Siberian Mathematical Journal, Vol. 55, *No.* 2, *pp.* 296–300, 2014
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THE SHARK TEETH IS A TOPOLOGICAL IFS-ATTRACTOR

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Abstract: We show that the space called the shark teeth is a topological IFS-attractor, that is, for every open cover of $X = \bigcup_{i=1}^{n} f_i(X)$, its image under every suitable large composition from the family of continuous functions $\{f_1, \dots, f_n\}$ lies in some set from the cover. In particular, there exists a space that is not homeomorphic to any IFS-attractor but is a topological IFS-attractor.

DOI: 10.1134/S0037446614020128

Keywords: fractal, iterated function system, IFS-attractor

The iterated function system (IFS) is one of the most popular and simple method of constructing fractal structures which has wide applications to data compression, computer graphics, medicine, economics, earthquake and weather prediction, and many others. A compact metric space X is called an *IFS-attractor* if $X = \bigcup_{i=1}^{n} f_i(X)$ for some contractions $f_1, \dots, f_n : X \to X$. In this case the family $\{f_1, \dots, f_n\}$ is called an *iterated function system*. We recall that a map $f : X \to X$ is a *contraction* if its Lipschitz constant

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

is less than 1.

The notion of iterated function system was introduced in 1981 by John Hutchinson [1] and popularized by Michael Barnsley [2]. The topological properties of IFS-attractors were studied in [3–5]. In particular the definition of topological IFS-attractor was proposed in [5]: a compact topological space X is a *topological IFS-attractor* if $X = \bigcup_{i=1}^{n} f_i(X)$ for some continuous maps $f_1, \dots, f_n : X \to X$ with the property that for every open cover $\mathscr{U}$ of X there is $m \in \mathbb{N}$, such that for all $g_1, \dots, g_m \in \{f_1, \dots, f_n\}$ the set $g_1 \circ \dots \circ g_m(X)$ lies in some $U \in \mathscr{U}$.

Note that every compact metric space X is a topological IFS-attractor if for its every open cover $\mathscr{U}$ the diameter of the set $g_1 \circ \dots \circ g_m(X)$ is less than the Lebesgue number of $\mathscr{U}$, for some $m \in \mathbb{N}$ and every $g_1, \dots, g_m \in \{f_1, \dots, f_n\}$.

It is easy to see that each IFS-attractor is a topological IFS-attractor but not the other way around. Moreover, we show that the space called the shark teeth and constructed in [5], which is not homeomorphic to the attractor of any iterated function system, is a topological IFS-attractor.

1. The Shark Teeth

Consider the piecewise linear periodic function

$$\varphi(t) = \begin{cases} t - n & \text{if } t \in [n, n + \frac{1}{2}] \text{ for some } n \in \mathbb{Z}, \\ n - t & \text{if } t \in [n - \frac{1}{2}, n] \text{ for some } n \in \mathbb{Z}. \end{cases}$$

Given $n \in \mathbb{N}$, consider the function $\varphi_n(t) = 2^{-n}\varphi(2^n t)$, which is a homothetic copy of $\varphi(t)$.

Some spaces called *shark teeth* are constructed in [6] and are parametrized by an infinite nondecreasing sequence $(n_k)_{k=1}^{\infty}$. Let $I = [0, 1] \times \{0\}$ be the *bone* of the shark teeth, and for every $k \in \mathbb{N}$ let $M_k =$

The authors were supported in part by the National Science Centre of Poland (Grant DEC–2012/07/B/ST1/03320 (the second author) and Grant DEC–2012/07/N/ST1/03551 (the first author)).

Kielce, Kraków, and Gdańsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 55, No. 2, pp. 364–369, March–April, 2014. Original article submitted May 24, 2013.

$\{(t, \frac{1}{k}\varphi_{n_k}(t)) : t \in [0,1]\}$ be the kth *row* of teeth. The space of shark teeth is given by the formula

$$M = I \cup \bigcup_{k=1}^{\infty} M_k.$$

In [5] is shown that the shark teeth constructed in the plane $\mathbb{R}^2$ with the nondecreasing sequence

$$n_k = \lfloor \log_2 \log_2 (k+1) \rfloor, \quad k \in \mathbb{N},$$

where $\lfloor x \rfloor$ is the integer part of x, is not homeomorphic to an IFS-attractor (see Fig. 1). In other words it is not an IFS-attractor in any metric.

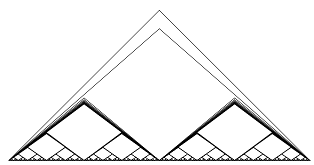

Fig. 1. The space M.

We show that

Theorem. *The space M from* [5] *is a topological IFS-attractor.*

2. Proof of the Theorem

For $k \in \mathbb{N}$ and the sets M_k, I, and M, by the same names we denote the functions

$$M_k : [0,1] \ni t \to \left(t, \frac{1}{k}\varphi_{n_k}(t)\right) \in M_k,$$

$$I : [0,1] \ni t \to (t, 0) \in I,$$

$$M : [0,1] \ni t \to I(t) \cup \bigcup_{k=1}^{\infty} M_k(t).$$

Note that for every $x \in M$ there exists a unique $t_x \in [0,1]$, such that $I(t_x) = x$ or $M_k(t_x) = x$ for some k. Therefore we can represent every point of M as an element from the unit interval and perhaps with positive parameter k. Note that for $k \neq l$ and every $x \in M_k \cap M_l$ we have $M_k(t_x) = M_l(t_x) = I(t_x)$, because then x belongs to I.

At three steps we will present the construction of a topological IFS and prove that M is its attractor.

Step 1. Let $\mathscr{F} = \{f_1, f_2, g_1, \ldots, g_4, h_1, \ldots, h_4\}$ be the collection of continuous functions on M to itself such that for every $x \in M$

$$g_1|_{M \setminus M_1}(x) = M_1(0), \quad g_1|_{M_1}(x) = M_1\left(\frac{\varphi(t_x)}{2}\right),$$

$$g_2|_{M \setminus M_1}(x) = M_1\left(\frac{1}{2}\right), \quad g_2|_{M_1}(x) = M_1\left(\frac{1}{2} - \frac{\varphi(t_x)}{2}\right),$$

$$g_3|_{M \setminus M_1}(x) = M_1\left(\frac{1}{2}\right), \quad g_3|_{M_1}(x) = M_1\left(\frac{1}{2} + \frac{\varphi(t_x)}{2}\right),$$

$$g_4|_{M \setminus M_1}(x) = M_1(1), \quad g_4|_{M_1}(x) = M_1\left(1 - \frac{\varphi(t_x)}{2}\right).$$

Thus the union of images of M under every function g_i fills up the first row of the teeth $M_1 = \bigcup_{i=1}^{4} g_i(M)$. Analogously we construct the functions h_i that fill up the second row M_2. Now we will construct functions f_1 and f_2 which cover the left and right sides of the rest of rows. Define $f_2(x) = f_1(x) + (\frac{1}{2}, 0)$, and so it only shifts values of f_1.

Given $i \in \mathbb{N}$, define $G_i = \bigcup\{M_k : n_k = i\}$ as ith *generation* of the shark teeth. We can also treat it like the function $G_i : [0,1] \ni t \to \bigcup\{M_k(t) : n_k = i\} \in G_i$. Note that every row in one generation contains the same number of teeth (2^i). By $k_i = \min\{k \mid n_k = i\}$ we denote the number of the first row of teeth in G_i, and by $N_i = |\{k \mid n_k = i\}|$, the number of rows in G_i. The function f_1 has to transform every generation into the left part of next generation, and so let $s_i = \frac{N_{i+1}}{N_i}$ be the number of rows from G_{i+1} filled by one row from G_i. In our case $N_i = 2^{2^{i+1}} - 2^{2^i}$ and $s_i = 2^{2^{i+1}} + 2^{2^i}$ for every $i \in \mathbb{N}$. We want the function f_1 to transform the whole row from G_i into s_i rows from $G_{i+1}([0, \frac{1}{2}])$. Therefore, the points $x, y \in M_k \cap I$ for $x \neq y$ and some positive k must have distinct values $f_1(x) \neq f_1(y)$ in the same order on I. To obtain this, every tooth from G_i must be divided into s_i+1 pieces, which each of them covers one tooth from G_{i+1} and the last fills small part of bone I. In other words for $j = 0, \ldots, 2^i - 1$ a tooth from $G_i\big([\frac{j}{2^i}, \frac{j+1}{2^i}]\big)$ is transformed by f_1 into s_i teeth from $G_{i+1}\big([\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}]\big)$ and bone $I\big([\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}]\big)$ (see Fig. 2).

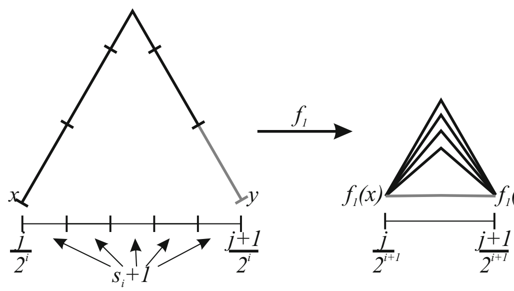

Fig. 2. The tooth from G_i is transformed to s_i teeth from G_{i+1} and a small part of the bone I.

Note that for $i, j \in \mathbb{N}$ and for similarity $p_{i,j}(t) = \frac{t}{2^i} + \frac{j}{2^i}$, we can write $[\frac{j}{2^i}, \frac{j+1}{2^i}] = p_{i,j}([0,1])$. Moreover, define $P_{ijk} = [p_{i,j}(\frac{k}{s_i+1}), p_{i,j}(\frac{k+1}{s_i+1})]$ for $k = 0, \ldots, s_i$. We can present the formula for f_1:

$$f_1|_I(x) = \frac{x}{2}$$

and for $i \in \mathbb{N}$, $l = 0, \ldots, N_i - 1$, and $j = 0, \ldots, 2^i - 1$ we have

$$f_1|_{M_{k_i+l}}(x) = \begin{cases} M_{k_{i+1}+ls_i+k}\big(p_{i+1,j}\big(2\varphi\big(\frac{s_i+1}{2}p_{i,j}^{-1}(t_x)\big)\big)\big), & t_x \in P_{ijk},\ k = 0, \ldots, s_i - 1, \\ I(p_{i+1,j}(2\varphi(\frac{s_i+1}{2}p_{i,j}^{-1}(t_x)))), & t_x \in P_{ijk},\ k = s_i. \end{cases}$$

We can write that $M = \bigcup_{f \in \mathscr{F}} f(M)$. Indeed,

$$\bigcup_{i=1}^{4}(g_i(M) \cup h_i(M)) = M_1 \cup M_2$$

and easy calculations can show that for every $i \in \mathbb{N}$ we have $f_1(G_i) = G_{i+1}([0, \frac{1}{2}]) \cup I([0, \frac{1}{2}])$ and $f_2(G_i) = G_{i+1}([\frac{1}{2}, 1]) \cup I([\frac{1}{2}, 1])$, and so

$$f_1(M) \cup f_2(M) = \bigcup_{i=1}^{\infty} G_i \cup I = \overline{M \setminus (M_1 \cup M_2)}.$$

Step 2. According to the definition of g_i and h_i we have the following property for $i = 0, \ldots, 4$:

$$\operatorname{diam} g_i(A) \leq \frac{\operatorname{diam}(A)}{2}, \quad \operatorname{diam}\ h_i(A) \leq \frac{\operatorname{diam}(A)}{2} \text{ for every connected set } A \subset M,$$

and so for every positive $m \in \mathbb{N}$ and a connected set $A \subset M$ we have

$$\operatorname{diam}\ g_{i_1} \circ \cdots \circ g_{i_m}(A) \leq \frac{1}{2^m} \operatorname{diam}(A) \tag{2.1}$$

where $i_1, \ldots, i_m \in \{1, \ldots, 4\}$ and analogously for functions h_i.

We also know the similar thing about f_i. For every positive $m \in \mathbb{N}$

$$\operatorname{diam}\ f_{i_1} \circ \cdots \circ f_{i_m}(M) \leq \frac{1}{2^m} \operatorname{diam}(M) \tag{2.2}$$

where $i_1, \ldots, i_m \in \{1, 2\}$. This arose due to the fact that for all $i \in \mathbb{N}$ and $j = 0, \ldots, 2^i - 1$

$$f_1\Big(G_i\Big(\Big[\frac{j}{2^i}, \frac{j+1}{2^i}\Big]\Big)\Big) = G_{i+1}\Big(\Big[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}\Big]\Big) \cup I\Big(\Big[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}\Big]\Big).$$

Step 3. Let $\mathscr{U}$ be an open cover of M. At the last step we will find a positive number m, such that the diameter of $\varphi_{i_1} \circ \cdots \circ \varphi_{i_m}(M)$ is less than the Lebesgue number λ of $\mathscr{U}$, where $\varphi_{i_1}, \ldots, \varphi_{i_m} \in \mathscr{F}$. Let us consider all possible compositions of functions from $\mathscr{F}$. We will study the diameter of the image of M under this composition. From step 2 we know that composition of functions only from $\{g_1, \ldots, g_4\}$, from $\{h_1, \ldots, h_4\}$, or from $\{f_1, f_2\}$ makes half the size of the space M (see (2.1) and (2.2)). Note also that for every connected set $A \subset M$ its images $g_i(A)$, $h_i(A)$, and $f_i(A)$ are contained in $\overline{M \setminus M_2}$, $\overline{M \setminus M_1}$ and $\overline{M \setminus (M_1 \cup M_2)}$ respectively, and so

$$\operatorname{diam}(g_i \circ f_j(A)) = 0, \quad \operatorname{diam}(g_i \circ h_j(A)) = 0,$$
$$\operatorname{diam}(h_i \circ f_j(A)) = 0, \quad \operatorname{diam}(h_i \circ g_j(A)) = 0$$

because they are all singletons. This means that if the functions g_i, h_i, and f_i appear in the composition in the above order, the diameter of the image will be 0. It only remains for us to consider the compositions of the form $f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)$ and analogously $f_{i_k} \circ \cdots \circ f_{i_1} \circ h_{j_1} \circ \cdots \circ h_{j_n}(M)$, where $i_1, \ldots, i_k \in \{1, 2\}$ and $j_1, \ldots, j_n \in \{1, \ldots, 4\}$. Let

$$\alpha(k) = \operatorname{Lip} f_1|_{G_k} = \operatorname{Lip} f_2|_{G_k}$$

be the Lipschitz constant of f_1 and f_2 restricted to kth generation. It is finite by the definition of f_1. Note that the set $f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)$ is contained in the generation G_{k-1}, and so we obtain

$$\begin{aligned}
&\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)) \\
&\leq \alpha(k-1) \cdot \operatorname{diam}(f_{i_{k-1}} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)) \\
&\cdots \leq \alpha(k-1) \cdot \ldots \cdot \alpha(0) \cdot \operatorname{diam}(g_{j_1} \circ \cdots \circ g_{j_n}(M)) \leq \prod_{i=0}^{k-1} \alpha(i) \cdot \frac{1}{2^n} \operatorname{diam}(M).
\end{aligned}$$

On the other hand,

$$\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)) \leq \operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1}(M)) \leq \frac{1}{2^k} \operatorname{diam}(M).$$

Fix $n_1 \in \mathbb{N}$ such that $\frac{1}{2^{n_1}} \operatorname{diam}(M) < \lambda$ and fix $n_2 \in \mathbb{N}$ such that

$$\prod_{i=0}^{n_1-1} \alpha(i) \cdot \frac{1}{2^{n_2}} \operatorname{diam}(M) < \lambda.$$

Then we claim the thesis holds for $m = n_1 + n_2$. Indeed, all images of M under compositions only from $\{g_1, \ldots, g_4\}$, from $\{h_1, \ldots, h_4\}$, or from $\{f_1, f_2\}$ have diameters less than λ by the definition of n_1.

Moreover, $\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_{m-k}}(M)) < \lambda$ for $i_1, \ldots, i_k \in \{1, 2\}$ and $j_1, \ldots, j_n \in \{1, \ldots, 4\}$ because

(1) if $k \le n_1$ then

$$\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_{m-k}}(M)) \le \prod_{i=0}^{k-1} \alpha(i) \cdot \frac{1}{2^{m-k}} \operatorname{diam}(M) \le \prod_{i=0}^{k-1} \alpha(i) \cdot \frac{1}{2^{n_2}} \operatorname{diam}(M) < \lambda;$$

(2) if $k > n_1$ then

$$\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_{m-k}}(M)) \le \frac{1}{2^k} \operatorname{diam}(M) \le \frac{1}{2^{n_1}} \operatorname{diam}(M) < \lambda.$$

Analogously we show that $\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ h_{j_1} \circ \cdots \circ h_{j_{m-k}}(M)) < \lambda$. The other compositions transform the whole space M to a point so the diameter of the image of M is $0 < \lambda$. This ends the proof.

3. Generalizations

In fact the above construction can be extended to all shark teeth. If we try to construct a topological IFS for the shark teeth with an arbitrary sequence $(n_k)_{k=1}^{\infty}$, we can meet the following problems:

(1) Some G_i are empty. Then we have to renumber G_i such that the empty sets are omitted.

(2) $s_i \notin \mathbb{Z}$. Then define $s_i = \lceil \frac{N_{i+1}}{N_i} \rceil$, where $\lceil x \rceil$ is a minimal integer greater than or equal to x. Consequently, the formula for the function f_1 changes slightly. The last row of teeth from every ith generation has to be transformed into less than s_i rows from G_{i+1}. It can be done by covering some rows form G_{i+1} once again.

(3) s_i is odd. Then we do not have to cover a small part of the bone under every tooth, and so we divide every tooth from G_i into s_i pieces, like in Fig. 3.

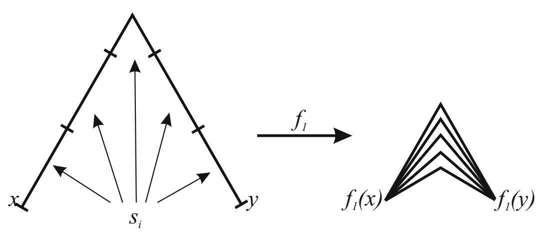

Fig. 3. When s_i is odd then the tooth from G_i is transformed only to s_i teeth from G_{i+1}.

Consequently, every shark teeth is a topological IFS-attractor.

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