Gauge symmetries

The quantum field theories which all together form the Standard Model of particles physics are the theories with a gauge symmetry. Therefore, the aim of the first lecture is to introduce the concept of gauge symmetry.

Gauge symmetry of electrodynamics

- The Lagrangian density of electrodynamics is

$$\mathcal{L}(x) = \frac{1}{4} F^{\mu\nu}(x)\, F_{\nu\mu}(x) + \overline{\psi}(x)\,\big(i\partial^\mu \gamma_\mu - m\big)\,\psi(x) - e\overline{\psi}(x)\gamma_\mu\psi(x)\,A^\mu(x), \tag{1}$$

  where the strength tensor $F^{\mu\nu}(x)$ is expressed through the four-potential $A^\mu(x)$ as

$$F^{\mu\nu}(x) \equiv \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x), \tag{2}$$

  and $\psi(x)$ is the spinor field and $\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma_0$. The first and the second term of Eq. (1) are the the Lagrangians of free electromagnetic and spinor fields. The third term represents the interaction of the spinor and electromagnetic fields.

- Since the action $\int d^4x \mathcal{L}(x)$ not the Lagrangian density really matters, one can perform the partial integration and the Lagrangian density of the free spinor field can be written as

$$\mathcal{L}(x) = \overline{\psi}(x)\,\big(- i \overleftarrow{\partial^\mu}\, \gamma_\mu - m\big)\,\psi(x). \tag{3}$$

- If one chooses as independent variables the fields ψ and A^μ, the Euler-Lagrange equations

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0, \tag{4}$$

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0, \tag{5}$$

  provide the equations of motion

$$\big[i\gamma_\mu\big(\partial^\mu + ieA^\mu(x)\big) - m\big]\psi(x) = 0, \tag{6}$$

$$\partial_\mu F^{\mu\nu}(x) = e\overline{\psi}(x)\gamma^\nu\psi(x). \tag{7}$$

- The Lagrangian (1) is invariant under the gauge transformation

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x), \qquad \psi(x) \to e^{-ie\Lambda(x)}\psi(x). \tag{8}$$

- It is easier to prove the invariance when the Lagrangian is written as

$$\mathcal{L}(x) = \frac{1}{4} F^{\mu\nu}(x)\, F_{\nu\mu}(x) + \overline{\psi}(x)\,\Big(i\big(\partial^\mu + ieA^\mu(x)\big)\gamma_\mu - m\Big)\,\psi(x). \tag{9}$$

  Exercise: Prove that the Lagrangian (9) is invariant under the gauge transformation (8).

- One observes that the Lagrangian of free spinor field is not invariant under the gauge transformation. The interaction term makes the Lagrangian invariant. Therefore, the gauge invariance dictates a structure of the interaction term.

- If the Lagrangian density includes the photon mass term ($m_\gamma^2 A^\mu A_\mu$) one gets the so-called Proca theory which is not gauge invariant. The mass term violates the gauge symmetry.

- Since the group of unitary matrices of the dimension $N \times N$ is called U(N), the symmetry group of electrodynamics is called U(1).

Noether theorem

- The Noether theorem states that there is a conserved charge if the action is invariant under a continuous transformation.
- Let us consider the action
$$S[\psi, A^\mu] = \int d^4x\, \mathcal{L}(\psi, A^\mu), \tag{10}$$
where the Lagrangian density is given by Eq. (1).
- Now, we perform the infinitesimal gauge transformation
$$\psi \to \psi + \delta\psi, \qquad A^\mu \to A^\mu + \delta A^\mu, \tag{11}$$
where
$$\delta\psi = -ie\Lambda\psi, \qquad \delta A^\mu = \partial^\mu\Lambda \tag{12}$$
and the function Λ is assumed to be small.
- The action changes as
$$S[\psi+\delta\psi, A^\mu+\delta A^\mu] = S[\psi, A^\mu] + \frac{\delta S[\psi, A^\mu]}{\delta\psi}\delta\psi + \frac{\delta S[\psi, A^\mu]}{\delta A^\mu}\delta A^\mu = S[\psi, A^\mu] + \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu\right]. \tag{13}$$
- Assuming that the action is invariant under the gauge symmetry that is $S[\psi + \delta\psi, A^\mu + \delta A^\mu] = S[\psi, A^\mu]$, we find that
$$\int d^4x \left[\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu\right] = 0. \tag{14}$$
- Using the Lagrangian density (3), one obtains
$$\frac{\partial\mathcal{L}}{\partial\psi} = -\overline{\psi}(x)\big[i\gamma_\mu\big(\overleftarrow{\partial^\mu} - ieA^\mu(x)\big) + m\big] = 0, \tag{15}$$
that is it vanishes because the fields are assumed to satisfy equation of motion (6).
- Since
$$\frac{\partial\mathcal{L}}{\partial A^\mu} = -e\overline{\psi}\gamma_\mu\psi, \tag{16}$$
Eq. (14) gives
$$\int d^4x\, \overline{\psi}\gamma_\mu\psi\, \partial^\mu\Lambda = 0, \tag{17}$$
where the result (12) is used.
- Performing the partial integration, one finally obtains
$$\int d^4x\, \big(\partial^\mu(\overline{\psi}\gamma_\mu\psi)\big)\Lambda = 0, \tag{18}$$
which means that the four-vector $\overline{\psi}\gamma_\mu\psi$ obeys the continuity equation
$$\partial^\mu(\overline{\psi}\gamma_\mu\psi) = 0, \tag{19}$$
and that the associated charge defined as
$$Q = \int d^3x\, \overline{\psi}\gamma_0\psi \tag{20}$$
is conserved.
- The current is traditionally multiplied by e.

SU(N) group

- The group of unitary $N \times N$ matrices of unit determinant is called SU(N).
- The matrix U, which belongs to the SU(N) group, can be parametrized as

$$U = e^{i\omega^a \tau^a}, \tag{21}$$

  where ω^a with $a = 1, 2, \ldots N^2 - 1$ are real numbers and τ^a are the generators of the fundamental representation of SU(N). We do not distinguish lower and upper indices a, b.
- The generators obey the commutation relations

$$[\tau^a, \tau^b] = i f^{abc} \tau^c, \tag{22}$$

  where f^{abc} are totally antisymmetric SU(N) structure constants.
- The generators are hermitian traceless matrices normalized in the canonical way as

$$\mathrm{Tr}[\tau^a \tau^b] = \frac{1}{2}\delta^{ab}. \tag{23}$$

- The matrix (29) is automatically unitary because

$$U^\dagger = e^{-i\omega^a \tau^a}. \tag{24}$$

- The matrix (29) is automatically of unit determinant because

$$\det U = \exp\big(i\omega^a \mathrm{Tr}[\tau^a]\big) = 1, \tag{25}$$

  where the formula $\det e^A = e^{\mathrm{Tr}[A]}$ is used.

  Exercise: Prove that $N^2 - 1$ real numbers uniquely define the matrix belonging to SU(N).
- The group SU(N) is nonAbelian *i.e.*

$$U_1 U_2 \neq U_2 U_1. \tag{26}$$

- In case of SU(2) group the generators are $\tau^a = \frac{1}{2}\sigma^a$ with σ^a known as the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{27}$$

  and the structure constants are the fully antisymmetric Levi-Civita symbols ϵ^{abc}.

Theory with SU(N) gauge symmetry

- Let $\psi_i(x)$ be a set of N spinor field that is $i = 1, 2, \ldots N$. We denote as $\psi(x)$ the column of N elements which are called color components.
- It is assumed that $\psi(x)$ transforms as

$$\psi(x) \to U(x)\psi(x), \tag{28}$$

  where $U(x)$ is a local SU(N) transformation that is

$$U(x) = e^{i\omega^a(x)\,\tau^a}, \tag{29}$$

  with the x-dependent parameters $\omega^a(x)$.

- The terms of the Lagrangian density, which include that of the free spinor field and of the interaction with the gauge potential $A^\mu(x)$, is expected to be

$$\mathcal{L} = \overline{\psi}\,\Big(i\big(\partial^\mu - igA^\mu\big)\gamma_\mu - m\Big)\,\psi. \tag{30}$$

It looks as the analogous term of electrodynamics but one should remember that the gauge potential $A^\mu(x)$ is $N \times N$ matrix.

- Assuming that the Lagrangian density (30) is invariant under the gauge transformation, we are going to find the gauge transformation of the potential $A^\mu(x)$. Knowing the transformation (28) and using $A^\mu \to A^{\mu\prime}$ the transformed Lagrangian (30) becomes

$$\mathcal{L}' = \overline{\psi}\,U^\dagger\Big(i\big(\partial^\mu - igA^{\mu\prime}\big)\gamma_\mu - m\Big)\,U\psi = \overline{\psi}\Big(i\big(\partial^\mu - ig\,U^\dagger A^{\mu\prime}U + U^\dagger(\partial^\mu U)\big)\gamma_\mu - m\Big)\,\psi, \tag{31}$$

where we have taken into account that $U^\dagger U = \mathbb{1}$ and $\partial^\mu U\psi = (\partial^\mu U)\psi + U\partial^\mu\psi$.

- $\mathcal{L}' = \mathcal{L}$ if

$$U^\dagger A^{\mu\prime}U + \frac{i}{g}U^\dagger(\partial^\mu U) = A^\mu, \tag{32}$$

which multiplied by U from the left and by $U^\dagger$ from the right gives the desired transformation law

$$A^\mu \to A^{\mu\prime} = UA^\mu U^\dagger - \frac{i}{g}(\partial^\mu U)U^\dagger. \tag{33}$$

- Let us note that $(\partial^\mu U)U^\dagger = -U\partial^\mu U^\dagger$ because $U^\dagger U = \mathbb{1}$.
- The strength tensor, which antisymmetric in the Lorentz indices, is defined as

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu], \tag{34}$$

where $[A^\mu, A^\nu] = A^\mu A^\nu - A^\nu A^\mu$ is the commutator of A^μ and A^ν.

<u>Exercise</u>: Prove that the strength tensor (34) transforms as

$$F^{\mu\nu} \to UF^{\mu\nu}U^\dagger. \tag{35}$$

- The term of Lagrangian which analogous to that of free electromagnetic field is

$$\frac{1}{2}\mathrm{Tr}[F^{\mu\nu}F_{\nu\mu}] \equiv \frac{1}{2}F^{\mu\nu}_{ij}F_{ji\,\nu\mu}. \tag{36}$$

- Keeping in mind that $\mathrm{Tr}[AB] = \mathrm{Tr}[BA]$, one immediately finds that the term $\mathrm{Tr}[F^{\mu\nu}F_{\nu\mu}]$ is invariant under gauge transformations.
- The complete Lagrangian of the theory is

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}[F^{\mu\nu}F_{\nu\mu}] + \overline{\psi}\,\big(iD^\mu\gamma_\mu - m\big)\,\psi. \tag{37}$$

where $D^\mu \equiv \partial^\mu\mathbb{1} - igA^\mu$ is the covariant derivative.

- Using the covariant derivative, the strength tensor can be written as

$$F^{\mu\nu} = D^\mu A^\nu - D^\nu A^\mu = \frac{i}{g}[D^\mu, D^\nu]. \tag{38}$$

- The quantities like A^μ, which are $N \times N$ matrices, can be expanded in the basis of generators as

$$A^\mu_{ij}(x) = A^\mu_a(x)\,\tau^a_{ij}, \quad i,j = 1,\,2,\ldots N, \quad a = 1,\,2,\ldots N^2-1. \tag{39}$$

A set of $N^2 - 1$ functions $A^\mu_a(x)$ is called the potential in the <u>adjoint representation</u> of the SU(N) group.

- Due to the commutation relation (22) a commutator $[A^\mu, A^\nu]$ can be written as

$$[A^\mu, A^\nu] = if^{abc} A^\mu_a A^\nu_b \tau^c. \tag{40}$$

- The strength tensor in the adjoint representation is

$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + gf^{abc} A^\mu_b A^\nu_c. \tag{41}$$

- The Lagrangian density can be rewritten as

$$\mathcal{L} = \frac{1}{4} F^{\mu\nu}_a F_{a\,\nu\mu} + \overline{\psi} \left(iD^\mu \gamma_\mu - m \right) \psi \tag{42}$$

- The equations of motion called Yang-Mills equations, which obtained from the Lagrange-Euler equations, are

$$\left[i\gamma_\mu D^\mu - m \right] \psi(x) = 0, \tag{43}$$

$$[D_\mu, F^{\mu\nu}(x)] = j^\mu(x), \tag{44}$$

where $j^\mu = j^\mu_a \tau_a$ with $j^\mu_a \equiv g\overline{\psi}\gamma^\nu \tau_a \psi$

<u>Exercise</u>: Derive the equations (43, 44) from the Lagrangian (42).

- Introducing the covariant derivative is the adjoint representation as

$$\mathcal{D}^{ab}_\mu = \partial^\mu \delta^{ab} - gf^{abc} A^c_\mu, \tag{45}$$

Eq. (44) can be rewritten as

$$\mathcal{D}^{ab}_\mu F^{\mu\nu}_b(x) = j^\mu_a(x), \tag{46}$$

- From the equation (44), one finds that

$$[D_\mu, j^\mu] = 0, \tag{47}$$

that is the current is not conserved but covariantly conserved.