b)$$

$$p^\mu \mathcal{D}_\mu G(p,x) + \frac{g}{2} p^\mu \{ \mathcal{F}_{\mu\nu}(x), \partial_p^\nu G(p,x) \} = C_g[Q, \bar{Q}, G], \quad (11c)$$

where g is the QCD coupling constant, $\{ \dots, \dots \}$ denotes the anticommutator and ∂_p^ν the four-momentum derivative; the covariant derivatives D_μ and \mathcal{D}_μ act as

$$D_\mu = \partial_\mu - ig[A_\mu(x), \dots], \quad \mathcal{D}_\mu = \partial_\mu - ig[\mathcal{A}_\mu(x), \dots],$$

with A_μ and \mathcal{A}_μ being four-potentials in the fundamental and adjoint representations, respectively:

$$A^\mu(x) = A_a^\mu(x) \tau^a, \quad \mathcal{A}^\mu(x) = T^a A_a^\mu(x).$$

The stress tensor in the fundamental representation is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, while $\mathcal{F}_{\mu\nu}$ denotes the field strength tensor in the adjoint representation. The collision terms C, \bar{C} , and C_g are discussed in detail in the next sections.

Let us finally mention that in the transport theory framework one can consider two different physical situations: (1) the gauge fields entering into the transport equations (11) are external, not due to the plasma constituents; and (2) the gauge fields can be generated self-consistently by the quarks and gluons. In the latter case, one also has to solve the Yang-Mills equation

$$D_\mu F^{\mu\nu}(x) = j^\nu(x), \quad (12)$$

where the color current is given by Eq. (5).

B. Decomposition of the distribution functions and associated transport equations

The parton distribution function N is essentially the statistical average of the Wigner transform of the product of two field operators representing quarks or gluons [16]. If the parton carries color charge in a representation R , then the distribution function N transforms under gauge transformations as $\bar{R} \otimes R$, where \bar{R} is the representation conjugate to R .

In the $SU(2)$ group, the products of the fundamental (2) and adjoint (3) representations decompose into irreducible representations as

$$2 \otimes 2 = 1 \oplus 3, \quad (13)$$

$$3 \otimes 3 = 1 \oplus 3 \oplus 5. \quad (14)$$

As known, the conjugate and direct fundamental representations of $SU(2)$ are equivalent to each other. The decomposi-

tion of the products of the fundamental (3) and adjoint (8) representations of the SU(3) group are

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}, \quad (15)$$

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \bar{\mathbf{10}} \oplus \mathbf{27}. \quad (16)$$

The above decompositions show that the distribution functions of quarks and antiquarks are uniquely specified by their singlet and adjoint components. Thus the functions can be written as

$$Q(p, x) = \frac{1}{N_c} q_0(p, x) + q^a(p, x) \tau^a, \quad (17a)$$

$$\bar{Q}(p, x) = \frac{1}{N_c} \bar{q}_0(p, x) + \bar{q}^a(p, x) \tau^a, \quad (17b)$$

where

$$q_0(p, x) = \text{Tr}[Q(p, x)], \quad q_a(p, x) = 2 \text{Tr}[\tau^a Q(p, x)], \quad (18a)$$

$$\bar{q}_0(p, x) = \text{Tr}[\bar{Q}(p, x)], \quad \bar{q}_a(p, x) = 2 \text{Tr}[\tau^a \bar{Q}(p, x)]. \quad (18b)$$

From Eq. (11a) it is possible to deduce a set of coupled equations for the colored and colorless components of the quark distribution function which read

$$p^\mu \partial_\mu q_0(p, x) + \frac{g}{2} p^\mu F_{\mu\nu}^a(x) \frac{\partial q_a(p, x)}{\partial p_\nu} = \text{Tr}[C], \quad (19a)$$

$$p^\mu D_\mu^{ab} q_b(p, x) + \frac{g}{2} d_{abc} p^\mu F_{\mu\nu}^b(x) \frac{\partial q_c(p, x)}{\partial p_\nu} + \frac{g}{N_c} p^\mu F_{\mu\nu}^a(x) \frac{\partial q_0(p, x)}{\partial p_\nu} = 2 \text{Tr}[\tau^a C], \quad (19b)$$

where d_{abc} are the totally symmetric structure constants of SU(N_c) and $D_\mu^{ac} = \partial_\mu \delta^{ac} + g f_{abc} A_\mu^b$. The projected equations, which can be also written for antiquarks, show that transport phenomena of colorless and colored components are coupled beyond the lowest order in the gauge coupling constant.

From the decompositions (14), (16) it is clear that the singlet and adjoint components are not enough to fully describe the gluon distribution function. For gluons one also needs components in higher dimensional representations. Below, we present a way to uniquely characterize the gluon distribution function in terms of its fully symmetric and antisymmetric components for the SU(2) gauge theory.

We first express $G(p, x)$ as

$$G(p, x) = \mathcal{G}_{ab}(p, x) T^a T^b, \quad (20)$$

which uses as a basis for 3×3 Hermitian matrices the set of nine independent matrices $T^a T^b$. We note that both G and \mathcal{G} are 3×3 matrices which are related to each other as

$$G_{ab}(p, x) = \delta^{ab} \mathcal{G}_{cc}(p, x) - \mathcal{G}_{ba}(p, x). \quad (21)$$

Expressing the product of $T^a T^b$ as

$$T^a T^b = \frac{1}{2} [T^a, T^b] + \frac{1}{2} \{T^a, T^b\}, \quad (22)$$

and taking into account that the commutator is proportional to T^c , instead of Eq. (20) we write

$$G(p, x) = \frac{1}{2} g_a(p, x) T^a + g_{ab}(p, x) \frac{1}{2} \{T^a, T^b\}, \quad (23)$$

where

$$g_a(p, x) = i f^{abc} \mathcal{G}_{cb}(p, x),$$

$$g_{ab}(p, x) = \frac{1}{2} [\mathcal{G}_{ab}(p, x) + \mathcal{G}_{ba}(p, x)]. \quad (24)$$

Equation (23) can be also written as

$$G_{ab}(p, x) = -i f^{abc} g_c(p, x) + \delta^{ab} g_{cc}(p, x) - g_{ab}(p, x). \quad (25)$$

Thus, according to the decomposition in Eq. (14), the anti-symmetric components of G correspond to the representation $\mathbf{3}$, while the six symmetric components correspond to the $\mathbf{5}$ and $\mathbf{1}$, the last one being the trace. Because of the Casimir constraint, $T^a T^a = 2$, the singlet component can be obtained from the symmetric part g_{ab} , i.e.,

$$g_0(p, x) \equiv \text{Tr}[G(p, x)] = 2 g_{aa}(p, x). \quad (26)$$

The transport equations obeyed by g_0 , g_a , and g_{ab} are found multiplying Eq. (11c) by the unity, T^a and $\{T^a, T^b\}/2$, respectively, and taking the trace. Using the relations (A12), we get

$$p^\mu \partial_\mu g_0(p, x) + g p^\mu F_{\mu\nu}^a(x) \frac{\partial g_a(p, x)}{\partial p_\nu} = \text{Tr}[C_g], \quad (27a)$$

$$p^\mu D_\mu^{ab} g_b(p, x) + g p^\mu F_{\mu\nu}^b(x) \left(\frac{1}{2} \delta^{ab} \frac{\partial g_0(p, x)}{\partial p_\nu} + \frac{\partial g_{ab}(p, x)}{\partial p_\nu} \right) = \text{Tr}[T^a C_g], \quad (27b)$$

$$p^\mu (D_\mu)_{bd}^{ac} g_{cd}(p, x) + \frac{g}{4} p^\mu \left(F_{\mu\nu}^a(x) \frac{\partial g_b(p, x)}{\partial p_\nu} + F_{\mu\nu}^b(x) \frac{\partial g_a(p, x)}{\partial p_\nu} \right) = \frac{1}{2} \text{Tr}[\{T^a, T^b\} C_g] - \frac{1}{2} \delta^{ab} \text{Tr}[C_g], \quad (27c)$$

where

$$(D_\mu)_{bd}^{ac} = \partial_\mu \delta^{ac} \delta^{bd} + g f^{aec} \delta^{bd} A_\mu^e + g f^{bed} \delta^{ac} A_\mu^e \quad (28)$$

is the covariant derivative acting on a tensor of rank 2. Note that multiplying the last equation by δ^{ab} , we get, as expected, the equation for g_0 .

For SU(3), or SU(N_c) in general, the decomposition of the gluon distribution function into irreducible representations and the equations obeyed by every component have a much more involved structure, and they will not be discussed here.

C. Waldmann-Snider collision terms

The transport equations for the quark-gluon plasma (11) have been written down without specifying the collision terms. Unfortunately, a complete derivation of C , \bar{C} , and \bar{C}_g is still lacking, as already mentioned in the Introduction. However, using the analogy with the spin systems one can justify the use of the Waldmann-Snider collision terms. The main characteristic of these collision terms is that they depend on the scattering amplitudes rather than on the collisional cross sections, as it happens in the usual Boltzmann equation.

Let us discuss the general structure of a collision term for a system of particles carrying quantum color charges. The most probable processes are binary collisions $(p, r; p_1, s) \leftrightarrow (p', t; p'_1, u)$ where p, p_1, p', p'_1 denote the momenta and r, s, t, u the colors, in the fundamental or adjoint representation, of interacting partons. We denote by $N(p, x)$ the generic distribution function of the partons—quarks or gluons. The Waldmann-Snider collision term, which enters the kinetic equation of N , is of the form [26]

$$C[N, N_1, N', N'_1] = \int dP' dP'_1 dP_1 (2\pi)^4 \delta^{(4)}(p + p_1 - p' - p'_1) \left[\frac{1}{2} \{1 \pm N, \mathcal{I}_+\} - \frac{1}{2} \{N, \mathcal{I}_-\} \right], \quad (29)$$

where we have used a rather common notation $N \equiv N(p, x)$, $N_1 \equiv N(p_1, x)$, $N' \equiv N(p', x)$, and $N'_1 \equiv N(p'_1, x)$. The first term, which represents a gain term, is given by

$$\mathcal{I}_+^{rr} = M_{rstu}(p, p_1; p', p'_1) M_{rstu}^*(p, p_1; p', p'_1) \times N^{tt}(p', x) N^{uu}(p'_1, x) [1 \pm N(p_1, x)]^{\bar{s}s}, \quad (30)$$

while the second one is a loss term defined as

$$\mathcal{I}_-^{rr} = M_{rstu}(p, p_1; p', p'_1) M_{rstu}^*(p, p_1; p', p'_1) N^{\bar{s}s}(p_1, x) \times [1 \pm N(p', x)]^{t\bar{t}} [1 \pm N(p'_1, x)]^{u\bar{u}}. \quad (31)$$

M_{rstu} represents the scattering amplitude associated with the collision process under consideration. The double sign \pm reflects the fermionic character of quarks and the bosonic character of gluons. We have used here the compact notation of Ref. [19].

For the consistency of the theory, it is necessary to prove that the Waldmann-Snider collision terms transform covariantly under a gauge transformation, in the same way as the left-hand sides of the transport equations (11) do. It is difficult to check this gauge covariance in full generality without specifying the scattering process and the corresponding scattering amplitudes. For all the cases we are going to consider, the gauge covariance of the Waldmann-Snider collision term holds as the distribution functions transform covariantly [see Eqs. (1) and (2)], and the scattering amplitudes, stripped of the color generators, are gauge invariant. We will briefly come back to this point in Sec. IV.

D. Conservation laws and entropy production

As well known, the collision terms should satisfy certain relations due to the conservation laws. In our case, the laws are: the baryon charge conservation

$$\partial_\mu b^\mu(x) = 0, \quad (32)$$

the energy-momentum conservation

$$\partial_\mu t^{\mu\nu}(x) + 2 \text{Tr}[j_\sigma(x) F^{\sigma\nu}(x)] = 0, \quad (33)$$

and the covariant conservation of the color current

$$D_\mu j^\mu(x) = 0. \quad (34)$$

Let us derive the relations constraining the collision terms which follow from Eqs. (32), (33), (34). Using the transport equation, one finds from the definition (7)

$$\begin{aligned} \partial_\mu b^\mu(x) &= \frac{1}{3} \int dP \text{Tr}[C - \bar{C}] \\ &\quad - \frac{g}{3} \int dP p^\sigma \text{Tr}[F_{\sigma\nu} \partial_p^\nu (Q + \bar{Q})]. \end{aligned}$$

Now, one performs partial integration of the second term in the right-hand side (rhs). Assuming that the distribution functions vanish at infinite momentum and observing that $g^{\mu\nu} F_{\mu\nu} = 0$, one finds that the term equals zero. Therefore the baryon current conservation (32) provides

$$\int dP \text{Tr}[C - \bar{C}] = 0. \quad (35)$$

In an analogous way, one finds that the energy-momentum conservation (33) implies

$$\int dP p^\mu \text{Tr}[C + \bar{C} + C_g] = 0, \quad (36)$$

while the covariant conservation of the color current leads to

$$\int dP [C - \bar{C} + 2\tau^a \text{Tr}[T^a C_g]] = 0, \quad (37)$$

where we have taken into account the relation (35).

Let us now discuss the entropy production. We neglect here the effects of quantum statistics, and consequently start with the definition (10). Following the derivation of Eqs. (35), (36), (37), one finds

$$\begin{aligned} \partial_\mu s^\mu(x) = & - \int dP \operatorname{Tr}[C \ln Q + \bar{C} \ln \bar{Q} + C_g \ln G] \\ & - \frac{g}{2} \int dP p^\mu \operatorname{Tr}[\{F_{\mu\nu}, Q\} \partial_p^\nu \ln Q \\ & - \{F_{\mu\nu}, \bar{Q}\} \partial_p^\nu \ln \bar{Q} + \{\mathcal{F}_{\mu\nu}, G\} \partial_p^\nu \ln G], \end{aligned} \quad (38)$$

where the partial integration has been once performed and it has been observed that

$$\operatorname{Tr}[A^\mu, Q] \ln Q = 0,$$

and that the analogous equalities hold for \bar{Q} and G . Assuming that Q and $\partial_p^\nu Q$ commute with each other, i.e.,

$$[Q, \partial_p^\nu Q] = 0, \quad (39)$$

one shows that $\partial_p^\nu \ln Q = Q^{-1} \partial_p^\nu Q$. Using the condition (39) and the similar ones for \bar{Q} and G , one proves that the second term in the rhs of Eq. (38) vanishes after one more partial integration. Then, we get

$$\partial_\mu s^\mu(x) = - \int dP \operatorname{Tr}[C \ln Q + \bar{C} \ln \bar{Q} + C_g \ln G]. \quad (40)$$

According to Eq. (40), the entropy of the quark-gluon system is produced due to the collisions. If the commutation condition (39) is relaxed, the second term in the rhs of Eq. (38) does not vanish, and we arrive to a paradoxical result that the mean-field dynamics does not conserve the entropy.

A local equilibrium configuration is achieved when there is no entropy production, i.e., $\partial_\mu s^\mu(x) = 0$. This equation is of very complicated structure and it has two classes of solutions. The first one cancels the collision terms but to get it the collision terms have to be specified. The second class appears due to the conservation laws, i.e., because of the relations (35), (36), and (37). In the remaining part of this paper, we will study the two sets of solutions.

III. LOCAL EQUILIBRIUM FROM THE CONSERVATION LAWS

In this section we discuss, following [4,6], consequences of the conservation laws (32), (33), (34). Specifically, we obtain the local equilibrium configuration which is found as a solution of the equation

$$\int dP \operatorname{Tr}[C \ln Q + \bar{C} \ln \bar{Q} + C_g \ln G] = 0, \quad (41)$$

due to the relations (35), (36), (37).

One easily constructs the local equilibrium distribution function out of the collision invariants. Indeed, one shows using Eqs. (32), (33), (34) that Eq. (41) is solved if

$$Q_{\text{eq}}(p, x) = \exp[-\beta(x)[u_\nu(x)p^\nu - \mu_b(x) - \tilde{\mu}(x)]], \quad (42a)$$

$$\bar{Q}_{\text{eq}}(p, x) = \exp[-\beta(x)[u_\nu(x)p^\nu + \mu_b(x) + \tilde{\mu}(x)]], \quad (42b)$$

$$G_{\text{eq}}(p, x) = \exp[-\beta(x)[u_\nu(x)p^\nu - \tilde{\mu}_g(x)]], \quad (42c)$$

where $\beta(x)$, $u^\nu(x)$, and $\mu_b(x)$ are, respectively, the inverse temperature, hydrodynamic velocity, and baryon chemical potential which are all scalars in color space. The color chemical potentials $\tilde{\mu}$ and $\tilde{\mu}_g$ are Hermitian matrices $N_c \times N_c$ for quarks and $(N_c^2 - 1) \times (N_c^2 - 1)$ for gluons. They are gauge dependent variables, which transform as

$$\tilde{\mu}(x) \rightarrow U(x) \tilde{\mu}(x) U^\dagger(x), \quad \tilde{\mu}_g(x) \rightarrow \mathcal{U}(x) \tilde{\mu}_g(x) \mathcal{U}^\dagger(x). \quad (43)$$

In general, $\tilde{\mu}$ can be expressed as $\tilde{\mu} = \mu_0 + \mu_a \tau^a$. However, the singlet component μ_0 is already singled out as a baryon chemical potential μ_b . Therefore we write down $\tilde{\mu} = \mu_a \tau^a$. Consequently, the color chemical potential $\tilde{\mu}$ is not only Hermitian but also traceless. The covariant conservation of the color current provides the relation

$$\tilde{\mu}_g = 2T^a \operatorname{Tr}[\tau^a \tilde{\mu}] = \mu_a T^a, \quad (44)$$

which implies that $\tilde{\mu}_g$ is also traceless. The baryon and color chemical potentials occur in Eq. (42) because of the conservation laws of baryon number and color charge, respectively. The temperature and hydrodynamic velocity are related to the energy-momentum conservation.

The local equilibrium state described by Eq. (42) is *not* color neutral. Substituting the distribution functions (42) into Eq. (5) one finds the color current as

$$\begin{aligned} j^\mu = & -g \frac{T^3}{\pi^2} u^\mu \left[N_f \left(e^{\beta \mu_b} \left(e^{\beta \tilde{\mu}} - \frac{1}{N_c} \operatorname{Tr}[e^{\beta \tilde{\mu}}] \right) \right) \right. \\ & \left. - e^{-\beta \mu_b} \left(e^{-\beta \tilde{\mu}} - \frac{1}{N_c} \operatorname{Tr}[e^{-\beta \tilde{\mu}}] \right) \right] + 2\tau^a \operatorname{Tr}[T^a e^{\beta \tilde{\mu}_g}], \end{aligned} \quad (45)$$

where T , u^μ , μ_b , $\tilde{\mu}$, and $\tilde{\mu}_g$ are functions of x . The fact that the color current is finite does not imply that the system as a whole carries a finite color charge. We note that the x dependence of the color chemical potentials, which enter the solution (42), is not specified. Therefore it can be always chosen in such a way that the total color charge defined as $\int d^3x j^0$ vanishes.

The derivation of the local distribution function based on the collisional invariants tells nothing about the time scales when the colorful configuration (42) exists. To get such information the collision terms have to be specified. This is discussed in the next sections.

The equilibrium solutions (42) are given in an arbitrary gauge. It is often useful to work in a gauge where the quark and antiquark chemical potentials are diagonal. Then,

$$\tilde{\mu} = \mu^d \tau^d, \quad \tilde{\mu}_g = \mu^d T^d, \quad (46)$$

where τ^d and T^d are the fundamental and adjoint generators of the Cartan subalgebra of $SU(N_c)$ [$d=3$ for $SU(2)$ and $d=3,8$ for $SU(3)$]. In this gauge one has, as will be seen below, well-defined numbers of quarks and antiquarks of a certain color; and then, the physical meaning of the color chemical potentials becomes transparent.

A. Diagonal gauge for SU(2)

Using the explicit form of $\tau^3 = \sigma^3/2$, where σ^3 is the Pauli matrix, the singlet and the (nonvanishing) adjoint components [see Eqs. (17), (18)] of the quark and antiquark distribution functions of local equilibrium (42) are found as

$$\begin{aligned} q_0(p,x) &= q_{\uparrow}(p,x) + q_{\downarrow}(p,x), \\ q_3(p,x) &= q_{\uparrow}(p,x) - q_{\downarrow}(p,x), \end{aligned} \quad (47a)$$

$$\begin{aligned} \bar{q}_0(p,x) &= \bar{q}_{\uparrow}(p,x) + \bar{q}_{\downarrow}(p,x), \\ \bar{q}_3(p,x) &= \bar{q}_{\uparrow}(p,x) - \bar{q}_{\downarrow}(p,x), \end{aligned} \quad (47b)$$

where the scalar functions $q_{\uparrow\downarrow}$ and $\bar{q}_{\uparrow\downarrow}$ are

$$q_{\uparrow\downarrow}(p,x) \equiv \exp\left[-\beta(x)\left(u^\mu(x)p_\mu - \mu_b(x) \mp \frac{1}{2}\mu_3(x)\right)\right], \quad (47c)$$

$$\bar{q}_{\uparrow\downarrow}(p,x) \equiv \exp\left[-\beta(x)\left(u^\mu(x)p_\mu + \mu_b(x) \pm \frac{1}{2}\mu_3(x)\right)\right]. \quad (47d)$$

While the generator τ^3 is diagonal, T^3 is not. To derive the expressions for gluons one has to observe that $(T^3)^2$ is the diagonal matrix with 1,1,0 on the diagonal. Consequently, $(T^3)^n = T^3$ when $n=1,3,5,\dots$ and $(T^3)^n = (T^3)^2$ when $n=2,4,6,\dots$. Thus the nonvanishing components (23) of the gluon distribution function (42c) are

$$g_0(p,x) = g_{\uparrow}(p,x) + g_{\Rightarrow}(p,x) + g_{\downarrow}(p,x), \quad (48a)$$

$$g_3(p,x) = g_{\uparrow}(p,x) - g_{\downarrow}(p,x), \quad (48b)$$

$$g_{11}(p,x) = g_{22}(p,x) = \frac{1}{2}g_{\Rightarrow}(p,x), \quad (48c)$$

$$g_{33}(p,x) = \frac{1}{2}(g_{\uparrow}(p,x) + g_{\downarrow}(p,x) - g_{\Rightarrow}(p,x)), \quad (48d)$$

where the functions $g_{\uparrow\downarrow}$ and g_{\Rightarrow} are

$$\begin{aligned} g_{\uparrow\downarrow}(p,x) &\equiv \exp\{-\beta(x)[u^\mu(x)p_\mu \mp \mu_3(x)]\}, \\ g_{\Rightarrow}(p,x) &\equiv \exp[-\beta(x)u^\mu(x)p_\mu]. \end{aligned} \quad (48e)$$

In the diagonal gauge, a finite value of the color chemical potential simply means that the populations of quarks, antiquarks, and gluons of different colors are not the same.

Using the distribution functions in the form (47), (48), the color current (45) can be written as

$$j^\mu = -4g \frac{T^3}{\pi^2} u^\mu [N_f \text{ch}(\beta\mu_b) \text{sh}(\beta\mu_3/2) + \text{sh}(\beta\mu_3)] \tau^3. \quad (49)$$

B. Diagonal gauge for SU(3)

The local equilibrium solutions for the $SU(3)$ plasma can be also written in the diagonal gauge. However, the formulas are not that simple as for the $SU(2)$ case. We take the generators in the fundamental representation as $\tau^a = \lambda^a/2$, where λ^a are the Gell-Mann matrices. The matrices λ^3 and λ^8 are diagonal with the elements 1, -1, 0 and $1/\sqrt{3}$, $1/\sqrt{3}$, $-2/\sqrt{3}$, respectively, along the diagonal. With a color chemical potential in the directions $a=3$ and $a=8$ one can then easily evaluate the singlet and (nonvanishing) adjoint components of Q_{eq} and \bar{Q}_{eq} , which we write in terms of the distributions functions of red, blue, and green quarks and antiquarks. Here, we have taken the convention to assign the first, second, and third rows/columns of the Gell-Mann matrices to the red, blue, and green colors, respectively. A simple evaluation leads to

$$\begin{aligned} q_0(p,x) &= q_{\text{red}}(p,x) + q_{\text{blue}}(p,x) + q_{\text{green}}(p,x), \\ q_3(p,x) &= q_{\text{red}}(p,x) - q_{\text{blue}}(p,x), \end{aligned} \quad (50a)$$

$$q_8(p,x) = \frac{1}{\sqrt{3}}(q_{\text{red}}(p,x) + q_{\text{blue}}(p,x) - 2q_{\text{green}}(p,x)), \quad (50b)$$

$$\begin{aligned} \bar{q}_0(p,x) &= \bar{q}_{\text{red}}(p,x) + \bar{q}_{\text{blue}}(p,x) + \bar{q}_{\text{green}}(p,x), \\ \bar{q}_3(p,x) &= \bar{q}_{\text{red}}(p,x) - \bar{q}_{\text{blue}}(p,x), \end{aligned} \quad (50c)$$

$$\bar{q}_8(p,x) = \frac{1}{\sqrt{3}}(\bar{q}_{\text{red}}(p,x) + \bar{q}_{\text{blue}}(p,x) - 2\bar{q}_{\text{green}}(p,x)), \quad (50d)$$

where the distribution functions of quarks and antiquarks of different colors are of the form (47c) and (47d), respectively, but with the following color chemical potentials:

$$\begin{aligned} \mu_{\text{red}}(x) &= \frac{1}{2}\left(\mu_3(x) + \frac{\mu_8(x)}{\sqrt{3}}\right), \\ \mu_{\text{blue}}(x) &= -\frac{1}{2}\left(\mu_3(x) - \frac{\mu_8(x)}{\sqrt{3}}\right), \\ \mu_{\text{green}}(x) &= -\frac{\mu_8(x)}{\sqrt{3}}. \end{aligned} \quad (51)$$

The computation of the singlet and adjoint components of the local equilibrium distribution function of gluons is much more involved. The evaluation of the traces requires one to

expand the exponentials and compute the traces of arbitrary powers of T_3 , of T_8 , and of $T_3 T_8$. With the help of MATHEMATICA, we have found the singlet and (nonvanishing) adjoint components as

$$g_0(p,x) = 2g_s(p,x) + g_{x+}(p,x) + g_{x-}(p,x) + g_{y+}(p,x) + g_{y-}(p,x) + g_{z+}(p,x) + g_{z-}(p,x), \quad (52)$$

$$g_3(p,x) = g_{z+}(p,x) - g_{z-}(p,x) + \frac{1}{2}(g_{x+}(p,x) + g_{x-}(p,x) + g_{y+}(p,x) + g_{y-}(p,x)), \quad (53)$$

$$g_8(p,x) = \sqrt{\frac{3}{2}}(g_{x+}(p,x) - g_{x-}(p,x) - g_{y+}(p,x) + g_{y-}(p,x)), \quad (54)$$

where the scalar functions g_s , $g_{x\pm}$, $g_{y\pm}$, and $g_{z\pm}$ are analogous to those from Eq. (48e) but their color chemical potentials are

$$\begin{aligned} \mu_s(x) = 0, \quad \mu_{x\pm}(x) &= \pm \frac{\mu_3(x)}{2} \pm \frac{\sqrt{3}\mu_8(x)}{2}, \\ \mu_{y\pm}(x) &= \pm \frac{\mu_3(x)}{2} \mp \frac{\sqrt{3}\mu_8(x)}{2}, \quad \mu_{z\pm}(x) = \pm \mu_3(x). \end{aligned} \quad (55)$$

Exactly as in the SU(2) case, we find that a finite value of the color chemical potential means that quarks, antiquarks, and gluons of different colors have different densities.

IV. LOCAL EQUILIBRIUM FROM VANISHING COLLISION TERMS

As follows from Eq. (40), there is no entropy production when the collision terms vanish. Thus local equilibrium is reached when the gain and loss terms compensate each other. Consequently, we will look for solutions of the equation $C = 0$. However, there are numerous scattering processes occurring in the quark-gluon plasma and, in general, the complete set of collision terms entering into the quark, antiquark, and gluon kinetic equations is rather large, even so we only consider the binary collisions. The most probable processes, i.e., those with the largest cross section, occur when two partons exchange a soft gluon in the t or u channels. The later possibility only happens for interaction of identical partons—quarks of the same flavor or gluons. In vacuum, the corresponding cross sections diverge as t^{-2} or u^{-2} when the four-momentum transfer t or u goes to zero. In the medium, these divergences are softened, as the gluon propagators are dressed by the interactions, and the electric and magnetic forces are either statically or dynamically screened. In the local equilibrium state, which is achieved at the shortest time scale, the collision terms associated with those processes, we call them “dominant,” have to vanish. Thus we will first consider the interactions: $qq \leftrightarrow qq$, $\bar{q}\bar{q} \leftrightarrow \bar{q}\bar{q}$, $q\bar{q} \leftrightarrow q\bar{q}$, $gq \leftrightarrow gg$, $qg \leftrightarrow qg$, and $\bar{q}g \leftrightarrow \bar{q}g$, and we will neglect all

other processes, as they are relevant for longer time scales.¹ These less probable processes drive the system either to a different local equilibrium or to the global equilibrium. We will also consider the subdominant processes with the soft quark in t or u channel which correspond to the vacuum cross sections diverging as t^{-1} or u^{-1} , respectively. These are the quark-antiquark annihilation and creation into and from two gluons in t or u channel which, as will be shown, have a qualitative effect on the local equilibrium state. With the subdominant processes, one should also consider all the channels and the respective crossing terms of the various binary collisions, plus another set of collisions that do not conserve the particle number. The complete analysis is very complex, and we will not carry it out here.

In this section we write down the relevant collision terms, and then we discuss the equations imposed by the vanishing of these terms. Finally, we solve the equations, showing that the nature of local equilibrium is fixed by the color structure of the scattering amplitudes.

A. Collision terms

The dominant parton-parton scattering amplitudes with one-gluon exchange in the t and u channels are of the form

$$M_{rsr's'}(p,p_1;p',p'_1) = \mathcal{M}(p,p_1;p',p'_1) T_{rr'}^a \tilde{T}_{ss'}^a \quad (56)$$

$$M_{rsr's'}(p,p_1;p',p'_1) = \mathcal{M}(p,p_1;p',p'_1) T_{rs}^a \tilde{T}_{s'r'}^a, \quad (57)$$

where T^a and \tilde{T}^a are the group generators of SU(N_c) of the two partons participating in the collision: $T^a = T^a$ for gluons, $T^a = \tau^a$ for quarks, and $T^a = -(\tau^a)^T$, where T means transposition, for antiquarks. With the t -channel amplitude (56), the collision term (29) equals

$$\begin{aligned} C[N,N_1,N',N'_1] &= \int dP' dP'_1 dP_1 (2\pi)^4 \delta^{(4)}(p+p_1 \\ &\quad - p' - p'_1) |\mathcal{M}|^2 \left(T^a N' T^b \text{Tr}[\tilde{T}^a N'_1 \tilde{T}^b] \right. \\ &\quad \left. - \frac{1}{2} \{ T^b T^a, N \} \text{Tr}[\tilde{T}^a N_1 \tilde{T}^b] \right), \end{aligned} \quad (58)$$

where we have neglected the effects of quantum statistics, and consequently the terms $1 \pm N$ have been replaced by unity. The collision term corresponding to the u -channel amplitude (57) can be found from Eq. (58) by means of the exchange $N \leftrightarrow N_1$ and $N' \leftrightarrow N'_1$ in the rhs of Eq. (58).

Using the identity (A1) given in the Appendix, we can write down Eq. (58) for the case of quark-quark scattering as

¹To estimate a mean free time associated with a given collision process one has to specify the distribution function. For a discussion of those mean free times in global equilibrium see Ref. [28].

$$\begin{aligned}
 & C[Q, Q_1, Q', Q'_1] \\
 &= \frac{1}{2} \int dP' dP'_1 dP_1 (2\pi)^4 \delta^{(4)}(p+p_1-p'-p'_1) |\mathcal{M}|^2 \\
 &\quad \times \left(\left(\text{Tr}[Q'] Q'_1 - \text{Tr}[Q] Q_1 \right) - \frac{1}{N_c^2} (Q' \text{Tr}[Q'_1] \right. \\
 &\quad \left. - Q \text{Tr}[Q_1]) - \frac{1}{N_c} (\{Q', Q'_1\} - \{Q, Q_1\}) \right). \quad (59)
 \end{aligned}$$

The collision term (58) for the quark-antiquark scattering is

$$\begin{aligned}
 & C[Q, \bar{Q}_1, Q', \bar{Q}'_1] \\
 &= \frac{1}{2} \int dP' dP'_1 dP_1 (2\pi)^4 \delta^{(4)}(p+p_1-p'-p'_1) |\mathcal{M}|^2 \\
 &\quad \times \left(\left(\text{Tr}[Q' \bar{Q}'_1] - \frac{N_c}{2} \{Q, \bar{Q}_1\} \right) - \frac{1}{N_c^2} (Q' \text{Tr}[\bar{Q}'_1] \right. \\
 &\quad \left. - Q \text{Tr}[\bar{Q}_1]) - \frac{1}{N_c} (\{Q', \bar{Q}'_1\} - \{Q, \bar{Q}_1\}) \right), \quad (60)
 \end{aligned}$$

where, as discussed previously, we have replaced \bar{Q}^T by \bar{Q} .

For the gluon-gluon scattering we have found a simplification of Eq. (58) only in the case of the SU(2) gauge group. Then, the collision term reads

$$\begin{aligned}
 & C[G, G_1, G', G'_1] \\
 &= \int dP' dP'_1 dP_1 (2\pi)^4 \delta^{(4)}(p+p_1-p'-p'_1) |\mathcal{M}|^2 \\
 &\quad \times \left(\left(\text{Tr}[G'^T G'_1] - \{G'^T, G'_1\} - \frac{1}{2} \{G, G_1\} \right) \right. \\
 &\quad \left. + (\text{Tr}[G'] G'_1 - G \text{Tr}[G_1]) \right). \quad (61)
 \end{aligned}$$

The scattering amplitudes of the subdominant processes with the quark exchange in the t and u channel have the following color structure:

$$M_{ijab}(p, p_1; p', p'_1) = \mathcal{M}(p, p_1; p', p'_1) \tau_{ik}^a \tau_{kj}^b, \quad (62)$$

$$M_{ijab}(p, p_1; p', p'_1) = \mathcal{M}(p, p_1; p', p'_1) \tau_{ik}^b \tau_{kj}^a. \quad (63)$$

The collision term associated with this t -channel annihilation processes is

$$\begin{aligned}
 & C[Q, \bar{Q}_1, G', G'_1] \\
 &= \int dP' dP'_1 dP_1 (2\pi)^4 \delta^{(4)}(p+p_1-p'-p'_1) \\
 &\quad \times |\mathcal{M}|^2 \left(\tau^a \tau^b \tau^{\bar{b}} \tau^{\bar{a}} G^{a\bar{a}}(p') G^{b\bar{b}}(p'_1) \right. \\
 &\quad \left. - \frac{1}{2} \{Q(p), \tau^a \tau^b \bar{Q}(p_1) \tau^b \tau^a\} \right). \quad (64)
 \end{aligned}$$

At the end of this section we call the attention of the reader to the structure of the collision terms (59), (60), (61). Because there are only objects like Q , G , G^T , which transform covariantly with respect to the gauge transformation (1), (2), and $\text{Tr}[Q]$, $\text{Tr}[G]$, and $\text{Tr}[Q\bar{Q}_1]$, which are gauge invariant, these collision terms transform covariantly, provided $|\mathcal{M}|^2$ is gauge invariant. The gauge covariance of the collision term (64) is evident when instead of C the projections $\text{Tr}[C]$ and $\text{Tr}[\tau^a C]$ are considered. As will be seen in the following section, these projections have the right gauge structure.

B. Conditions of local equilibrium

In this section we present the conditions for the cancellation of the collision terms associated with the processes discussed above.

1. $qq \leftrightarrow q\bar{q}$

The collision term (59) corresponding to the quark-quark scattering vanishes if

$$\begin{aligned}
 & \left(\text{Tr}[Q'] Q'_1 - \text{Tr}[Q] Q_1 \right) - \frac{1}{N_c^2} (Q' \text{Tr}[Q'_1] - Q \text{Tr}[Q_1]) \\
 & - \frac{1}{N_c} (\{Q', Q'_1\} - \{Q, Q_1\}) = 0, \quad (65)
 \end{aligned}$$

where $p+p_1=p'+p'_1$. Because the quark matrix transport equation can be uniquely characterized by its singlet and adjoint components [see Eq. (19)], the condition (65) requires

$$\text{Tr}[Q Q_1] = \text{Tr}[Q' Q'_1], \quad (66a)$$

$$\text{Tr}[Q] \text{Tr}[Q_1] = \text{Tr}[Q'] \text{Tr}[Q'_1], \quad (66b)$$

and

$$\text{Tr}[\tau^a \{Q, Q_1\}] = \text{Tr}[\tau^a \{Q', Q'_1\}], \quad (66c)$$

$$\text{Tr}[\tau^a Q] \text{Tr}[Q_1] = \text{Tr}[Q'] \text{Tr}[\tau^a Q'_1], \quad (66d)$$

$$\text{Tr}[\tau^a Q] \text{Tr}[Q_1] = \text{Tr}[\tau^a Q'] \text{Tr}[Q'_1]. \quad (66e)$$

The conditions for cancellation of the collision term for antiquark-antiquark scattering are totally analogous to those of the quark-quark case.

2. $q\bar{q} \leftrightarrow q\bar{q}$

The collision term (58) for the quark-antiquark scattering vanishes when

$$\left(\text{Tr}[Q' \bar{Q}'_1] - \frac{N_c}{2} \{Q, \bar{Q}_1\} \right) - \frac{1}{N_c^2} (Q' \text{Tr}[\bar{Q}'_1] - Q \text{Tr}[\bar{Q}_1]) - \frac{1}{N_c} (\{Q', \bar{Q}'_1\} - \{Q, \bar{Q}_1\}) = 0. \quad (67)$$

The conditions of cancellation of the projected matrix equation (67) read

$$\text{Tr}[Q \bar{Q}_1] = \text{Tr}[Q' \bar{Q}'_1], \quad (68a)$$

$$\text{Tr}[Q] \text{Tr}[\bar{Q}_1] = \text{Tr}[Q'] \text{Tr}[\bar{Q}'_1], \quad (68b)$$

and

$$\text{Tr}[\tau^a \{Q, \bar{Q}_1\}] = \text{Tr}[\tau^a \{Q', \bar{Q}'_1\}] = 0, \quad (68c)$$

$$\text{Tr}[\tau^a Q] \text{Tr}[\bar{Q}_1] = \text{Tr}[\tau^a Q'] \text{Tr}[\bar{Q}'_1]. \quad (68d)$$

The requirement that $\text{Tr}[\tau^a \{Q, \bar{Q}_1\}] = 0$ directly follows from the first term of Eq. (67).

3. $qg \leftrightarrow qg$

The collision term for quark-gluon scattering with one-gluon exchange in the t channel vanishes if

$$\tau^a Q' \tau^b \text{Tr}[T^a G'_1 T^b] - \frac{1}{2} \{\tau^b \tau^a, Q\} \text{Tr}[T^a G_1 T^b] = 0. \quad (69)$$

Requiring that $\text{Tr}[C] = 0$ and $\text{Tr}[\tau^a C] = 0$ provides the equations

$$\begin{aligned} & \text{Tr}[Q] \text{Tr}[G_1] - \frac{N_c}{2} \text{Tr}[\tau^a Q] \text{Tr}[T^a G_1] \\ & + \frac{1}{2} d^{abc} \text{Tr}[\tau^a Q] \text{Tr}[T^b T^c G_1] \\ & = \text{Tr}[Q'] \text{Tr}[G'] - \frac{N_c}{2} \text{Tr}[\tau^a Q'] \text{Tr}[T^a G'_1] \\ & + \frac{1}{2} d^{abc} \text{Tr}[\tau^a Q'] \text{Tr}[T^b T^c G'_1] \end{aligned} \quad (70a)$$

and

$$\begin{aligned} & \text{Tr}[\tau^c \tau^b Q \tau^a] \text{Tr}[T^b G_1 T^a] + R^c[Q, G_1] \\ & = \text{Tr}[\tau^c \tau^b Q' \tau^a] \text{Tr}[T^b G'_1 T^a], \end{aligned} \quad (70b)$$

where

$$\begin{aligned} R^c[Q, G_1] & \equiv \frac{i}{2} f_{cad} (\text{Tr}[\tau^d \tau^b Q] \text{Tr}[T^b G_1 T^a] \\ & - \text{Tr}[\tau^b \tau^d Q] \text{Tr}[T^a G_1 T^b]). \end{aligned} \quad (70c)$$

4. $gg \leftrightarrow gg$

The collision term of the gluon-gluon scattering equals zero when

$$T^a G' T^b \text{Tr}[T^a G'_1 T^b] - \frac{1}{2} \{T^b T^a, G\} \text{Tr}[T^a G_1 T^b] = 0. \quad (71)$$

For the SU(2) plasma the above condition can be simplified [see Eq. (61)] and it gives

$$\begin{aligned} & \left(\text{Tr}[G'^T G'_1] - \{G'^T, G'_1\} - \frac{1}{2} \{G, G_1^T\} \right) \\ & + (\text{Tr}[G'] G'_1 - G \text{Tr}[G_1]) = 0. \end{aligned} \quad (72)$$

We demand the cancellation of the totally symmetric and antisymmetric components of Eq. (72), see Eq. (27). Imposing $\text{Tr}[T_a C_g] = 0$ and $\text{Tr}[\{T_a, T_b\} C_g] = 0$, we get

$$\text{Tr}[T_a \{G, G_1^T\}] = \text{Tr}[T_a \{G', G'_1^T\}] = 0, \quad (73a)$$

$$\text{Tr}[T_a G] \text{Tr}[G_1] = \text{Tr}[G'] \text{Tr}[T_a G'_1], \quad (73b)$$

and

$$\begin{aligned} & \text{Tr}[\{T_a, T_b\} \{G, G_1^T\}] \\ & = 8 \delta^{ab} \text{Tr}[G' G'_1^T] - 2 \text{Tr}[\{T_a, T_b\} \{G', G'_1^T\}] \end{aligned} \quad (73c)$$

$$\text{Tr}[\{T_a, T_b\} G] \text{Tr}[G_1] = \text{Tr}[G'] \text{Tr}[\{T_a, T_b\} G'_1]. \quad (73d)$$

For $a \neq b$ Eq. (73c) requires that

$$\text{Tr}[\{T_a, T_b\} \{G, G_1^T\}] = \text{Tr}[\{T_a, T_b\} \{G', G'_1^T\}] = 0, \quad (73e)$$

while for $a = b$ ($T^a T^a = 2$) Eqs. (73c), (73d) imply

$$\text{Tr}[G G_1^T] = \text{Tr}[G' G'_1^T], \quad (73f)$$

$$\text{Tr}[G] \text{Tr}[G_1] = \text{Tr}[G'] \text{Tr}[G'_1]. \quad (73g)$$

5. $q\bar{q} \leftrightarrow qg$

With the scattering amplitude given in Eq. (62), the cancellation of the collision term corresponding to the quark-antiquark annihilation in the t channel demands

$$\text{Tr}[\tau^a Q \tau^a \tau^b \bar{Q}_1 \tau^b] = \text{Tr}[\tau^c \tau^d \tau^d \tau^c] G'^{c\bar{c}} G'^{d\bar{d}}, \quad (74a)$$

$$\frac{1}{2} \text{Tr}[\tau^e \{Q, \tau^a \tau^b \bar{Q}_1 \tau^b \tau^a\}] = \text{Tr}[\tau^e \tau^c \tau^d \tau^d \tau^c] G'^{c\bar{c}} G'^{d\bar{d}}. \quad (74b)$$

The left-hand side of the above equations can be simplified using the relation (A4) given in the Appendix and the formula

$$\tau^a \tau^b \bar{Q}_1 \tau^b \tau^a = \frac{1}{4N_c^2} \bar{Q}_1 + \frac{N_c^2 - 2}{4N_c} \text{Tr}[\bar{Q}_1]. \quad (75)$$

Furthermore, for the SU(2) plasma one finds, using the relations (A7), (A8) given in the Appendix, that

$$\begin{aligned} & \text{Tr}[Q\bar{Q}_1] + 4 \text{Tr}[Q] \text{Tr}[\bar{Q}_1] \\ &= 2 \text{Tr}[G'^T G'_1] - 2 \text{Tr}[G' G'_1] + 2 \text{Tr}[G'] \text{Tr}[G'_1], \end{aligned} \quad (76a)$$

$$\begin{aligned} & \text{Tr}[\tau^e \{Q, \bar{Q}_1\}] + 8 \text{Tr}[\tau^e Q] \text{Tr}[\bar{Q}_1] \\ &= 2G'^{ce} \text{Tr}[T^c G'_1] + 2 \text{Tr}[T^e G'^T_1 G'] \\ & \quad - 2 \text{Tr}[T^e G'_1 G'] + 2 \text{Tr}[T^e G'] \text{Tr}[G'_1]. \end{aligned} \quad (76b)$$

C. Local equilibrium solution for the SU(2) plasma

We find here the local equilibrium solutions that cancel all the collision terms discussed in the previous section for the SU(2) plasma. We start with the quark-quark scattering. Equations (66a), (66c) are solved by functions obeying

$$Q(p, x) Q(p_1, x) = Q(p', x) Q(p'_1, x), \quad (77)$$

for $p + p_1 = p' + p'_1$. Using standard arguments, see, e.g., [26], one finds that Eq. (77) is satisfied by exponential functions

$$Q(p, x) = \exp[-\beta(x)[\bar{u}_v(x)p^v - \mu_b(x) - \tilde{\mu}(x)]], \quad (78)$$

where $\bar{u}^\mu(x)$ and $\tilde{\mu}(x)$ are Hermitian and traceless matrices. Please note that the scalar chemical potential μ_b , which is interpreted as the baryon chemical potential, is already singled out. Because of Eq. (39), $\bar{u}^\mu(x)$ and $\tilde{\mu}(x)$ should obey the condition $[\bar{u}^\mu(x), \tilde{\mu}(x)] = 0$. Thus using the gauge freedom to rotate these quantities in color space, they can be chosen in diagonal form.

Equations (66b), (66d), (66e) require that the hydrodynamic velocity $\bar{u}^\mu(x)$ is proportional to the unit matrix. Otherwise different components of $\bar{u}^\mu(x)$ enter differently Eqs. (66b), (66d), (66e) and the constraint $p + p_1 = p' + p'_1$ is insufficient to satisfy these equations. Once $\bar{u}^\mu(x)$ is proportional to the unit matrix, the condition $[\bar{u}^\mu(x), \tilde{\mu}(x)] = 0$ is trivially satisfied, and there is no reason to require $\tilde{\mu}(x)$ to be diagonal. It is then an arbitrary traceless matrix, even so it can be diagonalized because of the gauge freedom. Since the hydrodynamic velocity is no longer a color matrix but a scalar, it is from now on denoted as u^μ not as \bar{u}^μ .

Repeating fully analogous considerations for the collision term of antiquark-antiquark scattering, we arrive at the antiquark distribution function

$$\bar{Q}(p, x) = \exp[-\bar{\beta}(x)[\bar{u}_v(x)p^v + \bar{\mu}_b(x) + \tilde{\bar{\mu}}(x)]]. \quad (79)$$

The conditions of cancellation for the quark-antiquark collision term provide the relations between the parameters of quark and antiquark distribution functions. Namely, Eqs. (68a), (68b), (68d) require

$$\beta(x)u^\mu(x) = \bar{\beta}(x)\bar{u}^\mu(x). \quad (80)$$

Because $u^\mu(x)u_\mu(x) = \bar{u}^\mu(x)\bar{u}_\mu(x) = 1$, we effectively have

$$u^\mu(x) = \bar{u}^\mu(x), \quad T(x) = \bar{T}(x). \quad (81)$$

Furthermore, Eq. (68c) imposes

$$\tilde{\mu}(x) = -\tilde{\bar{\mu}}(x), \quad (82)$$

but it leaves the baryon chemical potentials μ_b and $\bar{\mu}_b$ unrestricted.

Let us find now the distribution functions that cancel the gluon-gluon collision term. Condition (73f) is solved by those functions obeying

$$G(p, x)G^T(p_1, x) = G(p', x)G^T(p'_1, x), \quad (83)$$

which demands that

$$G(p, x) = \exp[-\beta_g(x)[\tilde{u}_g^v(x)p_v - \mu_g^0(x) - \tilde{\mu}_g(x)]] \quad (84)$$

where both $\tilde{u}_g^\mu(x)$ and $\tilde{\mu}_g(x)$ are Hermitian matrices while $\mu_g^0(x)$ is a scalar. Furthermore, $\tilde{u}_g^\mu(x) = [\tilde{u}_g^\mu(x)]^T$, which implies that $\tilde{u}_g^\mu(x)$ is a real symmetric matrix. However, the conditions (73g) and (73b) require that the gluon velocity matrix has to be proportional to the unit matrix, exactly as that of quarks and antiquarks.

The condition (73a) or (73e) implies that the product GG_1^T must be proportional to the unit matrix. Therefore the gluon color chemical potential must obey $\tilde{\mu}_g^T(x) = -\tilde{\mu}_g(x)$. Consequently, it contains only antisymmetric components and it can be uniquely expressed as $\tilde{\mu}_g(x) = \mu_g^a(x)T^a$.

Next, we analyze the conditions for cancellation of the quark-gluon collision term, i.e., Eqs. (70a), (70b). For SU(2) $d_{abc} = 0$, and then it is easy to check that Eq. (70a) imposes

$$T(x) = T_g(x), \quad u^\mu(x) = u_g^\mu(x). \quad (85)$$

Thus, the temperature, as well as the hydrodynamic velocity, are the same for the quark-antiquark and gluon components of the plasma.

Equation (70b) is a more complicated structure. Since it is fulfilled if $R^c = 0$, let us evaluate R^c . Taking into account that for SU(2)

$$\text{Tr}[\tau^a \tau^b Q(p, x)] = \frac{i}{4} f^{abc} q_c(p, x) + \frac{1}{4} \delta^{ab} q_0(p, x), \quad (86)$$

and using the relation (A9) given in the Appendix, we express R^c as

$$R^c = -\frac{1}{8} q_a(p, x) \text{Tr}[\{T^a, T^c\} G(p_1, x)] \\ + \frac{1}{2} q_c(p, x) g_0(p_1, x) - \frac{1}{4} q_0(p, x) g_c(p_1, x). \quad (87)$$

And now we refer to the diagonal gauge where the quark chemical potential is of the form $\tilde{\mu}(x) = \mu_3(x) \tau^3$. Requiring $R^1 = R^2 = 0$ implies $g_1 = g_2 = 0$, which, in turn, demands that the respective components of the gluon chemical potential vanish, i.e., $\mu_g^1 = \mu_g^2 = 0$. Demanding $R^3 = 0$ is only fulfilled if

$$\mu_3(x) = \mu_g^3(x). \quad (88)$$

Thus, Eq. (70b) is satisfied in arbitrary gauge if the relation (44) holds.

The dominant processes that have been considered up to now do not introduce any relation between the quark and antiquark baryon chemical potentials and they do not constrain the scalar gluon potential μ_g^0 . It is not surprising as these processes do not change the number of quarks, antiquarks, or gluons. To get the relation between μ_b , $\bar{\mu}_b$, and μ_g^0 , the subdominant process of quark-antiquark creation or annihilation has to be taken into account. Let us analyze this process. The color structure of Eqs. (76a), (76b) is rather complex. However, one checks that these equations are solved by the local equilibrium function (42) in the diagonal gauge (47), (48). In particular, one finds that

$$\text{Tr}[Q \bar{Q}_1] + 4 \text{Tr}[Q] \text{Tr}[\bar{Q}_1] \\ = e^{-\beta[u \cdot (p+p_1) - \mu_b + \bar{\mu}_b]} (10 + 4 e^{\beta\mu_3} + 4 e^{-\beta\mu_3}), \quad (89) \\ 2 \text{Tr}[G'^T G'_1] - 2 \text{Tr}[G' G'_1] + 2 \text{Tr}[G'] \text{Tr}[G'_1] \\ = e^{-\beta[u \cdot (p'+p'_1) - 2\mu_g^0]} (10 + 4 e^{\beta\mu_3} + 4 e^{-\beta\mu_3}). \quad (90)$$

Thus Eq. (76a) demands

$$\mu_b + \bar{\mu}_b = 2\mu_g^0. \quad (91)$$

While the checking is rather simple for Eq. (76a), it is much more difficult for Eq. (76b). To reach the goal we have expressed the (anti-)quark and gluon distribution functions through the projections (17) and (23), respectively, and we have used the formula (24). Then, one finds that Eq. (76b) is satisfied if the relation (91) holds.

To get the chemical potentials as in the local equilibrium function (42) the binary processes are insufficient. One easily

observes that the equilibrium with respect to the process $gg \leftrightarrow ggg$ implies $\mu_g^0 = 0$, and then, Eq. (91) provides $\mu_b = -\bar{\mu}_b$.

In summary, the requirement of equilibrium with respect to the dominant binary processes provides the local equilibrium state with the color structure as that in Eq. (42) which comes from the collisional invariants. The (scalar) chemical potentials of quarks, antiquarks, and gluons are, however, independent from each other. To get the relations $\mu_g^0 = 0$ and $\mu_b = -\bar{\mu}_b$, the multigluon processes and antiquark-quark annihilation into gluons must be taken into account. This means that the local *chemical* equilibrium is reached at longer time scale than the color equilibrium.

D. Local equilibrium solution for the $SU(N_c)$ plasma

We find here the local equilibrium solutions for the $SU(N_c)$ plasma. The quark-quark and antiquark-quark scattering processes are treated as for the $SU(2)$ case. The solutions of Eqs. (66), (68) read

$$Q(p, x) = \exp[-\beta(x)[\tilde{u}_v(x)p^v - \mu_b(x) - \tilde{\mu}(x)]], \quad (92)$$

$$\bar{Q}(p, x) = \exp[-\beta(x)[u_v(x)p^v + \bar{\mu}_b(x) + \tilde{\mu}(x)]]. \quad (93)$$

The conditions for cancellation of the collision terms discussed in Sec. IV A, which involve gluons, are much more complicated than those for $SU(2)$. Here, we will treat them perturbatively only. The requirement of vanishing of the collision term representing gluon-gluon scattering is expressed by Eq. (71). We first note that a distribution function proportional to the identity matrix, which is of the form $\exp[-\beta_g(u_g^v p_v - \mu_g^0)]$ satisfies this equation. We now look for more general solutions written as

$$G(p, x) = \exp[-\beta_g(x)(u_g^v(x)p_v - \mu_g^0(x))] F[\tilde{\alpha}(x)], \quad (94)$$

where we have factored out the $U(1)$ part of the distribution function; F is an arbitrary function of $\tilde{\alpha}(x) = \beta_g(x) \tilde{\mu}_g(x)$ with $\tilde{\mu}_g(x)$ being any Hermitian $(N_c^2 - 1) \times (N_c^2 - 1)$ matrix. From Eq. (71) one deduces that F should obey the quadratic equation

$$\left(T^a F[\tilde{\alpha}] T^b - \frac{1}{2} \{T^b T^a, F[\tilde{\alpha}]\} \right) \text{Tr}[F[\tilde{\alpha}] T^b T^a] = 0, \quad (95)$$

which is trivially satisfied by the unit matrix. We now assume that F allows for an infinitesimal expansion in $\tilde{\alpha}$ around the identity. Then,

$$F[\tilde{\alpha}] = 1 + \tilde{\alpha} + \dots, \quad (96)$$

and Eq. (95) imposes

$$T^a[\tilde{\alpha}, T^a] + \frac{1}{2} T^c \text{Tr}[T^c \tilde{\alpha}] = 0. \quad (97)$$

If $\tilde{\alpha}$ is proportional to the unit matrix, the equation is obviously satisfied. However, we exclude this possibility since a scalar chemical potential was already included in the U(1) part of Eq. (94). A different solution of the equation is given by $\tilde{\alpha} = \alpha_a T^a$. With the last option, we solve Eq. (95) to second order in $\tilde{\alpha}$, and find

$$F[\tilde{\alpha}] = 1 + \tilde{\alpha} + \frac{\tilde{\alpha}^2}{2} + \dots \quad (98)$$

In principle, one can solve the equation iteratively order by order in $\tilde{\alpha}$ but the procedure becomes more and more difficult with every order. We will not follow it but the above results suggest that the general solution is of the form

$$G(p, x) = \exp[-\beta_g(x)[u_g^\nu(x)p_\nu - \mu_g^0(x) - \tilde{\mu}_g(x)]], \quad (99)$$

as it should reduce to the $N_c = 2$ solution (84) with the scalar hydrodynamic velocity.

We now look for the quark-gluon scattering, and solve Eq. (69) perturbatively for small color chemical potentials of quarks and gluons. In 0th order, Eq. (69) imposes

$$T(x) = T_g(x), \quad u^\mu(x) = u_g^\mu(x). \quad (100)$$

In the first order in the color chemical potentials, we find that these should obey

$$\begin{aligned} & \left(\tau^a \tilde{\mu}(x) \tau^b - \frac{1}{2} \{ \tau^b \tau^a, \tilde{\mu}(x) \} \right) \text{Tr}[T^b T^a] \\ & + [\tau^a, \tau^b] \text{Tr}[\tilde{\mu}_g(x) T^b T^a] = 0, \end{aligned} \quad (101)$$

which is only satisfied if

$$\tilde{\mu}_g(x) = 2T^a \text{Tr}[\tilde{\mu}(x) \tau^a]. \quad (102)$$

One could go to higher orders in the expansion but the procedure becomes very tedious.

In the same way, one can treat the remaining processes such as the quark-antiquark annihilation. They lead to the same constraints as those for the SU(2) plasma expressed, in particular, by Eq. (91).

The perturbative treatment presented here is concluded as follows. At zeroth order, the various collision processes allow one to fix the variables which are scalar in color space—the temperature and hydrodynamic velocity. At first order, every collision term imposes restrictions on the form of the matrix chemical potentials. Solving the conditions to all orders should simply provide the solutions, which for classical statistics are exponential functions of color chemical potentials.

V. CHROMOHYDRODYNAMICS

The form of the local equilibrium distribution function determines the character of hydrodynamics obeyed by the system. Here, we are going to discuss the hydrodynamic equations corresponding to the local equilibrium state found

in the previous sections. As we have shown in Sec. IV C, the dominant processes, which are responsible for establishing the colorful equilibrium, do not equilibrate the system with respect to the scalar chemical potentials. The relations $\mu_g^0 = 0$ and $\mu_b = -\bar{\mu}_b$ are achieved at longer time scales. Since we are mostly interested here in the role of color charges in the hydrodynamic evolution, we neglect complications caused by the lack of chemical equilibrium and we use the distribution functions (42) where the relations $\mu_g^0 = 0$ and $\mu_b = -\bar{\mu}_b$ are built in.

The equations of hydrodynamics are provided by the macroscopic conservation laws of the baryon charge (32), energy-momentum (33), and of the color charge (34). Substituting the local equilibrium distribution functions (42) into Eqs. (7), (8), (5), one gets the baryon current, the energy-momentum tensor, and the color current which enter the equations of *ideal* hydrodynamics where dissipative effects are neglected. These quantities read

$$b^\mu(x) = b(x)u^\mu(x), \quad (103a)$$

$$t^{\mu\nu}(x) = [\varepsilon(x) + p(x)]u^\mu(x)u^\nu(x) - p(x)g^{\mu\nu}, \quad (103b)$$

$$j^\mu(x) = \rho(x)u^\mu(x), \quad (103c)$$

where b , ε , and ρ are the densities of, respectively, the baryon charge, energy, and color, while p denotes the pressure. In contrast to b , ε , and p which are color scalars, the color density ρ is a $N_c \times N_c$ matrix. All these thermodynamic quantities are given as

$$b = \frac{2N_f T^3}{3\pi^2} [e^{\beta\mu_b} \text{Tr}[e^{\beta\tilde{\mu}}] - e^{-\beta\mu_b} \text{Tr}[e^{-\beta\tilde{\mu}}]], \quad (104)$$

$$\begin{aligned} \varepsilon = 3p = & \frac{6T^4}{\pi^2} [N_f (e^{\beta\mu_b} \text{Tr}[e^{\beta\tilde{\mu}}] \\ & + e^{-\beta\mu_b} \text{Tr}[e^{-\beta\tilde{\mu}}]) + \text{Tr}[e^{\beta\tilde{\mu}_g}]], \end{aligned} \quad (105)$$

$$\begin{aligned} \rho = -g \frac{T^3}{\pi^2} & \left[N_f \left(e^{\beta\mu_b} \left(e^{\beta\tilde{\mu}} - \frac{1}{N_c} \text{Tr}[e^{\beta\tilde{\mu}}] \right) \right. \right. \\ & \left. \left. - e^{-\beta\mu_b} \left(e^{-\beta\tilde{\mu}} - \frac{1}{N_c} \text{Tr}[e^{-\beta\tilde{\mu}}] \right) \right) + 2\tau^a \text{Tr}[T^a e^{\beta\tilde{\mu}_g}] \right]. \end{aligned} \quad (106)$$

Now, we consider Eq. (33) representing the energy-momentum conservation. It is well known [29] that projecting the continuity equation of the energy-momentum tensor on the hydrodynamic velocity, one gets the condition of the entropy conservation during the fluid motion. Let us see how it works here. Multiplying Eq. (33) by u^μ , we get

$$u_\mu \partial_\nu t^{\mu\nu} = 0 \quad (107)$$

because $u_\mu u^\mu = 1$ and $u_\mu u_\nu F^{\mu\nu} = 0$. The latter equality holds due to the antisymmetry of $F^{\mu\nu}$. Equation (107) gives

$$u_\mu \partial^\mu \varepsilon + (\varepsilon + p) \partial^\mu u_\mu = 0, \quad (108)$$

which can be rewritten as

$$T \partial_\mu (s u^\mu) + \mu_b \partial_\mu (b u^\mu) + \text{Tr}[\tilde{\mu} \partial_\mu (\rho u^\mu)] = 0, \quad (109)$$

by means of the thermodynamic relations

$$d\varepsilon = T ds + \mu_b db + \text{Tr}[\tilde{\mu} d\rho], \quad (110)$$

$$\varepsilon + p = Ts + \mu_b b + \text{Tr}[\tilde{\mu} \rho], \quad (111)$$

where s is the (local) entropy density in the fluid rest frame. The second term in Eq. (109) vanishes due to the conservation of the ideal baryon flow (103a) and the third term also vanishes as

$$\text{Tr}[\tilde{\mu} \partial_\mu (\rho u^\mu)] = \text{Tr}[\tilde{\mu} D_\mu (\rho u^\mu)] = 0. \quad (112)$$

The first equality holds because $\tilde{\mu}$ and ρ commute with each other, and consequently $\text{Tr}[\tilde{\mu}[A^\mu, \rho]] = 0$. The last equality expresses the covariant conservation of the ideal color current (103c). Thus Eq. (109) finally gives the entropy conservation $\partial_\mu (s u^\mu) = 0$.

The analog of Euler's equation is obtained from Eq. (33), projecting it onto a direction perpendicular to u^μ . Equivalently, we consider the following combination of Eqs. (33), (107):

$$\partial_\nu t^{\mu\nu} - u^\mu u_\nu \partial_\sigma t^{\nu\sigma} = 2 \text{Tr}[j_\sigma F^{\nu\sigma}], \quad (113)$$

which gives

$$(\varepsilon + p) u^\nu \partial_\nu u^\mu = (\partial^\mu - u^\mu u_\nu \partial^\nu) p + 2 \text{Tr}[j_\nu F^{\mu\nu}]. \quad (114)$$

To get a better insight of the physical meaning of Eq. (114) we write it down in the three-vector notation where

$$u^\mu \equiv (\gamma c, \gamma \mathbf{v}), \quad j^\mu \equiv (c \rho, \mathbf{j}), \quad F^{0i} = E^i, \quad F^{ij} = \epsilon_{ijk} B^k, \quad (115)$$

with $\gamma \equiv (1 - v^2/c^2)^{-1/2}$ and \mathbf{E} , \mathbf{B} being the chromoelectric and chromomagnetic field, respectively. We have restored here the velocity of light c to facilitate taking the nonrelativistic limit of the derived hydrodynamic equation. Subtracting Eq. (114) for $\mu = 0$ multiplied by v^i/c from Eq. (114) for $\mu = i$, one gets

$$\begin{aligned} & \frac{\varepsilon + p}{1 - v^2/c^2} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} \\ &= - \left(\nabla + \frac{1}{c^2} \mathbf{v} \frac{\partial}{\partial t} \right) p - 2 \text{Tr} \left[\rho \mathbf{E} - \frac{1}{c^2} \mathbf{v} (\mathbf{j} \cdot \mathbf{E}) + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right], \end{aligned} \quad (116)$$

which in the nonrelativistic domain ($v^2 \ll c^2$) reads

$$(\varepsilon + p) \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = - \nabla p - 2 \text{Tr} \left[\rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right]. \quad (117)$$

We note that the nonrelativistic limit, which is taken for the sake of comparison with the analogous equation of the electron-ion plasma [1], is only applied to the hydrodynamic velocity. The motion of the fluid's constituents remains relativistic.

Although the quark-gluon plasma is composed of partons of several colors, the hydrodynamic equation (114) describes a single fluid. This happens because there is a unique hydrodynamic velocity in the local equilibrium state (42). Various color components, which enter the energy-momentum tensor, do not neutralize each other but they are "glued" together in the course of evolution. Such a single fluid chromohydrodynamics was briefly considered long ago [6] within kinetic theory. An equation very similar to Eq. (114) has been recently derived [10] directly from a postulated Lagrange density. The color current, which enters the Euler's equation discussed in [10], is of the form $Q \mathcal{J}^\mu$ where Q is the color charge and \mathcal{J}^μ is the conserved Abelian current. As seen in Eqs. (103), \mathcal{J}^μ can be identified with the baryon flow b^μ when we deal with a system of quarks only. In a multicomponent plasma, however, such an identification is not possible because vanishing of the baryon current does not imply vanishing of the color current.

VI. DISCUSSION AND SUMMARY

Local equilibrium is only a transient state of a nonequilibrium system in its course towards global equilibrium. Thus the question arises how fast such a state is achieved, and for how long it survives. We denote the two characteristic times of interest as τ_0 and τ_1 . As we have shown in Sec. IV, the dominant processes, those with the soft gluon exchange in the t or u channel, are responsible for establishing the colorful equilibrium. Since the electric forces are screened at momentum transfers smaller than the Debye mass (m_D) the largest contribution to these processes comes from the small angle scatterings due to the magnetic forces which are effectively screened at momentum transfers below m_D . We identify τ_0 with the relaxation time related to such interactions. Then, according to the estimate [19] found for the quark-gluon plasma in global (colorless) equilibrium where $m_D \sim gT$, we have

$$\frac{1}{\tau_0} \sim g^2 T \ln(1/g). \quad (118)$$

We note, however, that the relaxation time in the colorful plasma can significantly differ from Eq. (118) due to the interaction with the background chromodynamic field generated by the color current (45).

For how long does the colorful equilibrium exist? The answer crucially depends on the process which is responsible

for the plasma neutralization. We have explicitly shown that the dominant processes comply with the finite color chemical potentials. We have also checked that equilibration with respect to the process $q\bar{q} \leftrightarrow gg$ leaves the system colorful. We expect that the collisions, even those beyond binary approximation, do not demand vanishing of the color chemical potentials. The point is that in every collision process, which changes the particle momenta but not their “macroscopic” positions, the color current is conserved. Therefore the collisions do not alter the local color charge.

The plasma is presumably neutralized due to the collective phenomena: dissipative color currents and damp plasma waves both caused by uncompensated color charges. Then, the characteristic time of the system neutralization is controlled by the color conductivity which is again related to the estimate (118) [19]. Thus the two times of interest τ_0 and τ_1 are of the same order, and a much more careful analysis is needed to establish the domain of applicability of the local equilibrium solution found here. Such an analysis should take into account not only the interaction with the background fields present in the colorful equilibrium but the initial nonequilibrium configuration should be also specified.

At the end let us summarize the most important results of this study. The local equilibrium state dictated by the collisional invariants, which follow from the energy-momentum, baryon number, and color charge conservation, is colorful, i.e., there is a nonvanishing color current in the system. The baryon chemical potentials of quarks and of antiquarks and the scalar (colorless) chemical potential of gluons are constrained as in a global equilibrium: $\bar{\mu}_b = -\mu_b$ and $\mu_g^0 = 0$. The local equilibrium configuration resulting from the cancellation of collision terms, which represent the most probable binary parton interactions, is also colorful with the same color structure. The colorless chemical potentials, however, are unconstrained. The global equilibrium relations among them emerge when the subdominant processes are taken into account. It is conjectured that not only binary but even multiparton collisions comply with the finite color chemical potentials, thus suggesting that the color neutralization of the plasma occurs not due to the collisions but due to dissipative collective phenomena. Proper identification of these processes and their quantitative description will be very important for understanding of the whole equilibration scenario of the quark-gluon plasma.

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APPENDIX: EVALUATION OF TRACES

We collect here some useful formulas of the traces computed both in the fundamental and adjoint representation.

Due to the identity

$$\tau_{ij}^a \tau_{kl}^a = \frac{1}{2} \delta^{il} \delta^{jk} - \frac{1}{2N_c} \delta^{ij} \delta^{kl}, \quad (\text{A1})$$

we have the relations of the traces in the fundamental representation

$$\text{Tr}[\tau^a A \tau^a B] = -\frac{1}{2N_c} \text{Tr}[AB] + \frac{1}{2} \text{Tr}[A] \text{Tr}[B], \quad (\text{A2})$$

$$\text{Tr}[\tau^a A] \text{Tr}[\tau^a B] = \frac{1}{2} \text{Tr}[AB] - \frac{1}{2N_c} \text{Tr}[A] \text{Tr}[B]. \quad (\text{A3})$$

Furthermore, from Eq. (A1) one can deduce

$$\tau^a \tau^a = \frac{N_c^2 - 1}{2N_c}, \quad \tau^a \tau^b \tau^a = -\frac{1}{2N_c} \tau^b. \quad (\text{A4})$$

Taking into account

$$\tau_a \tau_b = \frac{1}{2N_c} \delta_{ab} + \frac{1}{2} d_{abc} \tau_c + \frac{i}{2} f_{abc} \tau_c \quad (\text{A5})$$

one evaluates traces of products of generators in the fundamental representation. In particular, one finds

$$\text{Tr}[\tau^a \tau^b \tau^c] = \frac{1}{4} (d^{abc} + i f^{abc}), \quad (\text{A6})$$

$$\begin{aligned} \text{Tr}[\tau^a \tau^b \tau^c \tau^d] &= \frac{1}{4N_c} (\delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \\ &+ \frac{1}{8} (d^{abr} d^{cdr} - d^{acr} d^{bdr} + d^{adr} d^{bcr}) \\ &+ \frac{i}{8} (d^{abr} f^{cdr} - d^{acr} f^{bdr} + d^{adr} f^{bcr}). \end{aligned} \quad (\text{A7})$$

For $N_c = 2$ one also has

$$\text{Tr}[\tau^a \tau^b \tau^c \tau^d \tau^e] = \frac{i}{16} (\delta^{ae} f^{abcd} + \delta^{cd} f^{abe} + \delta^{bd} f^{aec} + \delta^{bc} f^{ade}). \quad (\text{A8})$$

The identity analogous to Eq. (A1) for the adjoint representation of the SU(2) group is

$$T_{bc}^a T_{de}^a = \delta^{be} \delta^{cd} - \delta^{bd} \delta^{ce}, \quad (\text{A9})$$

and we have the following relations:

$$\text{Tr}[T^a A T^a B] = \text{Tr}[A] \text{Tr}[B] - \text{Tr}[AB^T], \quad (\text{A10})$$

$$\text{Tr}[T^a A] \text{Tr}[T^a B] = \text{Tr}[AB] - \text{Tr}[AB^T]. \quad (\text{A11})$$

Using the identity (A9) one also finds

$$\text{Tr}[T^a T^b T^c] = i f^{abc}, \quad (\text{A12a})$$

$$\text{Tr}[T^a T^b T^c T^d] = \delta^{ab} \delta^{cd} + \delta^{ad} \delta^{bc}, \quad (\text{A12b})$$

$$\text{Tr}[T^a T^b T^c T^d T^e] = \delta^{ad} f^{ecb} - \delta^{cd} f^{eab} - \delta^{ab} f^{ecd}. \quad (\text{A12c})$$

In the adjoint representation of SU(3), we have the identity

$$T_{bc}^a T_{de}^a = -\frac{2}{3}(\delta^{bd} \delta^{ce} - \delta^{be} \delta^{cd}) - (d_{bd} f_{cef} - d_{be} f_{cdf}), \quad (\text{A13})$$

which, in particular, allows one to compute the totally symmetric trace of four generators as

$$\begin{aligned} & \frac{1}{4} \text{Tr}[\{T_a, T_b\}\{T_c, T_d\}] \\ &= \frac{1}{2}(2\delta^{ab} \delta_{cd} + \delta^{ac} \delta_{bd} + \delta^{ad} \delta_{bc}) + \frac{3}{4} d_{abs} d_{c ds}. \quad (\text{A14}) \end{aligned}$$

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- [1] N.A. Krall and A.W. Trivelpiece, *Principles of Plasma Physics* (McGraw-Hill, New York, 1973).
- [2] J.P. Blaizot and E. Iancu, Phys. Rep. **359**, 355 (2002).
- [3] U.W. Heinz, Ann. Phys. (N.Y.) **161**, 48 (1985).
- [4] A. Dyrek and W. Florkowski, Phys. Rev. D **36**, 2172 (1987).
- [5] St. Mrówczyński, Phys. Lett. B **202**, 568 (1988).
- [6] St. Mrówczyński, Adv. Ser. Dir. High Energy Phys. **6**, 185 (1990).
- [7] D.D. Holm and B.A. Kupersmidt, Phys. Lett. **105A**, 225 (1984).
- [8] D.D. Holm and B.A. Kupersmidt, Phys. Rev. D **30**, 2557 (1984).
- [9] R. Jackiw, V.P. Nair, and S.Y. Pi, Phys. Rev. D **62**, 085018 (2000).
- [10] B. Bistrovic, R. Jackiw, H. Li, V.P. Nair, and S.Y. Pi, Phys. Rev. D **67**, 025013 (2003).
- [11] Proceedings of *Quark Matter 2001* [Nucl. Phys. **A698** (2003)].
- [12] STAR Collaboration, K.H. Ackermann *et al.*, Phys. Rev. Lett. **86**, 402 (2001).
- [13] J.Y. Ollitrault, Phys. Rev. D **46**, 229 (1992).
- [14] U.W. Heinz and P.F. Kolb, Nucl. Phys. **A702**, 269 (2002).
- [15] W. Broniowski, A. Baran, and W. Florkowski, Acta Phys. Pol. B **33**, 4235 (2002).
- [16] H.T. Elze and U.W. Heinz, Phys. Rep. **183**, 81 (1989).
- [17] A.V. Selikhov, Phys. Lett. B **268**, 263 (1991); **285**, 398(E) (1992).
- [18] D. Bodeker, Phys. Lett. B **426**, 351 (1998).
- [19] P. Arnold, D.T. Son, and L.G. Yaffe, Phys. Rev. D **59**, 105020 (1999).
- [20] J.P. Blaizot and E. Iancu, Nucl. Phys. **B557**, 183 (1999).
- [21] D.F. Litim and C. Manuel, Nucl. Phys. **B562**, 237 (1999).
- [22] St. Mrówczyński and P. Danielewicz, INS Report, IPJ/2049/PH, Swierk-Otwock, 1987 (unpublished).
- [23] L. Waldmann, Z. Naturforsch. A **12**, 661 (1957); R.F. Snider, J. Chem. Phys. **32**, 1051 (1960).
- [24] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. **11**, 001 (2000).
- [25] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. **05**, 051 (2003).
- [26] S.R. deGroot, W.A. Van Leeuwen, and Ch.G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).
- [27] St. Mrówczyński, Phys. Rev. D **39**, 1940 (1989).
- [28] P. Arnold, G.D. Moore, and L.G. Yaffe, J. High Energy Phys. **01**, 030 (2003).
- [29] L. D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1963).