PLASMA ELECTROMAGNETIC FLUCTUATIONS AS AN INITIAL VALUE PROBLEM

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Fluctuations of electric and magnetic fields in the collisionless plasma are found as a solution of the initial value linearized problem. The plasma initial state is on average stationary and homogeneous. When the state is stable, the initial fluctuations decay exponentially and in the long time limit a stationary spectrum of fluctuations is established. For the equilibrium plasma it reproduces the spectrum obtained from the fluctuation-dissipation relation. Fluctuations in the unstable two-stream system are also discussed.

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1. Introduction

Spectrum of electromagnetic fluctuations is an important plasma characteristics studied in various contexts. In terrestrial experiments the spectrum, which is observable through scattering measurements, signals, for example, an onset of plasma instability or turbulence. Electromagnetic fluctuations in primordial cosmological plasma are analyzed to explain an origin of magnetic fields in the Universe.

The fluctuations can be theoretically described using several methods reviewed in the classical monographs [1, 2]. Modern field-theory techniques developed for relativistic plasmas are worked out in [3, 4]. Physically most appealing seems to us the method proposed by Rostoker [5] and Klimontovich and Silin [6] which is clearly exposed in the handbook [7]. The method, which is applicable to both equilibrium and nonequilibrium plasmas, provides the spectrum of fluctuations as a solution of the initial value (linearized) problem. The initial plasma state is assumed to be on average stationary and homogeneous. When the state is stable, the initial fluctuations are explicitly shown to exponentially decay and in the long time limit one finds a stationary spectrum of fluctuations. In this way one obtains for the equilibrium plasma the spectrum which is alternatively provided by the fluctuation-dissipation relation. When the initial state is unstable, the memory of initial fluctuations is not lost, as the unstable modes, which are present in the initial fluctuation spectrum, exponentially grow.

The fluctuations of the distribution function, electric charge or longitudinal electric field can be found rather easily, see [5–7]. Analytic computation of the magnetic field fluctuations appear to be quite lengthy and tedious while the computation of fluctuation spectrum of the electric field, which is not constrained to be purely longitudinal, is a real challenge even in the collionless plasma, as one has to take into account and sort out numerous terms. Up to our knowledge such calculations have not been published. In this article we study the fluctuation spectrum of electric and magnetic fields in detail. In the case of equilibrium, we reproduce the spectrum usually provided by the fluctuation-dissipation relation. Fluctuations in unstable systems are also discussed and, as an example, we compute the fluctuation spectrum of longitudinal field in the two-stream system.

The method under consideration, although physically appealing, is certainly not the most effective to analyze equilibrium plasmas. And our actual goal is to set a stage for nonequilibrium calculations similar to those of the two-stream system. Our particular interests is focused on the quark-gluon plasma — a highly relativistic system governed by non-Abelian dynamics which, in spite of important differences, manifests profound similarities to electromagnetic plasmas discussed at length in [8]. The quark-gluon plasma produced in relativistic heavy-ion collisions is presumably unstable to chromomagnetic modes, see the review [9]. The instability growth is associated with generation of chromomagnetic fields which in turn strongly influence transport properties of the plasma [10]. The fluctuation spectrum of chromomagnetic fields is an important issue to be settled.

Our paper is organized as follows. In Sec. 2 we present the theoretical framework to be used in our further considerations. The linearized kinetic equation are solved together with Maxwell equations by means of the one-sided Fourier transformation in Sec. 3. The electric and magnetic fields are expressed through the initial values of the fields and electron distribution function. Sec. 4 deals with the initial fluctuations. Those of the distribution function are identified with the fluctuations in a classical system of noninteracting particles. The initial fluctuations of fields are expressed through the particle fluctuations using the Maxwell equations. The well-known fluctuation spectrum of longitudinal electric field is obtained in Sec. 5 while the fluctuation spectra of magnetic and electric fields are derived in Secs. 6 and 7, respectively. It then becomes clear why the analysis of longitudinal electric field is much easier than that of the general case. In Sec. 8 we extend our calculations to nonequilibrium anisotropic plasma, discussing fluctuations of longitudinal electric field in the unstable two-stream system. Our results are summarized and concluded in Sec. 9. Throughout the article we use the CGS natural units with $c = k_{\rm B} = 1$.

2. Preliminaries

We consider a classical plasma where ions are assumed to be a passive background which merely compensate the charge of electrons. However, the ions can be easily included in the considerations. The time scale of fluctuations of interest is much shorter than that of inter-particle collisions, and consequently the starting point of our analysis is the collisionless transport equation of electrons

$$\left(\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) - e \left(\boldsymbol{E}(t,\boldsymbol{r}) + \boldsymbol{v} \times \boldsymbol{B}(t,\boldsymbol{r}) \right) \cdot \nabla_p \right) f(t,\boldsymbol{r},\boldsymbol{p}) = 0 \,, \tag{1}$$

where $f(t,\boldsymbol{r},\boldsymbol{p})$ is the distribution function; $\boldsymbol{E}(t,\boldsymbol{r})$ and $\boldsymbol{B}(t,\boldsymbol{r})$ denote the electric and magnetic fields in the plasma.

The transport equation (1) is supplemented by the Maxwell equations

$$\begin{aligned} \nabla \cdot \boldsymbol{E}(t,\boldsymbol{r}) &= 4\pi\rho(t,\boldsymbol{r}) \,, \qquad \nabla \cdot \boldsymbol{B}(t,\boldsymbol{r}) = 0 \,, \\ \nabla \times \boldsymbol{E}(t,\boldsymbol{r}) &= -\frac{\partial \boldsymbol{B}(t,\boldsymbol{r})}{\partial t} \,, \qquad \nabla \times \boldsymbol{B}(t,\boldsymbol{r}) = 4\pi \boldsymbol{j}(t,\boldsymbol{r}) + \frac{\partial \boldsymbol{E}(t,\boldsymbol{r})}{\partial t} \,, \end{aligned} \tag{2}$$

with the electric charge density and current given as

$$\rho(t,\boldsymbol{r}) = -e \int \frac{d^3p}{(2\pi)^3} f(t,\boldsymbol{r},\boldsymbol{p}) + e\, n_{\rm ions} \,, \tag{3}$$

$$\boldsymbol{j}(t,\boldsymbol{r}) = -e \int \frac{d^3p}{(2\pi)^3} \boldsymbol{v}\, f(t,\boldsymbol{r},\boldsymbol{p}) \,. \tag{4}$$

The distribution function is assumed to be of the form

$$f(t,\boldsymbol{r},\boldsymbol{p}) = f^0(\boldsymbol{p}) + \delta f(t,\boldsymbol{r},\boldsymbol{p}) \,, \tag{5}$$

with

$$f^0(\boldsymbol{p}) \gg \delta f(t,\boldsymbol{r},\boldsymbol{p}) \,, \qquad |\nabla_p f^0(\boldsymbol{p})| \gg |\nabla_p \delta f(t,\boldsymbol{r},\boldsymbol{p})| \,, \tag{6}$$

and

$$\int \frac{d^3p}{(2\pi)^3} f^0(\boldsymbol{p}) - n_{\rm ions} = 0\,, \qquad \int \frac{d^3p}{(2\pi)^3}\, \boldsymbol{v}\, f^0(t,\boldsymbol{r},\boldsymbol{p}) = 0\,. \tag{7}$$

The transport equation linearized in δf is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\delta f(t,\boldsymbol{r},\boldsymbol{p}) - e(\boldsymbol{E}(t,\boldsymbol{r}) + \boldsymbol{v}\times\boldsymbol{B}(t,\boldsymbol{r}))\cdot\nabla_p f^0(\boldsymbol{p}) = 0\,. \tag{8}$$

and

$$\rho(t,\boldsymbol{r}) \;=\; -e\int \frac{d^3p}{(2\pi)^3}\,\delta f(t,\boldsymbol{r},\boldsymbol{p})\,, \tag{9}$$

$$\boldsymbol{j}(t,\boldsymbol{r}) \;=\; -e\int \frac{d^3p}{(2\pi)^3}\,\boldsymbol{v}\,\delta f(t,\boldsymbol{r},\boldsymbol{p})\,. \tag{10}$$

3. Initial value problem

We are going to solve the linearized transport equation (8) and Maxwell equations (2) with the initial conditions

$$\begin{aligned} \delta f(t=0,\boldsymbol{r},\boldsymbol{p}) \;&=\; \delta f_0(\boldsymbol{r},\boldsymbol{p})\,,\\ \boldsymbol{E}(t=0,\boldsymbol{r}) \;&=\; \boldsymbol{E}_0(\boldsymbol{r})\,, \qquad \boldsymbol{B}(t=0,\boldsymbol{r}) = \boldsymbol{B}_0(\boldsymbol{r})\,. \end{aligned} \tag{11}$$

We apply to the equations the one-sided Fourier transformation defined as

$$f(\omega,\boldsymbol{k}) = \int_0^\infty dt \int d^3r e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} f(t,\boldsymbol{r})\,. \tag{12}$$

The inverse transformation is

$$f(t,\boldsymbol{r}) = \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi}\int \frac{d^3k}{(2\pi)^3} e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} f(\omega,\boldsymbol{k})\,, \tag{13}$$

where the real parameter $\sigma > 0$ is chosen is such a way that the integral over ω is taken along a straight line in the complex ω-plane, parallel to the real axis, above all singularities of $f(\omega,\boldsymbol{k})$.

We note that

$$\int_0^\infty dt \int d^3r e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \frac{\partial f(t,\boldsymbol{r})}{\partial t} = -i\omega f(\omega,\boldsymbol{k}) - f(t=0,\boldsymbol{k})\,. \tag{14}$$

The linearized transport (8) and Maxwell (2) equations, which are transformed by means of the one-sided Fourier transformation, read

$$-i(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\delta f(\omega,\boldsymbol{k},\boldsymbol{p})-e(\boldsymbol{E}(\omega,\boldsymbol{k})+\boldsymbol{v}\times\boldsymbol{B}(\omega,\boldsymbol{k}))\cdot\nabla_p f^0(\boldsymbol{p})=\delta f_0(\boldsymbol{k},\boldsymbol{p}) \,, \quad (15)$$

$$\begin{aligned} i\boldsymbol{k}\cdot\boldsymbol{E}(\omega,\boldsymbol{k}) &= 4\pi\rho(\omega,\boldsymbol{k})\,, \\ i\boldsymbol{k}\cdot\boldsymbol{B}(\omega,\boldsymbol{k}) &= 0\,, \\ i\boldsymbol{k}\times\boldsymbol{E}(\omega,\boldsymbol{k}) &= i\omega\boldsymbol{B}(\omega,\boldsymbol{k})+\boldsymbol{B}_0(\boldsymbol{k})\,, \\ i\boldsymbol{k}\times\boldsymbol{B}(\omega,\boldsymbol{k}) &= 4\pi\boldsymbol{j}(\omega,\boldsymbol{k})-i\omega\boldsymbol{E}(\omega,\boldsymbol{k})-\boldsymbol{E}_0(\boldsymbol{k})\,. \end{aligned} \quad (16)$$

One finds the solution of the transport equation as

$$\delta f(\omega,\boldsymbol{k},\boldsymbol{p})=\frac{i}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\Big(e(\boldsymbol{E}(\omega,\boldsymbol{k})+\boldsymbol{v}\times\boldsymbol{B}(\omega,\boldsymbol{k}))\cdot\nabla_p f^0(\boldsymbol{p})+\delta f_0(\boldsymbol{k},\boldsymbol{p})\Big)\,. \quad (17)$$

3.1. Electric field

Substituting the solution (17) into the Fourier transformed current (10) and using the third Maxwell equation (16) to express the magnetic field through the electric one, the current gets the form

$$\begin{aligned} &\boldsymbol{j}(\omega,\boldsymbol{k}) = \\ &-\ ie^2\int\frac{d^3p}{(2\pi)^3}\,\frac{\boldsymbol{v}}{\omega-\boldsymbol{v}\cdot\boldsymbol{k}}\left(\left(1-\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)\boldsymbol{E}(\omega,\boldsymbol{k})+\frac{1}{\omega}(\boldsymbol{v}\cdot\boldsymbol{E}(\omega,\boldsymbol{k}))\boldsymbol{k}\right)\cdot\nabla_p f^0(\boldsymbol{p}) \\ &+\ e^2\int\frac{d^3p}{(2\pi)^3}\,\frac{\boldsymbol{v}}{\omega-\boldsymbol{v}\cdot\boldsymbol{k}}\left(\frac{1}{\omega}\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k})\right)\cdot\nabla_p f^0(\boldsymbol{p}) \\ &-\ ie\int\frac{d^3p}{(2\pi)^3}\,\frac{\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\,\delta f_0(\boldsymbol{k},\boldsymbol{p})\,. \end{aligned} \quad (18)$$

Since the dielectric tensor $\varepsilon^{ij}(\omega,\boldsymbol{k})$ in the collisionless limit equals [1]

$$\begin{aligned} \varepsilon^{ij}(\omega,\boldsymbol{k}) &= \delta^{ij}+\frac{4\pi e^2}{\omega}\int\frac{d^3p}{(2\pi)^3}\frac{v^i}{\omega-\boldsymbol{v}\cdot\boldsymbol{k}+i0^+} \\ &\quad\times\left(\left(1-\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right)\delta^{jk}+\frac{v^jk^k}{\omega}\right)\nabla_p^k f^0(\boldsymbol{p})\,, \end{aligned} \quad (19)$$

the current can be written as

$$\begin{aligned} j^i(\omega,\boldsymbol{k}) &= \frac{-i\omega}{4\pi}(\varepsilon^{ij}(\omega,\boldsymbol{k})-\delta^{ij})E^j(\omega,\boldsymbol{k})+e^2\!\int\frac{d^3p}{(2\pi)^3}\frac{v^i}{\omega-\boldsymbol{v}\cdot\boldsymbol{k}} \\ &\quad\times\left(\frac{1}{\omega}\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k})\right)^j\nabla_p^j f^0(\boldsymbol{p})-ie\int\frac{d^3p}{(2\pi)^3}\frac{v^i}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\delta f_0(\boldsymbol{k},\boldsymbol{p})\,. \end{aligned} \quad (20)$$

Combing the third and fourth Maxwell equations (16), one finds

$$\left[(\omega^2 - \boldsymbol{k}^2)\delta^{ij} + k^i k^j\right] E^j(\omega,\boldsymbol{k}) \;=\; -4\pi i\omega\, j^i(\omega,\boldsymbol{k}) + i\omega E_0^i(\boldsymbol{k}) - i(\boldsymbol{k}\times\boldsymbol{B}_0(\boldsymbol{k}))^i\,. \quad (21)$$

Substituting the current (20) into Eq. (21), one obtains

$$\begin{aligned}&\left[-\boldsymbol{k}^2\delta^{ij} + k^i k^j + \omega^2\varepsilon^{ij}(\omega,\boldsymbol{k})\right] E^j(\omega,\boldsymbol{k})\\ &= -\,4\pi i e^2 \int \frac{d^3p}{(2\pi)^3}\,\frac{v^i}{\omega - \boldsymbol{v}\cdot\boldsymbol{k}}(\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k}))^j\nabla_p^j f^0(\boldsymbol{p})\\ &-\,4\pi e\omega \int \frac{d^3p}{(2\pi)^3}\,\frac{v^i}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\delta f_0(\boldsymbol{k},\boldsymbol{p}) + i\omega E_0^i(\boldsymbol{k}) - i(\boldsymbol{k}\times\boldsymbol{B}_0(\boldsymbol{k}))^i\,. \quad (22)\end{aligned}$$

Denoting the matrix in left-hand-side of Eq. (22) as

$$\Sigma^{ij}(\omega,\boldsymbol{k}) \equiv -\boldsymbol{k}^2\delta^{ij} + k^i k^j + \omega^2\varepsilon^{ij}(\omega,\boldsymbol{k})\,, \quad (23)$$

the electric field given by Eq. (22) can be written down as

$$\begin{aligned}E^i(\omega,\boldsymbol{k}) \;=\; &-4\pi e \int \frac{d^3p}{(2\pi)^3}\,\frac{(\Sigma^{-1})^{ij}(\omega,\boldsymbol{k})v^j}{\omega - \boldsymbol{v}\cdot\boldsymbol{k}}\\ &\times\left[ie(\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k}))\cdot\nabla_p f^0(\boldsymbol{p}) + \omega\delta f_0(\boldsymbol{k},\boldsymbol{p})\right]\\ &+ i\omega(\Sigma^{-1})^{ij}(\omega,\boldsymbol{k})E_0^j(\boldsymbol{k}) - i(\Sigma^{-1})^{ij}(\omega,\boldsymbol{k})(\boldsymbol{k}\times\boldsymbol{B}_0(\boldsymbol{k}))^j\,, \quad (24)\end{aligned}$$

which is the main result of this section.

When the plasma stationary state described by $f^0(\boldsymbol{p})$ is isotropic, the dielectric tensor can be expressed through its longitudinal and transverse components

$$\varepsilon^{ij}(\omega,\boldsymbol{k}) = \varepsilon_{\rm L}(\omega,\boldsymbol{k})\,\frac{k^i k^j}{\boldsymbol{k}^2} + \varepsilon_{\rm T}(\omega,\boldsymbol{k})\left(\delta^{ij} - \frac{k^i k^j}{\boldsymbol{k}^2}\right), \quad (25)$$

where $\varepsilon_{\rm L}(\omega,\boldsymbol{k})$ and $\varepsilon_{\rm T}(\omega,\boldsymbol{k})$ are well known [7] to be equal to

$$\varepsilon_{\rm L}(\omega,\boldsymbol{k}) \;=\; 1 + \frac{4\pi e^2}{\boldsymbol{k}^2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i0^+}\boldsymbol{k}\cdot\frac{\partial f^0(\boldsymbol{p})}{\partial\boldsymbol{p}}\,, \quad (26)$$

$$\begin{aligned}\varepsilon_{\rm T}(\omega,\boldsymbol{k}) \;=\; &1 + \frac{2\pi e^2}{\omega}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i0^+}\\ &\times\left[\boldsymbol{v}\cdot\frac{\partial f^0(\boldsymbol{p})}{\partial\boldsymbol{p}} - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\boldsymbol{k}^2}\boldsymbol{k}\cdot\frac{\partial f^0(\boldsymbol{p})}{\partial\boldsymbol{p}}\right]. \quad (27)\end{aligned}$$

The matrix $\Sigma^{ij}(\omega,\boldsymbol{k})$, which then equals

$$\Sigma^{ij}(\omega,\boldsymbol{k}) = \omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})\frac{k^ik^j}{\boldsymbol{k}^2} + (\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2)\Big(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\Big)\,, \tag{28}$$

can be inverted as

$$(\Sigma^{-1})^{ij}(\omega,\boldsymbol{k}) = \frac{1}{\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\frac{k^ik^j}{\boldsymbol{k}^2} + \frac{1}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\Big(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\Big)\,. \tag{29}$$

When the momentum distribution $f^0(\boldsymbol{p})$ is isotropic, $\nabla_p f^0(\boldsymbol{p}) \sim \boldsymbol{p}$, and consequently $(\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k}))\cdot\nabla_p f^0(\boldsymbol{p}) = 0$. Therefore, the first term in the right-hand-side of Eq. (24) vanishes and the electric field is found as

$$\begin{aligned} E^i(\omega,\boldsymbol{k}) \;=\; & -4\,\pi e\omega\left(\frac{1}{\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\frac{k^ik^j}{\boldsymbol{k}^2} + \frac{1}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\Big(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\Big)\right) \\ & \times \int \frac{d^3p}{(2\pi)^3}\,\frac{v^j}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\delta f_0(\boldsymbol{k},\boldsymbol{p}) \\ & + i\omega\left(\frac{1}{\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\frac{k^ik^j}{\boldsymbol{k}^2} + \frac{1}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\Big(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\Big)\right)E_0^j(\boldsymbol{k}) \\ & - \frac{i(\boldsymbol{k}\times\boldsymbol{B}_0(\boldsymbol{k}))^i}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\,. \end{aligned} \tag{30}$$

If the field is purely longitudinal,

$$\boldsymbol{E}(\omega,\boldsymbol{k}) = (\boldsymbol{k}\cdot\boldsymbol{E}(\omega,\boldsymbol{k}))\,\frac{\boldsymbol{k}}{\boldsymbol{k}^2}\,, \qquad \boldsymbol{E}_0(\boldsymbol{k}) = (\boldsymbol{k}\cdot\boldsymbol{E}_0(\boldsymbol{k}))\,\frac{\boldsymbol{k}}{\boldsymbol{k}^2}\,,$$

Eq. (30) gives

$$\boldsymbol{k}\cdot\boldsymbol{E}(\omega,\boldsymbol{k}) = -\frac{4\pi e}{\omega\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\int\frac{d^3p}{(2\pi)^3}\,\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\delta f_0(\boldsymbol{k},\boldsymbol{p}) + \frac{i\boldsymbol{k}\cdot\boldsymbol{E}_0(\boldsymbol{k})}{\omega\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\,. \tag{31}$$

Taking into account that

$$i\boldsymbol{k}\cdot\boldsymbol{E}_0(\boldsymbol{k}) = 4\pi\rho_0(\boldsymbol{k}) = -4\pi e\int\frac{d^3p}{(2\pi)^3}\,\delta f_0(\boldsymbol{k},\boldsymbol{p})\,,$$

Eq. (31) can be rewritten as

$$\boldsymbol{k}\cdot\boldsymbol{E}(\omega,\boldsymbol{k}) = -\frac{4\pi e}{\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\int\frac{d^3p}{(2\pi)^3}\,\frac{\delta f_0(\boldsymbol{k},\boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,. \tag{32}$$

Eq. (32) can be obtained directly by substituting the solution of transport equation (17) (with $\boldsymbol{B} = 0$) into the first Maxwell equation. Then, the initial electric field does not show up.

3.2. Magnetic field

Using again the third Maxwell equation (16) to express the magnetic field through the electric one, Eq. (24) immediately provides

$$\begin{aligned} B^i(\omega,\boldsymbol{k}) \; = \; & \frac{1}{\omega}\,\epsilon^{ijk}k^j(\Sigma^{-1})^{kl}(\omega,\boldsymbol{k}) \\ & \bigg(-4\pi i e^2 \int \frac{d^3p}{(2\pi)^3}\,\frac{v^l}{\omega - \boldsymbol{v}\cdot\boldsymbol{k}}(\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k}))\cdot\nabla_p f^0(\boldsymbol{p}) \\ & - 4\pi e\omega \int \frac{d^3p}{(2\pi)^3}\,\frac{v^l}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\delta f_0(\boldsymbol{k},\boldsymbol{p}) + i\omega E_0^l(\boldsymbol{k}) - i(\boldsymbol{k}\times\boldsymbol{B}_0(\boldsymbol{k}))^l\bigg) \\ & + \frac{i}{\omega}B_0^i(\boldsymbol{k})\,. \end{aligned} \tag{33}$$

When the plasma stationary state is isotropic and $(\Sigma^{-1})^{ij}(\omega,\boldsymbol{k})$ is given by Eq. (29), one finds

$$\epsilon^{ijk}k^j(\Sigma^{-1})^{kl}(\omega,\boldsymbol{k}) = \frac{\epsilon^{ijl}k^j}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\,. \tag{34}$$

The first term in the right-hand side of Eq. (33) vanishes, because $(\boldsymbol{v}\times\boldsymbol{B}_0(\boldsymbol{k}))\cdot\nabla_p f^0(\boldsymbol{p}) = 0$, and thus

$$\begin{aligned} \boldsymbol{B}(\omega,\boldsymbol{k}) = \; & -\frac{4\pi e}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\int \frac{d^3p}{(2\pi)^3}\,\frac{\boldsymbol{k}\times\boldsymbol{v}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\delta f_0(\boldsymbol{k},\boldsymbol{p}) \\ & + \frac{i\boldsymbol{k}\times\boldsymbol{E}_0(\boldsymbol{k})}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2} + \frac{i\omega\varepsilon_{\rm T}(\omega,\boldsymbol{k})}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2}\boldsymbol{B}_0(\boldsymbol{k})\,. \end{aligned} \tag{35}$$

4. Initial fluctuations

The correlation functions of electric or magnetic fields, $\langle E^i(t_1,\boldsymbol{r}_1) E^j(t_2,\boldsymbol{r}_2)\rangle$, $\langle B^i(t_1,\boldsymbol{r}_1)B^j(t_2,\boldsymbol{r}_2)\rangle$ ($\langle\cdots\rangle$ denotes averaging over statistical ensemble), are determined by the fields $\boldsymbol{E}(t,\boldsymbol{r})$, $\boldsymbol{B}(t,\boldsymbol{r})$ found in the previous section and the initial correlations $\langle\delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{r}_2,\boldsymbol{p}_2)\rangle$, $\langle E_0^i(\boldsymbol{r}_1)E_0^j(\boldsymbol{r}_2)\rangle$, $\langle B_0^i(\boldsymbol{r}_1)B_0^j(\boldsymbol{r}_2)\rangle$, $\langle\delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)\, E_0^j(\boldsymbol{r}_2)\rangle$, $\langle\delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)B_0^j(\boldsymbol{r}_2)\rangle$ and $\langle E_0^i(\boldsymbol{r}_1)B_0^j(\boldsymbol{r}_2)\rangle$ which are discussed in this section.

We identify the initial correlation function $\langle\delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)\rangle$ with the correlation function $\langle\delta f(t_1,\boldsymbol{r}_1,\boldsymbol{p}_1)\delta f(t_2,\boldsymbol{r}_2,\boldsymbol{p}_2)\rangle_{\rm free}$ taken at $t_1 = t_2 = 0$ of the system of free classical particles (obeying Boltzmann statistics) in a stationary homogeneous state described by the distribution function $f^0(\boldsymbol{p})$.

As well known [7],

$$\langle \delta f(t_1,\boldsymbol{r}_1,\boldsymbol{p}_1)\delta f(t_2,\boldsymbol{r}_2,\boldsymbol{p}_2)\rangle_{\rm free} = (2\pi)^3\delta^{(3)}(\boldsymbol{p}_2-\boldsymbol{p}_1) \times \delta^{(3)}(\boldsymbol{r}_2-\boldsymbol{r}_1-\boldsymbol{v}_1(t_2-t_1))f^0(\boldsymbol{p}_1)\,. \quad (36)$$

Then,

$$\begin{aligned}\langle \delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)\rangle &= \langle \delta f(t_1=0,\boldsymbol{r}_1,\boldsymbol{p}_1)\delta f(t_2=0,\boldsymbol{r}_2,\boldsymbol{p}_2)\rangle_{\rm free} \\ &= (2\pi)^3\delta^{(3)}(\boldsymbol{p}_2-\boldsymbol{p}_1)\delta^{(3)}(\boldsymbol{r}_2-\boldsymbol{r}_1)f^0(\boldsymbol{p}_1)\,, \quad (37)\end{aligned}$$

and

$$\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle = (2\pi)^3\delta^{(3)}(\boldsymbol{p}_2-\boldsymbol{p}_1)(2\pi)^3\delta^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_1)f^0(\boldsymbol{p}_1)\,. \quad (38)$$

The correlation function $k_2^j\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)E_0^j(\boldsymbol{k}_2)\rangle$ can be also expressed through $\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle$. Using the first Maxwell equation, one finds

$$\begin{aligned}k_2^j\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)E_0^j(\boldsymbol{k}_2)\rangle &= -4\pi i\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)\rho_0(\boldsymbol{k}_2)\rangle \\ &= 4\pi i e\int \frac{d^3p_2}{(2\pi)^3}\,\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle \\ &= 4\pi i e(2\pi)^3\delta^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_1)\; f^0(\boldsymbol{p}_1)\,. \quad (39)\end{aligned}$$

And finally,

$$\begin{aligned}k_1^ik_2^j\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle &= -16\pi^2\langle \rho_0(\boldsymbol{k}_1)\rho_0(\boldsymbol{k}_2)\rangle \\ &= -16\pi^2e^2\int \frac{d^3p_1}{(2\pi)^3}\,\frac{d^3p_2}{(2\pi)^3}\,\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle \\ &= -16\pi^2e^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_1)\int \frac{d^3p}{(2\pi)^3}\, f^0(\boldsymbol{p})\,. \quad (40)\end{aligned}$$

When the electric field is not purely longitudinal, the computation of the initial correlations $\langle E_0^i(\boldsymbol{r}_1)E_0^j(\boldsymbol{r}_2)\rangle$, $\langle \delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)E_0^j(\boldsymbol{r}_2)\rangle$ is more complicated, as the electric field $\boldsymbol{E}_0(\boldsymbol{r})$ is not fully determined by $\delta f_0(\boldsymbol{r},\boldsymbol{p})$ but $\delta f(t,\boldsymbol{r},\boldsymbol{p})$ enters here. To compute $\langle E_0^i(\boldsymbol{r}_1)E_0^j(\boldsymbol{r}_2)\rangle$, $\langle \delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)E_0^j(\boldsymbol{r}_2)\rangle$ as well as $\langle B_0^i(\boldsymbol{r}_1)B_0^j(\boldsymbol{r}_2)\rangle$, $\langle \delta f_0(\boldsymbol{r}_1,\boldsymbol{p}_1)B_0^j(\boldsymbol{r}_2)\rangle$, and $\langle E_0^i(\boldsymbol{r}_1)B_0^j(\boldsymbol{r}_2)\rangle$, we use the Maxwell equations transformed using the Fourier transformation not the one-sided Fourier transformation. Actually, the Fourier transformed Maxwell equations are very similar to the one-sided Fourier transformed Maxwell equations (16). The initial electric and magnetic fields are simply absent in the former ones. However, it should be clearly stated that the one-sided Fourier transformation is *not* mixed up with the Fourier transformation. The latter is used to compute only the initial fluctuations which are independent of ω.

Combing the third and the fourth Maxwell equation, one gets the equation as Eq. (22) but the terms with $\boldsymbol{E}_0(\boldsymbol{k})$ and $\boldsymbol{B}_0(\boldsymbol{k})$ are absent. Inverting the matrix in the right-hand side of the equation, we get the electric field expressed through the current

$$E^i(\omega,\boldsymbol{k}) = -4\pi i\omega\left[\frac{1}{\omega^2}\frac{k^ik^j}{\boldsymbol{k}^2} + \frac{1}{\omega^2-\boldsymbol{k}^2}\left(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\right)\right]j^j(\omega,\boldsymbol{k})\,. \tag{41}$$

The magnetic field is given as

$$\boldsymbol{B}(\omega,\boldsymbol{k}) = -\frac{4\pi i}{\omega^2-\boldsymbol{k}^2}\;\boldsymbol{k}\times\boldsymbol{j}(\omega,\boldsymbol{k})\,. \tag{42}$$

The correlation function $\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle$ is derived as

$$\begin{aligned}
\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle \;&=\; \int\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\langle E^i(\omega_1,\boldsymbol{k}_1)E^j(\omega_2,\boldsymbol{k}_2)\rangle \\
&=\; -(4\pi)^2\int\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\left[\frac{1}{\omega_1}\frac{k_1^ik_1^k}{\boldsymbol{k}_1^2} + \frac{\omega_1}{\omega_1^2-\boldsymbol{k}_1^2}\left(\delta^{ik} - \frac{k_1^ik_1^k}{\boldsymbol{k}_1^2}\right)\right] \\
&\quad\times\left[\frac{1}{\omega_2}\frac{k_2^jk_2^l}{\boldsymbol{k}_2^2} + \frac{\omega_2}{\omega_2^2-\boldsymbol{k}_2^2}\left(\delta^{jl} - \frac{k_1^jk_1^l}{\boldsymbol{k}_1^2}\right)\right] \\
&\quad\times\langle j^k(\omega_1,\boldsymbol{k}_1)j^j(\omega_2,\boldsymbol{k}_2)\rangle \\
&=\; -(4\pi e)^2\int\frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\,v_1^kv_2^l \\
&\quad\times\left[\frac{1}{\omega_1}\frac{k_1^ik_1^k}{\boldsymbol{k}_1^2} + \frac{\omega_1}{\omega_1^2-\boldsymbol{k}_1^2}\left(\delta^{ik} - \frac{k_1^ik_1^k}{\boldsymbol{k}_1^2}\right)\right] \\
&\quad\times\left[\frac{1}{\omega_2}\frac{k_2^jk_2^l}{\boldsymbol{k}_2^2} + \frac{\omega_2}{\omega_2^2-\boldsymbol{k}_2^2}\left(\delta^{jl} - \frac{k_1^jk_1^l}{\boldsymbol{k}_1^2}\right)\right] \\
&\quad\times\langle\delta f(\omega_1,\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f(\omega_2,\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle\,.
\end{aligned} \tag{43}$$

As previously, we identify $\langle\delta f(\omega_1,\boldsymbol{k}_1,\boldsymbol{p}_1)\,\delta f(\omega_2,\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle$ with $\langle\delta f(\omega_1,\boldsymbol{k}_1,\boldsymbol{p}_1)\;\delta f(\omega_2,\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle_{\rm free}$ which equals

$$\begin{aligned}\langle \delta f(\omega_1,\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f(\omega_2,\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle_{\rm free} \;=\;& (2\pi)^3\delta^{(3)}(\boldsymbol{p}_2-\boldsymbol{p}_1)\,2\pi\delta(\omega_1+\omega_2)\\ &\times(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\\ &\times 2\pi\delta\Big(\frac{\omega_1-\omega_2}{2}-\frac{\boldsymbol{k}_1-\boldsymbol{k}_2}{2}\boldsymbol{v}_1\Big)f^0(\boldsymbol{p}_1)\,. \end{aligned} \tag{44}$$

Then, after performing trivial integrations, $\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle$ equals

$$\begin{aligned}\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle \;=\;& -(4\pi e)^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\\ &\times\int\frac{d^3p}{(2\pi)^3}f^0(\boldsymbol{p})\frac{((\boldsymbol{k}_1\cdot\boldsymbol{v})v^i-k_1^i)((\boldsymbol{k}_2\cdot\boldsymbol{v})v^j-k_2^j)}{((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}. \end{aligned}\tag{45}$$

Computing $k_1^ik_2^j\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle$, one reproduces the result (40). Analogously to the correlation function $\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle$, one finds

$$\langle E_0^i(\boldsymbol{k}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle = 4\pi ie\,(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\,f^0(\boldsymbol{p}_2)\,\frac{(\boldsymbol{k}_1\cdot\boldsymbol{v}_2)v_2^i-k_1^i}{(\boldsymbol{k}_1\cdot\boldsymbol{v}_2)^2-\boldsymbol{k}_1^2}\,. \tag{46}$$

Starting with Eq. (42), we obtain

$$\begin{aligned}\langle B_0^i(\boldsymbol{k}_1)B_0^j(\boldsymbol{k}_2)\rangle \;=\;& -(4\pi e)^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\,\epsilon^{ikl}\epsilon^{jmn}k_1^kk_2^m\\ &\times\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\frac{v^lv^n}{((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}\,, \end{aligned}\tag{47}$$

and

$$\langle B_0^i(\boldsymbol{k}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle = 4\pi ie\,(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\,f^0(\boldsymbol{p}_2)\,\frac{\epsilon^{ijk}k_1^jv_2^k}{(\boldsymbol{k}_1\cdot\boldsymbol{v}_2)^2-\boldsymbol{k}_1^2}\,. \tag{48}$$

Finally, one computes

$$\begin{aligned}\langle E_0^i(\boldsymbol{k}_1)B_0^j(\boldsymbol{k}_2)\rangle \;=\;& -(4\pi e)^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\\ &\times\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\frac{((\boldsymbol{k}_1\cdot\boldsymbol{v})v^i-k_1^i)\epsilon^{jkl}k_2^kv^l}{((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}\,. \end{aligned}\tag{49}$$

5. Fluctuations of longitudinal electric field in isotropic plasma

We first consider a special case of purely longitudinal electric field in the isotropic plasma when the electric field is given by Eq. (31). Then,

$$k_1^i k_2^j \langle E^i(\omega_1,\boldsymbol{k}_1) E^j(\omega_2,\boldsymbol{k}_2)\rangle = \frac{1}{\omega_1\omega_2\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)\,\varepsilon_{\rm L}(\omega_2,\boldsymbol{k}_2)}$$
$$\times \bigg[16\pi^2 e^2 \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}_1}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}_1}\frac{\boldsymbol{k}_2\cdot\boldsymbol{v}_2}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}_2}\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle$$
$$-4\pi i e\int \frac{d^3p_1}{(2\pi)^3}\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}_1}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}_1} k_2^j\langle \delta f_0(\boldsymbol{k}_1,\boldsymbol{p}_1) E_0^j(\boldsymbol{k}_2)\rangle$$
$$-4\pi i e\int \frac{d^3p_2}{(2\pi)^3}\frac{\boldsymbol{k}_2\cdot\boldsymbol{v}_2}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}_2} k_1^i\langle E_0^i(\boldsymbol{k}_1)\delta f_0(\boldsymbol{k}_2,\boldsymbol{p}_2)\rangle - k_1^i k_2^j\langle E_0^i(\boldsymbol{k}_1)E_0^j(\boldsymbol{k}_2)\rangle\bigg]. \quad (50)$$

Substituting the formulas of initial fluctuations (38, 39, 40) into Eq. (50), one finds

$$k_1^i k_2^j \langle E^i(\omega_1,\boldsymbol{k}_1) E^j(\omega_2,\boldsymbol{k}_2)\rangle = 16\pi^2 e^2\frac{(2\pi)^3\delta^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_1)}{\omega_1\omega_2\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)\,\varepsilon_{\rm L}(\omega_2,\boldsymbol{k}_2)}$$
$$\times\int\frac{d^3p}{(2\pi)^3}\left[\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}\frac{\boldsymbol{k}_2\cdot\boldsymbol{v}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}}+\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}+\frac{\boldsymbol{k}_2\cdot\boldsymbol{v}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}}+1\right] f^0(\boldsymbol{p}). \quad (51)$$

It appears that

$$\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}\frac{\boldsymbol{k}_2\cdot\boldsymbol{v}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}}+\frac{\boldsymbol{k}_1\cdot\boldsymbol{v}}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}+\frac{\boldsymbol{k}_2\cdot\boldsymbol{v}}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}}+1=\frac{\omega_1}{\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v}}\frac{\omega_2}{\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v}},$$

and consequently Eq. (51) simplifies to

$$k_1^i k_2^j \langle E^i(\omega_1,\boldsymbol{k}_1) E^j(\omega_2,\boldsymbol{k}_2)\rangle = 16\pi^2 e^2\frac{(2\pi)^3\delta^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_1)}{\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)\,\varepsilon_{\rm L}(\omega_2,\boldsymbol{k}_2)}$$
$$\times\int\frac{d^3p}{(2\pi)^3}\frac{f^0(\boldsymbol{p})}{(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})}. \quad (52)$$

This equation could be easily obtained directly from Eq. (32) where the initial electric field is already eliminated.

Taking into account that the electric fields are parallel to their wave vectors, and consequently $k_1^i k_1^j \langle E^i(\omega_1,\boldsymbol{k}_1)\, E^j(\omega_2,-\boldsymbol{k}_1)\rangle = \boldsymbol{k}_1^2\langle E^i(\omega_1,\boldsymbol{k}_1)\times E^i(\omega_2,-\boldsymbol{k}_1)\rangle$, one finally finds

$$\langle E^i(\omega_1,\boldsymbol{k}_1)E^i(\omega_2,\boldsymbol{k}_2)\rangle = -16\pi^2e^2\frac{(2\pi)^3\delta^{(3)}(\boldsymbol{k}_2+\boldsymbol{k}_1)}{\boldsymbol{k}_1^2\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)\,\varepsilon_{\rm L}(\omega_2,-\boldsymbol{k}_1)} \times\int\frac{d^3p}{(2\pi)^3}\frac{f^0(\boldsymbol{p})}{(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})(\omega_2+\boldsymbol{k}_1\cdot\boldsymbol{v})}\,. \quad (53)$$

Let us now compute $\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle$ given by

$$\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle = \int\limits_{-\infty+i\sigma}^{\infty+i\sigma}\frac{d\omega_1}{2\pi}\int\limits_{-\infty+i\sigma}^{\infty+i\sigma}\frac{d\omega_2}{2\pi}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3} \times e^{-i(\omega_1t_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1+\omega_2t_2-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)}\langle E^i(\omega_1,\boldsymbol{k}_1)E^i(\omega_2,\boldsymbol{k}_2)\rangle\,. \quad (54)$$

Zeros of $\varepsilon_{\rm L}(\omega_i,\boldsymbol{k}_i)$ and of the denominators $(\omega_i-\boldsymbol{k}_i\cdot\boldsymbol{v}+i0^+)$ with $i=1,2$ contribute to the integrals over ω_1 and ω_2. However, once the plasma system under consideration is stable with respect to longitudinal modes, all zeros of $\varepsilon_{\rm L}$ lie in the lower half-plane of complex ω. Consequently, the contributions associated with these zeros exponentially decay in time and they vanish in the long time limit of both t_1 and t_2.

We are further interested in the long time limit of $\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle$ and then, the only non-vanishing contribution corresponds to the poles at $\omega_1=\boldsymbol{k}_1\cdot\boldsymbol{v}$ and $\omega_2=\boldsymbol{k}_2\cdot\boldsymbol{v}$. This contribution reads

$$\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle_\infty = 16\pi^2e^2\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}e^{-i\boldsymbol{k}\cdot\big(\boldsymbol{v}(t_1-t_2)-(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)} \times\frac{f^0(\boldsymbol{p})}{\boldsymbol{k}^2\varepsilon_{\rm L}(\boldsymbol{k}\cdot\boldsymbol{v},\boldsymbol{k})\,\varepsilon_{\rm L}(-\boldsymbol{k}\cdot\boldsymbol{v},-\boldsymbol{k})}\,. \quad (55)$$

Keeping in mind that $\varepsilon_{\rm L}(-\omega,-\boldsymbol{k})=\varepsilon^*_{\rm L}(\omega,\boldsymbol{k})$ for real ω and $\boldsymbol{k}$, it can be rewritten as

$$\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle_\infty = 32\pi^3e^2\int\frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\frac{e^{-i\big(\omega(t_1-t_2)-\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)}}{\boldsymbol{k}^2|\varepsilon_{\rm L}(\omega,\boldsymbol{k})|^2} \times\int\frac{d^3p}{(2\pi)^3}\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})f^0(\boldsymbol{p})\,. \quad (56)$$

As seen, $\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle_\infty$ given by Eq. (56) depends on t_1,t_2 and $\boldsymbol{r}_1,\boldsymbol{r}_2$ only through (t_1-t_2) and $(\boldsymbol{r}_1-\boldsymbol{r}_2)$ and it can be written as

$$\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle_\infty = \int\frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}e^{-i\big(\omega(t_1-t_2)-\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)}\langle E^iE^i\rangle_{\omega\boldsymbol{k}}\,, \quad (57)$$

where the fluctuation spectrum $\langle E^i E^i \rangle_{\omega\boldsymbol{k}}$ is

$$\langle E^i E^i \rangle_{\omega\boldsymbol{k}} = \frac{32\pi^3 e^2}{\boldsymbol{k}^2 |\varepsilon_{\rm L}(\omega,\boldsymbol{k})|^2} \int \frac{d^3p}{(2\pi)^3}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) f^0(\boldsymbol{p})\,. \tag{58}$$

In the case of equilibrium plasma, the formula (58) provides the result which can be obtained directly by means of the fluctuation-dissipation theorem. Let us derive the result. Due to the identity

$$\frac{1}{x \pm i0^+} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x)\,, \tag{59}$$

the imaginary part of $\varepsilon_{\rm L}(\omega,\boldsymbol{k})$, which is given by Eq. (26), equals

$$\mathrm{Im}\,\varepsilon_{\rm L}(\omega,\boldsymbol{k}) = -\frac{4\pi^2 e^2}{\boldsymbol{k}^2} \int \frac{d^3p}{(2\pi)^3}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, \boldsymbol{k}\cdot\frac{\partial f^0(\boldsymbol{p})}{\partial \boldsymbol{p}}\,. \tag{60}$$

In equilibrium $f^0(\boldsymbol{p}) \sim e^{-\beta E_p}$ and $\partial f^0(\boldsymbol{p})/\partial\boldsymbol{p} = -\beta \boldsymbol{v} f^0(\boldsymbol{p})$. Therefore, $\mathrm{Im}\,\varepsilon_{\rm L}$ equals

$$\begin{aligned}\mathrm{Im}\,\varepsilon_{\rm L}(\omega,\boldsymbol{k}) &= \frac{4\pi^2 e^2}{T\boldsymbol{k}^2} \int \frac{d^3p}{(2\pi)^3}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, (\boldsymbol{k}\cdot\boldsymbol{v})\, f^0(\boldsymbol{p}) \\ &= \frac{4\pi^2 e^2 \omega}{T\boldsymbol{k}^2} \int \frac{d^3p}{(2\pi)^3}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, f^0(\boldsymbol{p})\,. \end{aligned} \tag{61}$$

Using the expression (61), the formula (58) is rewritten as

$$\langle E^i E^i \rangle_{\omega\boldsymbol{k}} = 8\pi \frac{T}{\omega} \frac{\mathrm{Im}\,\varepsilon_{\rm L}(\omega,\boldsymbol{k})}{|\varepsilon_{\rm L}(\omega,\boldsymbol{k})|^2}\,, \tag{62}$$

which agrees with Eq. (51.25) from [7] which is obtained there in essentially the same way.

6. Fluctuations of magnetic field in isotropic plasma

As seen in Eq. (35), the magnetic field in isotropic plasma is given by three terms. Therefore, nine terms enter the correlation function $\langle B^i(\omega_1,\boldsymbol{k}_1)\, B^j(\omega_2,\boldsymbol{k}_2)\rangle$. Substituting into these terms the initial fluctuations derived in Sec. 4, one finds after an elementary but lengthy and tedious analysis the following expression:

$$\langle B^i(\omega_1,\boldsymbol{k}_1)B^j(\omega_2,\boldsymbol{k}_2)\rangle = \frac{(4\pi e)^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\epsilon^{ikl}\epsilon^{jmn}k_1^k k_2^m}{(\omega_1^2\varepsilon_{\rm T}(\omega_1,\boldsymbol{k}_1)-\boldsymbol{k}_1^2))(\omega_2^2\varepsilon_{\rm T}(\omega_2,\boldsymbol{k}_2)-\boldsymbol{k}_2^2))}$$
$$\times\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\frac{v^l v^n}{(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}$$
$$\times\Big[(\omega_1(\boldsymbol{k}_1\cdot\boldsymbol{v})-\boldsymbol{k}_1^2)+\omega_1\varepsilon_{\rm T}(\omega_1,\boldsymbol{k}_1)(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\Big]$$
$$\times\Big[(\omega_2(\boldsymbol{k}_2\cdot\boldsymbol{v})-\boldsymbol{k}_2^2)+\omega_2\varepsilon_{\rm T}(\omega_2,\boldsymbol{k}_2)(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})\Big]\,. \tag{63}$$

We now compute $\langle B^i(t_1,\boldsymbol{r}_1)B^j(t_2,\boldsymbol{r}_2)\rangle$ given by

$$\langle B^i(t_1,\boldsymbol{r}_1)B^j(t_2,\boldsymbol{r}_2)\rangle = \int\limits_{-\infty+i\sigma}^{\infty+i\sigma}\frac{d\omega_1}{2\pi}\int\limits_{-\infty+i\sigma}^{\infty+i\sigma}\frac{d\omega_2}{2\pi}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}$$
$$\times\, e^{-i(\omega_1t_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1+\omega_2t_2-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)}\langle B^i(\omega_1,\boldsymbol{k}_1)B^j(\omega_2,\boldsymbol{k}_2)\rangle\,. \tag{64}$$

Zeros of $(\omega_i^2\varepsilon_{\rm T}(\omega_i,\boldsymbol{k}_i)-\boldsymbol{k}_i^2)$ and of $\omega_i-\boldsymbol{k}_i\cdot\boldsymbol{v}+i0^+)$ with $i=1,2$ contribute to the integrals over ω_1 and ω_2. However, once the plasma system under consideration is stable with respect to transverse modes, all zeros of $(\omega_i^2\varepsilon_{\rm T}(\omega_i,\boldsymbol{k}_i)-\boldsymbol{k}_i^2)$ lie in the lower half-plane of complex ω. Consequently, the contributions associated with these zeros exponentially decay in time and they vanish in the long time limit of both t_1 and t_2.

We further consider the long time limit of $\langle B^i(t_1,\boldsymbol{r}_1)B^j(t_2,\boldsymbol{r}_2)\rangle$ and then, the only non-vanishing contribution corresponds to the poles at $\omega_1=\boldsymbol{k}_1\cdot\boldsymbol{v}$ and $\omega_2=\boldsymbol{k}_2\cdot\boldsymbol{v}$. This contribution reads

$$\langle B^i(t_1,\boldsymbol{r}_1)B^j(t_2,\boldsymbol{r}_2)\rangle_\infty = -\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})$$
$$\times\;e^{-i(\omega_1t_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1+\omega_2t_2-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)}$$
$$\times\;\frac{(4\pi e)^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\,\epsilon^{ikl}\epsilon^{jmn}k_1^k k_2^m}{(\omega_1^2\varepsilon_{\rm T}(\omega_1,\boldsymbol{k}_1)-\boldsymbol{k}_1^2))(\omega_2^2\varepsilon_{\rm T}(\omega_2,\boldsymbol{k}_2)-\boldsymbol{k}_2^2))}$$
$$\times\;\frac{v^l v^n}{((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}$$
$$\times\;(\omega_1(\boldsymbol{k}_1\cdot\boldsymbol{v})-\boldsymbol{k}_1^2)(\omega_2(\boldsymbol{k}_2\cdot\boldsymbol{v})-\boldsymbol{k}_2^2)\Big|_{\omega_1=\boldsymbol{k}_1\cdot\boldsymbol{v},\;\omega_2=\boldsymbol{k}_2\cdot\boldsymbol{v}} \tag{65}$$

and it can be easily expressed as

$$\langle B^i(t_1,\boldsymbol{r}_1)B^j(t_2,\boldsymbol{r}_2)\rangle_\infty = \int \frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3} \times e^{-i\big(\omega(t_1-t_2)-\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)}\langle B^iB^j\rangle_{\omega\,\boldsymbol{k}}\,, \quad (66)$$

where the fluctuation spectrum is

$$\langle B^iB^j\rangle_{\omega\,\boldsymbol{k}} = \frac{32\pi^3e^2\epsilon^{ikl}\epsilon^{jmn}k^kk^m}{(\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k})-\boldsymbol{k}^2)(\omega^2\varepsilon_{\rm T}(-\omega,-\boldsymbol{k})-\boldsymbol{k}^2)} \times \int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,v^lv^n\,. \quad (67)$$

When both ω and $\boldsymbol{k}$ are real $\varepsilon_{\rm T}(-\omega,-\boldsymbol{k})=\varepsilon^*_{\rm T}(\omega,\boldsymbol{k})$. Therefore, the fluctuation spectrum can be rewritten as

$$\langle B^iB^j\rangle_{\omega\,\boldsymbol{k}} = \frac{32\pi^3e^2\epsilon^{ikl}\epsilon^{jmn}k^kk^m}{|\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k})-\boldsymbol{k}^2|^2} \times \int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})v^lv^n\,. \quad (68)$$

One observes that the matrix function

$$M^{ij}(\omega,\boldsymbol{k}) \equiv \int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\,v^iv^j\,, \quad (69)$$

which enters the correlation function (68), can be decomposed as

$$M^{ij}(\omega,\boldsymbol{k}) = M_{\rm L}(\omega,\boldsymbol{k})\,\frac{k^ik^j}{\boldsymbol{k}^2} + M_{\rm T}(\omega,\boldsymbol{k})\left(\delta^{ij}-\frac{k^ik^j}{\boldsymbol{k}^2}\right), \quad (70)$$

because the plasma is assumed to be isotropic. Comparing Eq. (70) to Eq. (69), one finds

$$M_{\rm L}(\omega,\boldsymbol{k}) \equiv \int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{\boldsymbol{k}^2}\,, \quad (71)$$

$$M_{\rm T}(\omega,\boldsymbol{k}) \equiv \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\left[\boldsymbol{v}^2-\frac{(\boldsymbol{k}\cdot\boldsymbol{v})^2}{\boldsymbol{k}^2}\right]. \quad (72)$$

Using the decomposition (70), the correlation function (68) can be written down as

$$\langle B^iB^j\rangle_{\omega\,\boldsymbol{k}} = \frac{32\pi^3e^2(\delta^{ij}\boldsymbol{k}^2-k^ik^j)}{|\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k})-\boldsymbol{k}^2|^2}M_{\rm T}(\omega,\boldsymbol{k})\,. \quad (73)$$

For equilibrium plasma the correlation function $\langle B^i B^j \rangle_{\omega \boldsymbol{k}}$ can be expressed in the form of fluctuation-dissipation relation. One first observes that due to the identity (59), the imaginary part of $\varepsilon_{\rm T}(\omega, \boldsymbol{k})$, which is given by Eq. (27), is

$$\begin{aligned} \mathrm{Im}\, \varepsilon_{\rm T}(\omega, \boldsymbol{k}) \;&=\; -\frac{2\pi^2 e^2}{\omega} \int \frac{d^3p}{(2\pi)^3}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \\ &\quad \times \left[\boldsymbol{v}\cdot \frac{\partial f^0(\boldsymbol{p})}{\partial \boldsymbol{p}} - \frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\boldsymbol{k}^2}\, \boldsymbol{k}\cdot \frac{\partial f^0(\boldsymbol{p})}{\partial \boldsymbol{p}} \right]. \qquad (74) \end{aligned}$$

With the equilibrium distribution function, $\mathrm{Im}\, \varepsilon_{\rm T}$ equals

$$\mathrm{Im}\, \varepsilon_{\rm T}(\omega, \boldsymbol{k}) = \frac{2\pi^2 e^2}{T\omega\, \boldsymbol{k}^2} \int \frac{d^3p}{(2\pi)^3}\, \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\, \left(\boldsymbol{k}^2\boldsymbol{v}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2\right) f^0(\boldsymbol{p})\,. \quad (75)$$

Consequently, the function $M_{\rm T}$ (72) can be expressed through $\mathrm{Im}\, \varepsilon_{\rm T}$ (75) as

$$M_{\rm T}(\omega, \boldsymbol{k}) = \frac{T\omega}{4\pi^2 e^2}\, \mathrm{Im}\, \varepsilon_{\rm T}(\omega, \boldsymbol{k})\,, \qquad (76)$$

and finally,

$$\langle B^i B^j \rangle_{\omega\, \boldsymbol{k}} = \frac{8\pi\, T}{\omega^3}\, (\delta^{ij}\boldsymbol{k}^2 - k^i k^j)\, \frac{\mathrm{Im}\, \varepsilon_{\rm T}(\omega, \boldsymbol{k})}{|\varepsilon_{\rm T}(\omega, \boldsymbol{k}) - \frac{\boldsymbol{k}^2}{\omega^2}|^2}\,. \qquad (77)$$

Eq. (77) coincides with the formula (11.2.2.7) from [1] obtained there directly from the fluctuation-dissipation theorem.

7. Fluctuations of electric field in isotropic plasma

The analysis of electric field fluctuations is much more complicated than that of the magnetic field. First of all, there are five terms which enter the formula of electric field given by Eq. (30), and consequently, the correlation function $\langle E^i(\omega_1, \boldsymbol{k}_1) E^j(\omega_2, \boldsymbol{k}_2) \rangle$ includes 25 terms. The magnetic field is purely transverse and some terms automatically drop out but the electric fields have longitudinal and transverse components. Using the formulas of initial fluctuations, which are derived in Sec. 4, and patiently analyzing term by term, one obtains after an elementary but very lengthy calculation the correlation function of the form:

$$\langle E^i(\omega_1,\boldsymbol{k}_1)E^j(\omega_2,\boldsymbol{k}_2)\rangle = 16\pi^2 e^2 (2\pi)^3 \delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)\int \frac{d^3p}{(2\pi)^3}\, f^0(\boldsymbol{p})$$

$$\times\Bigg\{ \frac{k_1^i}{\omega_1^2\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)}\,\frac{k_2^j}{\omega_2^2\varepsilon_{\rm L}(\omega_2,\boldsymbol{k}_2)}\,\frac{\omega_1^2\omega_2^2}{\boldsymbol{k}_1^2(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\,\boldsymbol{k}_2^2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})}$$

$$+\frac{k_1^i}{\omega_1^2\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)}\frac{v^j\boldsymbol{k}_2^2-k_2^j(\boldsymbol{k}_2\cdot\boldsymbol{v})}{\omega_2^2\varepsilon_{\rm T}(\omega_2,\boldsymbol{k}_2)-\boldsymbol{k}_2^2}$$

$$\times\frac{\omega_1^2[\omega_2(\omega_2(\boldsymbol{k}_2\cdot\boldsymbol{v})-\boldsymbol{k}_2^2)-\boldsymbol{k}_2^2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})]}{\boldsymbol{k}_1^2(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\,\boldsymbol{k}_2^2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})\,((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}$$

$$+\frac{v^i\boldsymbol{k}_1^2-k_1^i(\boldsymbol{k}_1\cdot\boldsymbol{v})}{\omega_1^2\varepsilon_{\rm T}(\omega_1,\boldsymbol{k}_1)-\boldsymbol{k}_1^2}\,\frac{k_2^j}{\omega_2^2\varepsilon_{\rm L}(\omega_2,\boldsymbol{k}_2)}$$

$$\times\frac{\omega_2^2[\omega_1(\omega_1(\boldsymbol{k}_1\cdot\boldsymbol{v})-\boldsymbol{k}_1^2)-\boldsymbol{k}_1^2(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})]}{\boldsymbol{k}_1^2(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\,((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)\,\boldsymbol{k}_2^2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})}$$

$$+\frac{k_1^i(\boldsymbol{k}_1\cdot\boldsymbol{v})-v^i\boldsymbol{k}_1^2}{\omega_1^2\varepsilon_{\rm T}(\omega_1,\boldsymbol{k}_1)-\boldsymbol{k}_1^2}\,\frac{k_2^j(\boldsymbol{k}_2\cdot\boldsymbol{v})-v^j\boldsymbol{k}_2^2}{\omega_2^2\varepsilon_{\rm T}(\omega_2,\boldsymbol{k}_2)-\boldsymbol{k}_2^2}$$

$$\times\frac{\omega_1(\omega_1(\boldsymbol{k}_1\cdot\boldsymbol{v})-\boldsymbol{k}_1^2)-\boldsymbol{k}_1^2(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})}{\boldsymbol{k}_1^2(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})((\boldsymbol{k}_1\cdot\boldsymbol{v})^2-\boldsymbol{k}_1^2)}$$

$$\times\frac{\omega_2(\omega_2(\boldsymbol{k}_2\cdot\boldsymbol{v})-\boldsymbol{k}_2^2)-\boldsymbol{k}_2^2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})}{\boldsymbol{k}_2^2(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{v})((\boldsymbol{k}_2\cdot\boldsymbol{v})^2-\boldsymbol{k}_2^2)}\Bigg\}. \tag{78}$$

We now compute $\langle E^i(t_1,\boldsymbol{r}_1)E^j(t_2,\boldsymbol{r}_2)\rangle$ given by

$$\langle E^i(t_1,\boldsymbol{r}_1)E^j(t_2,\boldsymbol{r}_2)\rangle = \int\limits_{-\infty+i\sigma}^{\infty+i\sigma}\frac{d\omega_1}{2\pi}\int\limits_{-\infty+i\sigma}^{\infty+i\sigma}\frac{d\omega_2}{2\pi}\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}$$

$$\times\, e^{-i(\omega_1 t_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1+\omega_2 t_2-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)}\langle E^i(\omega_1,\boldsymbol{k}_1)E^j(\omega_2,\boldsymbol{k}_2)\rangle\,. \tag{79}$$

Zeros of $(\omega_i^2\varepsilon_{\rm T}(\omega_i,\boldsymbol{k}_i)-\boldsymbol{k}_i^2)$, $\omega_i^2\varepsilon_{\rm L}(\omega_i,\boldsymbol{k}_i)$ and of $(\omega_i-\boldsymbol{k}_i\cdot\boldsymbol{v}+i0^+)$ with $i=1,2$ contribute to the integrals over ω_1 and ω_2. However, once the plasma system under consideration is stable, all zeros of $(\omega_i^2\varepsilon_{\rm T}(\omega_i,\boldsymbol{k}_i)-\boldsymbol{k}_i^2)$ and $\omega_i^2\varepsilon_{\rm L}(\omega_i,\boldsymbol{k}_i)$ lie in the lower half-plane of complex ω. Consequently, the contributions associated with these zeros exponentially decay in time and they vanish in the long time limit of both t_1 and t_2.

We further consider the long time limit of $\langle E^i(t_1,\boldsymbol{r}_1)E^j(t_2,\boldsymbol{r}_2)\rangle$ and then, the only non-vanishing contribution corresponds to the poles at $\omega_1 = \boldsymbol{k}_1\cdot\boldsymbol{v}$ and $\omega_2 = \boldsymbol{k}_2\cdot\boldsymbol{v}$. This contribution reads

$$\langle E^i(t_1,\boldsymbol{r}_1)E^j(t_2,\boldsymbol{r}_2)\rangle_\infty = -16\pi^2e^2\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)$$
$$\times\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,e^{-i(\omega_1t_1-\boldsymbol{k}_1\cdot\boldsymbol{r}_1+\omega_2t_2-\boldsymbol{k}_2\cdot\boldsymbol{r}_2)}\frac{\omega_1\omega_2}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}$$
$$\times\left[\frac{\omega_1k_1^i}{\omega_1^2\varepsilon_{\rm L}(\omega_1,\boldsymbol{k}_1)}+\frac{k_1^i(\boldsymbol{k}_1\cdot\boldsymbol{v})-v^i\boldsymbol{k}_1^2}{\omega_1^2\varepsilon_{\rm T}(\omega_1,\boldsymbol{k}_1)-\boldsymbol{k}_1^2}\right]$$
$$\times\left[\frac{\omega_2k_2^j}{\omega_2^2\varepsilon_{\rm L}(\omega_2,\boldsymbol{k}_2)}+\frac{v^j\boldsymbol{k}_2^2-k_2^j(\boldsymbol{k}_2\cdot\boldsymbol{v})}{\omega_2^2\varepsilon_{\rm T}(\omega_2,\boldsymbol{k}_2)-\boldsymbol{k}_2^2}\right]\Bigg|_{\omega_1=\boldsymbol{k}_1\cdot\boldsymbol{v},\ \omega_2=\boldsymbol{k}_2\cdot\boldsymbol{v}}. \qquad (80)$$

The correlation function (80) can be rewritten as

$$\langle E^i(t_1,\boldsymbol{r}_1)E^j(t_2,\boldsymbol{r}_2)\rangle_\infty=\int\frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}e^{-i\big(\omega(t_1-t_2)-\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)\big)}\langle E^iE^j\rangle_{\omega\boldsymbol{k}}\,, \qquad (81)$$

where the fluctuation spectrum is

$$\langle E^iE^j\rangle_{\omega\boldsymbol{k}} = 16\pi^2e^2\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\frac{\omega^2}{\boldsymbol{k}^4}$$
$$\times\left\{\frac{k^i}{\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\,\frac{k^j}{\omega^2\varepsilon_{\rm L}(-\omega,\boldsymbol{k}^2)}\,\omega^2+\frac{k^i}{\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})}\,\frac{v^j\boldsymbol{k}^2-k^j(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega^2\varepsilon_{\rm T}(-\omega,\boldsymbol{k})-\boldsymbol{k}^2}\,\omega\right.$$
$$\left.+\frac{v^i\boldsymbol{k}^2-k^i(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k})-\boldsymbol{k}^2}\,\frac{k^j}{\omega^2\varepsilon_{\rm L}(-\omega,\boldsymbol{k})}\,\omega+\frac{k^i(\boldsymbol{k}\cdot\boldsymbol{v})-v^i\boldsymbol{k}^2}{\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k})-\boldsymbol{k}^2}\,\frac{k^j(\boldsymbol{k}\cdot\boldsymbol{v})-v^j\boldsymbol{k}^2}{\omega^2\varepsilon_{\rm T}(-\omega,\boldsymbol{k})-\boldsymbol{k}^2}\right\}. \qquad (82)$$

One easily proves that the second and third contribution to the fluctuation spectrum (82) vanish due to the plasma isotropy. Taking into account that for real ω and $\boldsymbol{k}$, the dielectric function obeys $\varepsilon_s(-\omega,-\boldsymbol{k}) = \varepsilon_s^*(\omega,\boldsymbol{k})$ with $s = L, T$, the fluctuation spectrum (82) can be written as

$$\langle E^iE^j\rangle_{\omega\boldsymbol{k}} \;=\; 16\pi^2e^2\int\frac{d^3p}{(2\pi)^3}\,f^0(\boldsymbol{p})\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\frac{\omega^2}{\boldsymbol{k}^4}$$
$$\times\;\left\{\frac{\omega^2k^ik^j}{|\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})|^2}+\frac{(k^i(\boldsymbol{k}\cdot\boldsymbol{v})-v^i\boldsymbol{k}^2)(k^j(\boldsymbol{k}\cdot\boldsymbol{v})-v^j\boldsymbol{k}^2)}{|\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k})-\boldsymbol{k}^2|^2}\right\}. \qquad (83)$$

Due to the plasma isotropy, the expression, which enters the transverse contribution, can be further rewritten as

$$\int \frac{d^3p}{(2\pi)^3} f^0(\boldsymbol{p}) 2\pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})(k^i(\boldsymbol{k}\cdot\boldsymbol{v}) - v^i\boldsymbol{k}^2)(k^j(\boldsymbol{k}\cdot\boldsymbol{v}) - v^j\boldsymbol{k}^2)$$
$$= \frac{1}{2}\Big(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\Big)\boldsymbol{k}^2 \int \frac{d^3p}{(2\pi)^3} f^0(\boldsymbol{p}) 2\pi\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v})((\boldsymbol{k}^2\boldsymbol{v}^2 - (\boldsymbol{k}\cdot\boldsymbol{v})^2)\,. \quad (84)$$

In the equilibrium plasma, the imaginary parts of $\varepsilon_{\rm L}(\omega,\boldsymbol{k})$ and $\varepsilon_{\rm T}(\omega,\boldsymbol{k})$ are given by the formulas (61, 75) and the fluctuation spectrum (83) can be expressed as

$$\langle E^iE^j\rangle_{\omega\,\boldsymbol{k}} = 8\pi T\omega^3 \times\Big[\frac{k^ik^j}{\boldsymbol{k}^2}\frac{{\rm Im}\,\varepsilon_{\rm L}(\omega,\boldsymbol{k})}{|\omega^2\varepsilon_{\rm L}(\omega,\boldsymbol{k})|^2} + \Big(\delta^{ij} - \frac{k^ik^j}{\boldsymbol{k}^2}\Big)\frac{{\rm Im}\,\varepsilon_{\rm T}(\omega,\boldsymbol{k})}{|\omega^2\varepsilon_{\rm T}(\omega,\boldsymbol{k}) - \boldsymbol{k}^2|^2}\Big], \quad (85)$$

which for the longitudinal fields reproduces the formula (62). The result (85) agrees with Eq. (11.2.2.6) from [1] derived using the fluctuation-dissipation relation.

8. Fluctuations of longitudinal electric field in the two-stream system

Nonequlibrium calculations are usually much more difficult than the equilibrium ones. The first problem is to invert the matrix $\Sigma^{ij}(\omega,\boldsymbol{k})$ defined by Eq. (23). In the case of longitudinal electric field, which is discussed here, it is solved trivially. We start with Eq. (22) projecting it on $\boldsymbol{k}$ and assuming that $\boldsymbol{E}$ and $\boldsymbol{E}_0$ are purely longitudinal. Then, the matrix (23) is replaced by the scalar function.

Further, we neglect the first term in the r.h.s. of Eq. (22). This term vanishes in isotropic systems; it is of order e^2 higher than the second term; it is also expected to be small in nonrelativistic regime due to the smallness of particle velocity. So, there are good reasons to neglect it. Eliminating $\boldsymbol{E}_0$ by means of the first Maxwell equation we obtain Eq. (32) which was previously derived for the case of isotropic plasma. In the following we consider fluctuations of longitudinal electric fields in the nonrelativistic two-stream system.

The distribution function of the two-stream system is chosen to be

$$f^0(\boldsymbol{p}) = (2\pi)^3 n\Big[\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) + \delta^{(3)}(\boldsymbol{p} + \boldsymbol{q})\Big]\,, \quad (86)$$

where n is the electron density in a single stream. To compute $\varepsilon_{\rm L}(\omega,\boldsymbol{k})$ we first perform integration by parts in Eq. (26) and then, substituting the distribution function (86) into the resulting formula, we obtain in the nonrelativistic approximation

$$\varepsilon_{\rm L}(\omega,\boldsymbol{k}) = \frac{\big(\omega^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2\big)^2 - 2\mu^2\big(\omega^2+(\boldsymbol{k}\cdot\boldsymbol{u})^2\big)}{\big(\omega^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2\big)^2}\,, \tag{87}$$

where $\boldsymbol{u}$ is the stream velocity (nonrelativistically $\boldsymbol{u}=\boldsymbol{q}/m$ with m being the electron mass) and $\mu^2 \equiv 4\pi e^2 n/m$. There are four roots $\pm\omega_\pm(\boldsymbol{k})$ of the dispersion equation $\varepsilon_{\rm L}(\omega,\boldsymbol{k})=0$ which read

$$\omega_\pm^2(\boldsymbol{k}) = \mu^2 + (\boldsymbol{k}\cdot\boldsymbol{u})^2 \pm \mu\sqrt{\mu^2+4(\boldsymbol{k}\cdot\boldsymbol{u})^2}\,. \tag{88}$$

As seen, $0<\omega_+(\boldsymbol{k})\in R$ for any $\boldsymbol{k}$ but $\omega_-(\boldsymbol{k})$ is imaginary for $(\boldsymbol{k}\cdot\boldsymbol{u})^2<2\mu^2$ when it represents the well-known two-stream electrostatic instability. For $(\boldsymbol{k}\cdot\boldsymbol{u})^2>2\mu^2$, the mode is stable, $0<\omega_-(\boldsymbol{k})\in R$.

The correlation function $\langle E^i(\omega_1,\boldsymbol{k}_1)E^i(\omega_2,\boldsymbol{k}_2)\rangle$ as given by Eq. (53) equals

$$\begin{aligned}
\langle E^i(\omega_1,\boldsymbol{k}_1)E^i(\omega_2,\boldsymbol{k}_2)\rangle &= -16\pi^2e^2n\frac{(2\pi)^3\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)}{\boldsymbol{k}_1^2}\\
&\times\Big[(\omega_1+\boldsymbol{k}_1\cdot\boldsymbol{u})(\omega_2+\boldsymbol{k}_2\cdot\boldsymbol{u})+(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{u})(\omega_2-\boldsymbol{k}_2\cdot\boldsymbol{u})\Big]\\
&\times\frac{\omega_1^2-(\boldsymbol{k}_1\cdot\boldsymbol{u})^2}{(\omega_1-\omega_-(\boldsymbol{k}_1))(\omega_1+\omega_-(\boldsymbol{k}_1))(\omega_1-\omega_+(\boldsymbol{k}_1))(\omega_1+\omega_+(\boldsymbol{k}_1))}\\
&\times\frac{\omega_2^2-(\boldsymbol{k}_2\cdot\boldsymbol{u})^2}{(\omega_2-\omega_-(\boldsymbol{k}_2))(\omega_2+\omega_-(\boldsymbol{k}_2))(\omega_2-\omega_+(\boldsymbol{k}_2))(\omega_2+\omega_+(\boldsymbol{k}_2))}\,.
\end{aligned} \tag{89}$$

One observes that the poles of the correlation function $\langle E^i(\omega_1,\boldsymbol{k}_1)E^i(\omega_2,\boldsymbol{k}_2)\rangle$ at $\omega_1=\boldsymbol{k}_1\boldsymbol{v}$ and $\omega_2=\boldsymbol{k}_2\boldsymbol{v}$, which give the stationary contribution to the equilibrium fluctuation spectrum, have disappeared in Eq. (89) as the inverse dielectric functions vanish at these points.

The correlation function $\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle$ is given by Eq. (54) with $\langle E^i(\omega_1,\boldsymbol{k}_1)E^i(\omega_2,\boldsymbol{k}_2)\rangle$ defined by Eq. (89). Performing the trivial integration over $\boldsymbol{k}_2$ and taking into account that $\omega_\pm(-\boldsymbol{k})=\omega_\pm(\boldsymbol{k})$, one finds:

$$\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\rangle = 32\pi^2e^2n \int\limits_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega_1}{2\pi i} \int\limits_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega_2}{2\pi i}$$
$$\times \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\left(\omega_1 t_1+\omega_2 t_2-\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)}}{\boldsymbol{k}^2}[\omega_1\omega_2-(\boldsymbol{k}\cdot\boldsymbol{u})^2]$$
$$\times \frac{\omega_1^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2}{(\omega_1-\omega_-(\boldsymbol{k}))(\omega_1+\omega_-(\boldsymbol{k}))(\omega_1-\omega_+(\boldsymbol{k}))(\omega_1+\omega_+(\boldsymbol{k}))}$$
$$\times \frac{\omega_2^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2}{(\omega_2-\omega_-(\boldsymbol{k}))(\omega_2+\omega_-(\boldsymbol{k}))(\omega_2-\omega_+(\boldsymbol{k}))(\omega_2+\omega_+(\boldsymbol{k}))}\,. \tag{90}$$

There are 16 contributions to the integrals over ω_1 and ω_2 in Eq. (90) related to the poles at $\pm\omega_\pm$. Summing up the contributions, we get after lengthy calculation

$$\big\langle E^i(t_1,\boldsymbol{r}_1)E^i(t_2,\boldsymbol{r}_2)\big\rangle = 16\pi^2e^2n \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)}}{\boldsymbol{k}^2} \frac{1}{(\omega_+^2-\omega_-^2)^2}$$
$$\times \Bigg\{ \frac{\big(\omega_+^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2\big)^2}{\omega_+^2}$$
$$\times \Big[(\omega_+^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos\left(\omega_+(t_1+t_2)\right) + (\omega_+^2+(\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos\left(\omega_+(t_1-t_2)\right)\Big]$$
$$- \frac{(\omega_+^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2)(\omega_-^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2)}{\omega_+\omega_-}\Big[(\omega_+\omega_- - (\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos(\omega_+t_1+\omega_-t_2)$$
$$+ (\omega_+\omega_- + (\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos(\omega_+t_1-\omega_-t_2)$$
$$+ (\omega_+\omega_- - (\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos(\omega_-t_1+\omega_+t_2)$$
$$+ (\omega_+\omega_- + (\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos(\omega_-t_1-\omega_+t_2)\Big] + \frac{(\omega_-^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2)^2}{\omega_-^2}$$
$$\times \Big[(\omega_-^2-(\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos\left(\omega_-(t_1+t_2)\right) + (\omega_-^2+(\boldsymbol{k}\cdot\boldsymbol{u})^2)\cos\left(\omega_-(t_1-t_2)\right)\Big]\Bigg\}\,. \tag{91}$$

Let us now consider the domain of wave vectors $(\boldsymbol{k}\cdot\boldsymbol{u})^2 < 2\mu^2$ when $\omega_-(\boldsymbol{k})$ is imaginary and it represents the unstable electrostatic mode.

We write down $\omega_-(\boldsymbol{k})$ as $i\gamma_{\boldsymbol{k}}$ with $0 < \gamma_{\boldsymbol{k}} \in R$,

$$\gamma_{\boldsymbol{k}} \equiv \sqrt{\mu\sqrt{\mu^2 + 4(\boldsymbol{k}\cdot\boldsymbol{u})^2} - \mu^2 - (\boldsymbol{k}\cdot\boldsymbol{u})^2}\,. \tag{92}$$

We are interested in the contributions to the correlation function coming from the unstable modes. The contributions, which are the fastest growing functions of $(t_1 + t_2)$ and $(t_1 - t_2)$, correspond to the last term in Eq. (91). The contributions provide

$$\begin{aligned}
&\langle E^i(t_1, \boldsymbol{r}_1)E^i(t_2, \boldsymbol{r}_2)\rangle_{\text{unstable}} = 4\pi^2 e^2 n \\
&\times \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\boldsymbol{k}(\boldsymbol{r}_1-\boldsymbol{r}_2)}}{\boldsymbol{k}^2\mu^2(\mu^2 + 4(\boldsymbol{k}\cdot\boldsymbol{u})^2)} \frac{(\gamma_{\boldsymbol{k}}^2 + (\boldsymbol{k}\cdot\boldsymbol{u})^2)^2}{\gamma_{\boldsymbol{k}}^2} \\
&\times \Big[(\gamma_{\boldsymbol{k}}^2 + (\boldsymbol{k}\cdot\boldsymbol{u})^2)\cosh\left(\gamma_{\boldsymbol{k}}(t_1+t_2)\right) + (\gamma_{\boldsymbol{k}}^2 - (\boldsymbol{k}\cdot\boldsymbol{u})^2)\cosh\left(\gamma_{\boldsymbol{k}}(t_1-t_2)\right)\Big]\,,
\end{aligned} \tag{93}$$

where we have taken into account that $\omega_+^2 - \omega_-^2 = 2\mu\sqrt{\mu^2 + 4(\boldsymbol{k}\cdot\boldsymbol{u})^2}$. As seen, the correlation function (93) is invariant with respect to space translations — it depends on the difference $(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ only. The initial plasma state is on average homogeneous and it remains like this in course of the system's temporal evolution. The time dependence of the correlation function (93) is very different from the space dependence. The electric fields exponentially grow and so does the correlation function both in (t_1+t_2) and (t_1-t_2). The fluctuation spectrum also evolves in time as the growth rate of unstable modes is wave-vector dependent. After sufficiently long times the fluctuation spectrum is dominated by the fastest growing modes. It should be remembered, however, that our results hold for times which are not too long. Otherwise, the perturbation, which exponentially grows, violates the condition (6) justifying the linearization procedure.

9. Summary and conclusion

The calculations presented here show how to obtain a spectrum of electromagnetic fluctuations in equilibrium or nonequilibrium plasmas as a solution of an initial value problem. We first linearize the transport equation around the state which is on average stationary and homogenous. The linearized transport equation is solved together with the Maxwell equations by means of the one-sided Fourier transformation. The time dependent fluctuation spectrum is expressed through the fluctuations in the initial state. Electromagnetic initial fluctuations are determined by the initial fluctuations of the distribution function. The later are identified with the fluctuations in a classical system of noninteracting particles. We compute fluctuation spectrum of longitudinal electric fields in isotropic plasma, and then there are considered fluctuations of magnetic and electric fields. Our equilibrium results coincide with those obtained by means of the fluctuation-dissipation theorem. However, the method adopted here clearly shows how the system looses its memory and how the stationary equilibrium spectrum of fluctuations emerges. As an example of unstable systems, the fluctuations of longitudinal electric field in the two-stream system are considered. The fluctuation spectrum appears to be qualitatively different than that of the equilibrium plasma.

The scheme of calculation, which is worked our here in detail, can be applied to a variety of plasma nonequilibrium states. Our actual objective is, however, to generalize the scheme to study fluctuations in the quark-gluon plasma mentioned in the Introduction. Such a generalization is not quite trivial. The problem is that chromodynamic fields are, in contrast to their electromagnetic counterparts, gauge dependent, but physically meaningful correlation functions have to be gauge independent. A treatment of color charges needs to be different as well.

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