PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 3, March 2013, Pages 931–935
S 0002-9939(2012)11737-X
Article electronically published on July 16, 2012

A 1-DIMENSIONAL PEANO CONTINUUM WHICH IS NOT AN IFS ATTRACTOR

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(Communicated by Alexander N. Dranishnikov)

Abstract. Answering an old question of M. Hata, we construct an example of a 1-dimensional Peano continuum which is not homeomorphic to an attractor of IFS.

A compact metric space X is called an *IFS-attractor* if $X = \bigcup_{i=1}^n f_i(X)$ for some contracting self-maps $f_1, \dots, f_n : X \to X$. In this case the family $\{f_1, \dots, f_n\}$ is called an *iterated function system* (briefly, an IFS); see [3]. We recall that a map $f : X \to X$ is *contracting* if its Lipschitz constant

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)}$$

is less than 1.

Topological properties of IFS-attractors were studied in [6], [7], [9], [1], [11]. In particular, it was observed that each connected IFS-attractor X is locally connected. The reason is that X has property S. We recall [10, 8.2] that a metric space X has *property S* if for every $\varepsilon > 0$ the space X can be covered by a finite number of connected subsets of diameter $< \varepsilon$. It is well known [10, 8.4] that a connected compact metric space X is locally connected if and only if it has property S if and only if X is a *Peano continuum* (which means that X is the continuous image of the interval $[0, 1]$). Therefore, a compact space X is not homeomorphic to an IFS-attractor whenever X is connected but not locally connected. Now it is natural to ask if there is a Peano continuum homeomorphic to no IFS-attractor. An easy answer is “Yes” as every IFS-attractor has finite topological dimension; see [5]. Consequently, no infinite-dimensional compact topological space is homeomorphic to an IFS-attractor. In such a way we arrive at the following question posed by M. Hata in Remarks to Theorem 4.6 [6].

Problem 1. Is each finite-dimensional Peano continuum homeomorphic to an IFS-attractor?

In this paper we shall give a negative answer to this question. Our counterexample is a rim-finite plane Peano continuum. A topological space X is called *rim-finite*

Received by the editors July 19, 2011.

2010 *Mathematics Subject Classification.* Primary 28A80, 54D05, 54F50, 54F45.

Key words and phrases. Fractal, Peano continuum, Iterated Function System, IFS-attractor.

The first author has been partially financed by NCN means, granted by decision DEC-2011/01/B/ST1/01439.

The second author was supported by the ESF Human Capital Operational Programme grant 6/1/8.2.1./POKL/2009.

if it has a base of the topology consisting of open sets with finite boundaries. It follows that each compact rim-finite space X has dimension $\dim(X) \le 1$.

Theorem 1. *There is a rim-finite plane Peano continuum homeomorphic to no IFS-attractor.*

It should be mentioned that examples of Peano continua $K \subset \mathbb{R}^n$, which are not *isometric* to IFS-attractors, were constructed by Kwieciński [8] and Sanders [13]. However these continua are *homeomorphic* to IFS-attractors, so they do not answer Problem 1.

Theorem 1 contrasts with a result of Duvall and Husch [4] saying that a finite-dimensional compact metrizable space X containing an open zero-dimensional subspace without isolated points is homeomorphic to an IFS-attractor.

1. S-dimension of IFS-attractors

In order to prove Theorem 1 we shall observe that each connected IFS-attractor has finite S-dimension. This dimension was introduced and studied in [2].

The *metric S-dimension* $\text{S-Dim}(X,d)$ is defined for each metric space (X,d) with property S. For each $\varepsilon > 0$ denote by $S_\varepsilon(X)$ the smallest number of connected subsets of diameter $< \varepsilon$ that cover the space X and let

$$\text{S-Dim}(X,d) = \overline{\lim_{\varepsilon\to+0}} -\frac{\ln S_\varepsilon(X)}{\ln \varepsilon}.$$

The metric S-dimension is greater than or equal to the standard box-counting dimension

$$\text{Dim}(X,d) = \overline{\lim_{\varepsilon\to+0}} -\frac{\ln N_\varepsilon(X)}{\ln \varepsilon},$$

where $N_\varepsilon(X)$ stands for the smallest number of subsets of diameter $< \varepsilon$ that cover X. By a classical result of Pontrjagin and Schnirelmann [12], for each compact metrizable space X the infimum

$$\dim(X) = \inf\{\text{Dim}(X,d) : d \text{ is a continuous metric on } X\}$$

coincides with the covering topological dimension of X.

In contrast, for a Peano continuum X its *S-dimension*

$$\text{S-dim}(X) = \inf\{\text{S-Dim}(X,d) : d \text{ is a continuous metric on } X\}$$

can be strictly larger than the topological dimension $\dim(X)$ of X; see [2, 7.1].

Theorem 2. *Assume that a connected compact metric space (X,d) is an attractor of an IFS $f_1, f_2, \dots, f_n : X \to X$ with contracting constant $\lambda = \max_{i\le n} \text{Lip}(f_i) < 1$. Then X has finite S-dimensions*

$$\text{S-dim}(X) \le \text{S-Dim}(X,d) \le -\frac{\ln(n)}{\ln(\lambda)}.$$

Proof. The inequality $\text{S-dim}(X) \le \text{S-Dim}(X,d)$ follows from the definition of the S-dimension $\text{S-dim}(X)$. The inequality $\text{S-Dim}(X,d) \le -\frac{\ln(n)}{\ln(\lambda)}$ will follow as soon as for every $\delta > 0$ we find $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ we get

$$-\frac{\ln S_\varepsilon(X)}{\ln \varepsilon} < -\frac{\ln(n)}{\ln(\lambda)} + \delta.$$

Let $D = \mathrm{diam}(X)$ be the diameter of the metric space X. Since

$$\lim_{k\to\infty} \frac{\ln(n^k)}{\ln(\lambda^{k-1}D)} = \lim_{k\to\infty} \frac{k\ln(n)}{(k-1)\ln(\lambda)+\ln D} = \frac{\ln(n)}{\ln(\lambda)},$$

there is $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ we get

$$-\frac{\ln(n^k)}{\ln(\lambda^{k-1}D)} < -\frac{\ln(n)}{\ln(\lambda)} + \delta.$$

We claim that the number $\varepsilon_0 = \lambda^{k_0-1}D$ has the required property. Indeed, given any $\varepsilon \in (0, \varepsilon_0]$ we can find $k \geq k_0$ with $\lambda^k D < \varepsilon \leq \lambda^{k-1}D$ and observe that

$$\mathcal{C}_k = \big\{ f_{i_1} \circ \cdots \circ f_{i_k}(X) : i_1, \dots, i_k \in \{1, \dots, n\} \big\}$$

is a cover of X by compact connected subsets having diameter $\leq \lambda^k D < \varepsilon$. Then $S_\varepsilon(X) \leq |\mathcal{C}_k| \leq n^k$ and

$$-\frac{\ln(S_\varepsilon(X))}{\ln(\varepsilon)} \leq -\frac{\ln(n^k)}{\ln(\lambda^{k-1}D)} < -\frac{\ln(n)}{\ln(\lambda)} + \delta.$$

□

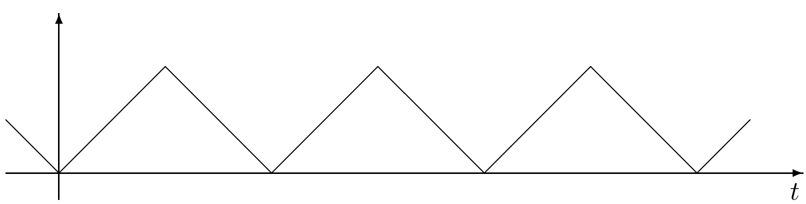
In the next section we shall construct an example of a rim-finite plane Peano continuum M with infinite S-dimension S-dim(M). Theorem 2 implies that the space M is not homeomorphic to an IFS-attractor and this proves Theorem 1.

2. The space M

Our space M is a partial case of the spaces constructed in [2] and called “shark teeth”. Consider the piecewise linear periodic function

$$\varphi(t) = \begin{cases} t - n & \text{if } t \in [n, n+\frac{1}{2}] \text{ for some } n \in \mathbb{Z}, \\ n - t & \text{if } t \in [n-\frac{1}{2}, n] \text{ for some } n \in \mathbb{Z}, \end{cases}$$

whose graph looks as follows:



For every $n \in \mathbb{N}$ consider the function

$$\varphi_n(t) = 2^{-n}\varphi(2^n t),$$

which is a homothetic copy of the function $\varphi(t)$.

Consider the nondecreasing sequence

$$n_k = \lfloor \log_2 \log_2(k+1) \rfloor, \quad k \in \mathbb{N},$$

where $\lfloor x \rfloor$ is the integer part of x. Our example is the continuum

$$M = [0,1] \times \{0\} \cup \bigcup_{k=1}^{\infty} \big\{ \big(t, \tfrac{1}{k}\varphi_{n_k}(t)\big) : t \in [0,1] \big\}$$

in the plane $\mathbb{R}^2$, shown in Figure 1.

The following theorem yields Theorem 1 as a corollary.

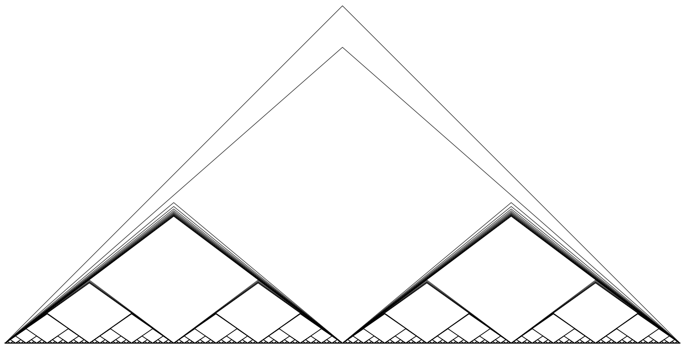


FIGURE 1. The space M

Theorem 3. *The space M has the following properties:*

(1) *M is a rim-finite plane Peano continuum;*
(2) $\dim(M) = 1$ *and* $\text{S-dim}(X) = \infty$*;*
(3) *M is not homeomorphic to an IFS attractor.*

Proof. It is easy to see that X is a rim-finite plane Peano continuum. The rim-finiteness of M implies that $\dim(M) = 1$.

To show that $\text{S-dim}(M) = \infty$, fix any continuous metric d on M. Let $R = d\big((0,0),(1,0)\big)$ be the d-distance between the end-points of the "bone" $I = [0,1] \times \{0\} \subset M$ of the "shark teeth" M.

Given $\varepsilon > 0$, consider any cover $\mathcal{C}$ of M by connected subsets of d-diameter $< \varepsilon$ with $|\mathcal{C}| = S_\varepsilon(M)$. For every $k \geq 1$ let $M_k = \{\big(t, \frac{1}{k}\varphi_{n_k}(t)\big) : t \in [0,1]\}$ be the kth generation of "teeth" and $\mathcal{C}_k = \{C \in \mathcal{C} : C \cap M_k \neq \emptyset \text{ and } C \cap I = \emptyset\}$. It is easy to see that each $C \in \mathcal{C}_k$ lies in $M_k \setminus I$ and hence the families $\mathcal{C}_k$, $k \geq 1$, are disjoint.

We claim that $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(2^{n_k} + 1)$ for every $k \geq 1$. Indeed, note that each element $C \in \mathcal{C}$ meeting the set $M_k \cap I$ at some point $x \in M_k \cap I$ lies in the ε-ball $B_\varepsilon(x) = \{y \in M : d(x,y) < \varepsilon\}$. Then the family $\mathcal{C}_k \cup \{B_\varepsilon(x) : x \in M_k \cap I\}$ covers the kth generation of "teeth" M_k and

$$R \leq \operatorname{diam} M_k \leq \sum_{C \in \mathcal{C}_k} \operatorname{diam} C + \sum_{x \in M_k \cap I} \operatorname{diam} B_\varepsilon(x) \leq \varepsilon |\mathcal{C}_k| + 2\varepsilon(2^{n_k} + 1).$$

Consequently, $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(2^{n_k} + 1)$.

Taking into account that for any $\alpha > 0$ there exists $\sup_{k \geq 1} \frac{2^{n_k}}{k^\alpha} = A < \infty$, we note that $2^{n_k} \leq Ak^\alpha$ for each $k \geq 1$. This implies the lower bound $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(Ak^\alpha + 1)$. Let $k_0 = (\frac{R - 4\varepsilon}{4A\varepsilon})^{\frac{1}{\alpha}}$ and note that for any $k \leq k_0$, we get $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(Ak_0^\alpha + 1) = \frac{R}{2\varepsilon}$. Then

$$S_\varepsilon(M) = |\mathcal{C}| \geq \sum_{k \leq k_0} |\mathcal{C}_k| \geq \frac{R}{2\varepsilon}\lfloor k_0 \rfloor \geq \frac{R}{2\varepsilon}(k_0 - 1) = \frac{R}{2\varepsilon}\left(\left(\frac{R}{4A\varepsilon} - \frac{1}{A}\right)^{\frac{1}{\alpha}} - 1\right)$$

and there exist $D > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we get $S_\varepsilon(M) \geq D\varepsilon^{-(1+\frac{1}{\alpha})}$. This implies that $\text{S-Dim}(M,d) \geq 1 + \frac{1}{\alpha}$ for any $\alpha > 0$. Consequently, $\text{S-Dim}(M,d) = \infty$ for any continuous metric d on M and $\text{S-dim}(M) = \infty$. $\square$

3. Some open questions

We shall say that a compact topological space X is a *topological IFS-attractor* if $X = \bigcup_{i=1}^n f_i(X)$ for some continuous maps $f_1, \dots, f_n : X \to X$ such that for any open cover $\mathcal{U}$ of X there is $m \in \mathbb{N}$ such that for any functions $g_1, \dots, g_m \in \{f_1, \dots, f_n\}$ the set $g_1 \circ \cdots \circ g_m(X)$ lies in some set $U \in \mathcal{U}$. It is easy to see that each IFS-attractor is a topological IFS-attractor and each connected topological IFS-attractor is metrizable and locally connected.

Problem 2. Is each (finite-dimensional) Peano continuum a topological IFS-attractor? In particular, is the space M constructed in Theorem 3 a topological IFS-attractor?

Acknowledgement

The authors would like to express their sincere thanks to Wiesław Kubiś for posing problems and useful comments.

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