I. Introduction, two-dimensional Hilbert space

- We consider a two-dimensional Hilbert-space with the states

$$\boxed{|1\rangle \quad \text{and} \quad |2\rangle .}$$

- The system is described by the Hamiltonian $H = H_0$ of the following form

$$\boxed{H_0 = m_1 |1\rangle\langle 1| + m_2 |2\rangle\langle 2| .}$$

It is obvious, that H_0 is diagonal in this basis $\{|1\rangle, |2\rangle\}$, so that $|1\rangle$ and $|2\rangle$ are eigenstates of H_0 with the energies/masses m_1 and m_2.

- The time-development of the states is of course given by the action of the unitary time-evolution operator, so that

$$\boxed{\begin{aligned} |1(t)\rangle &= e^{-iH_0 t}|1\rangle = e^{-im_1 t}|1\rangle, \\ |2(t)\rangle &= e^{-iH_0 t}|2\rangle = e^{-im_2 t}|2\rangle . \end{aligned}}$$

- Now one can define the amplitude for a given $|i\rangle$, prepared at $t=0$, to be still there at a time $t>0$. This amplitude is called "survival probability amplitude" $a(t)$:

$$\boxed{a_i(t) = \langle i | e^{-iH_0 t} | i \rangle}$$

- Then the probability to find the state $|i\rangle$ at time $t>0$ is given by

$$\boxed{P_i(t) = |a_i(t)|^2}$$

- For the system described by H_0 this probability is one, because

$$\boxed{\begin{aligned} a_1(t) &= e^{-im_1 t} \quad \Rightarrow \quad P_1(t) = |e^{-im_1 t}|^2 = 1 \\ a_2(t) &= e^{-im_2 t} \quad \Rightarrow \quad P_2(t) = |e^{-im_2 t}|^2 = 1 \end{aligned}}$$

$\Rightarrow$ of course, this result is expected, because there is no interaction in our system.

II. Two-dimensional Hilbert-space vol. II

- Now, the Hamiltonian expanded by an interaction term, so that

$$\boxed{\begin{aligned} H &= H_0 + H_I \\ &= \underbrace{m_1 |1\rangle\langle 1| + m_2 |2\rangle\langle 2|}_{\equiv H_0} + \underbrace{\lambda \{ |1\rangle\langle 2| + |2\rangle\langle 1| \}}_{\equiv H_I} . \end{aligned}}$$

=> Thus, $\{|1\rangle, |2\rangle\}$ are not eigenstates of the system anymore.

=> Again, we're interested in the survival probability $p(t)$ and therefore also in the survival probability amplitude $a(t)$.

=> For the calculation of $a(t)$ and $p(t)$, one needs to find the eigenvalues / eigenstates of $H = H_0 + H_I$

=> To find these states one has to rotate the system into a new basis, in which H is diagonal.

- To find the new basis, one can define the vector

$$\boxed{\underline{\xi} = \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}}$$

and its hermitian conjugate

$$\boxed{\underline{\xi}^+ = (|1\rangle, |2\rangle) .}$$

- Now the Hamiltonian H can be rewritten as a bilinear form

$$\boxed{\begin{aligned} H &= \underline{\xi}^+ \,\underline{\Omega}\, \underline{\xi} \\ &= (|1\rangle, |2\rangle) \begin{pmatrix} m_1 & \lambda \\ \lambda & m_2 \end{pmatrix} \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix} \\ &= (|1\rangle, |2\rangle) \begin{pmatrix} m_1 \langle 1| + \lambda \langle 2| \\ \lambda \langle 1| + m_2 \langle 2| \end{pmatrix} \\ &= m_1 |1\rangle\langle 1| + \lambda |1\rangle\langle 2| + \lambda |2\rangle\langle 1| + m_2 |2\rangle\langle 2| \\ &= m_1 |1\rangle\langle 1| + m_2 |2\rangle\langle 2| + \lambda \{ |1\rangle\langle 2| + |2\rangle\langle 1| \} , \end{aligned}}$$

where we defined the symmetric matrix

$$\boxed{\Omega \equiv \begin{pmatrix} m_1 & \lambda \\ \lambda & m_2 \end{pmatrix}.}$$

- Now for a symmetric matrix, one can always find an orthogonal transformation O, so that $O^T \Omega O$ is diagonal.
- In this case, the transformation O is an element of the group $SO(2)$:

$$O \equiv \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

- Then we have

$$\boxed{H = \underline{\xi}^{+} O \underbrace{O^T \Omega O}_{\equiv D} \underbrace{O^T \underline{\xi}}_{\underline{\xi}'}} ,$$

where the eigenvalues M_1, M_2 are given by

$$0 \overset{!}{=} \underset{\det(\Omega - \mathbb{1}M)}{\det} \begin{pmatrix} m_1 - M & \lambda \\ \lambda & m_2 - M \end{pmatrix} = (m_1 - M)(m_2 - M) - \lambda^2$$

$$= M^2 - M(m_1 + m_2) + m_1 m_2 - \lambda^2$$

$$\Rightarrow \quad M_{1/2} = \frac{m_1 + m_2}{2} \pm \sqrt{\frac{(m_1 + m_2)^2}{4} - \frac{4(m_1 m_2 - \lambda^2)}{4}}$$

$$= \frac{m_1 + m_2}{2} \pm \frac{1}{2}\sqrt{(m_1 - m_2)^2 + 4\lambda^2}$$

$\Rightarrow$ For $\lambda = 0$: $M_1 = m_1$, $M_2 = m_2$!

- The eigenstates are given by

$$\boxed{\underline{\xi}' = O^T \underline{\xi} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix} \iff \begin{cases} \langle 1'| = \langle 1| \cos\phi - \langle 2| \sin\phi \\ \langle 2'| = \langle 1| \sin\phi + \langle 2| \cos\phi \end{cases}}$$

$O\xi'$

- Now we can calculate inverse relation

$$\boxed{\begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix} = O\,\xi' = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \langle 1'| \\ \langle 2'| \end{pmatrix} \iff \begin{cases} \langle 1| = \langle 1'|\cos\phi + \langle 2'|\sin\phi \\ \langle 2| = -\langle 1'|\sin\phi + \langle 2'|\cos\phi \end{cases}}$$

- Now we have the right incredients to calculate the survival probability $p_1(t)$:

$$a_1(t) = \langle 1|e^{-iHt}|1\rangle = \{\langle 1'|\cos\phi + \langle 2'|\sin\phi\}e^{-iHt}\{\cos\phi|1'\rangle + \sin\phi|2'\rangle\}$$
$$= \cos^2\phi\, e^{-iM_1t} + \sin^2\phi\, e^{-iM_2t}$$

$$p_1(t) = |a_1(t)|^2 = \{\cos^2\phi\, e^{-iM_1t} + \sin^2\phi\, e^{-iM_2t}\}\{\cos^2\phi\, e^{iM_1t} + \sin^2\phi\, e^{iM_2t}\}$$
$$= \cos^4\phi + \sin^4\phi + \sin^2\phi\cos^2\phi\{e^{-i(M_1-M_2)t} + e^{i(M_1-M_2)t}\}$$
$$= \cos^4\phi + \sin^4\phi + \sin^2\phi\cos^2\phi \cdot 2\cos((M_1-M_2)t)$$

- For $\phi = \frac{\pi}{4}$ we obtain

$$p_1(t) = \underbrace{\cos^4(\tfrac{\pi}{4})}_{\frac{1}{4}} + \underbrace{\sin^4(\tfrac{\pi}{4})}_{\frac{1}{4}} + \underbrace{\sin^2(\tfrac{\pi}{4})}_{\frac{1}{2}}\underbrace{\cos^2(\tfrac{\pi}{4})}_{\frac{1}{2}} \cdot 2\cos((M_1-M_2)t)$$
$$= \frac{1}{4} + \frac{1}{4} + 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\cos((M_1-M_2)t)$$
$$= \frac{1}{2}\left(1 + \cos((M_1-M_2)t)\right)$$
$$= \cos^2\left(\frac{M_1-M_2}{2}t\right).$$

Then $p_2(t)$ is given by

$$\boxed{p_2(t) = 1 - p_1(t) = \sin^2\left(\frac{M_1-M_2}{2}t\right).}$$

III Short time behaviour

- For small times $t \ll 1$ the survival probability $P_1(t)$ can be expanded in a Taylor polynomial

$$\boxed{\begin{aligned} P_1(t) &\simeq \cos^4\phi + \sin^4\phi + \sin^2\phi\cos^2\phi \cdot 2\left\{1 - \frac{(M_1-M_2)^2}{2}t^2 + \mathcal{O}(t^4)\right\} \\ &= (\cos^2\phi + \sin^2\phi)^2 - \sin^2\phi\cos^2\phi\,(M_1-M_2)^2 t^2 \\ &= 1 - \frac{t^2}{\tau_Z^2}, \end{aligned}}$$

in which

$$\boxed{\tau_Z = \left(\sin^2\phi\cos^2\phi\,(M_1-M_2)^2\right)^{-1}}$$

is the so-called Zeno-time.

$\Rightarrow$ The Zeno-time τ_Z gives the regime, in which the decay is not exponentially.

- In case of $\phi = \frac{\pi}{4}$, we obtain

$$\boxed{P_1(t) \simeq 1 - \frac{(M_1-M_2)^2}{4}t^2, \quad P_2(t) \simeq \frac{(M_1-M_2)^2}{4}t^2.}$$

IV Zeno-effect

- Suppose a measurement, in which the apparatus destroys the state $|2\rangle$ and leaves the state $|1\rangle$ unaffected.

- If we now make a measurement at $t=\tau \ll \tau_Z$ the survival probability is given by $P_1(\tau) = 1 - \frac{(M_1-M_2)^2}{4}\tau^2$

- If we now make a second measurement at $t=2\tau \ll \tau_Z$ the survival probability is given by $P_1(\tau)^2$.

- If we now make N measurements at $t = N\tau$ we have

$$P_1^N(\tau) = \left(1 - \frac{(M_1-M_2)^2}{4}\tau^2\right)^N$$

$$= \left(1 - \frac{(M_1-M_2)^2}{4}\frac{t^2}{N^2}\right)^N$$

$$\overset{\text{large } N}{=} e^{-\frac{(M_1-M_2)^2}{4N}t^2}$$

$$\overset{N\to\infty}{\longrightarrow} 1$$

$\Rightarrow$ We obtain, that the system stays in its original state $|1\rangle$ as a consequence of many measurements!

$\Rightarrow$ This is the so-called Zeno-effect.

V Exponential and non-exponential decay law

- The exponential decay law
$$\boxed{N(t) = N(t_0)\, e^{-\Gamma(t-t_0)}}$$
is well known.

- Now setting $t_0 = 0$ (as before) and dividing by $N(t_0)$, we obtain the survival probability
$$\boxed{p(t) = e^{-\Gamma t}.}$$

$\Longrightarrow$ One has to be careful with this consideration, because in quantum mechanics the probabilities are derived from fundamental probability amplitudes!

- Therefore we start with the definition of $p(t)$ as the square of the modulus of the survival probability amplitude:
$$\boxed{p(t) = |a(t)|^2 \quad , \quad a(t) = \langle i | e^{-iHt} | i \rangle .}$$
Note, that we prepared the initial state $|i\rangle$ at $t_0 = 0$!

- In the first part (Zeno-effect), we saw that the unstable states aren't eigenstates of the systems Hamiltonian, so that we can insert a complete set of energy/mass eigenstates

$$\boxed{\begin{aligned} a(t) &= \sum_j \langle i | e^{-iHt} | M_j \rangle \langle M_j | i \rangle \\ &\overset{\text{continuous case}}{=} \int dM\, \underbrace{|\langle i | M \rangle|^2}\, e^{-iMt} \\ &= \int dM\, \rho(M)\, e^{-iMt} . \end{aligned}}$$

① $\rho(M) \geq 0$ in $\mathbb{R}$, $\langle M | i \rangle$ is the mass/energy-representation of the wavefunction, therefore $\rho(M) \equiv |\langle M | i \rangle|^2$ is the probability density to find/measure the energy/mass M of the state!!!

② We found a Fourier-transform of $a(t)$.

- Now to approximate the probability density $\rho(M)$, we take the Breit-Wigner-limit, which is just a Cauchy distribution:

$$\boxed{\rho(M) \approx d(M) = \frac{N}{(M-m)^2 + (\Gamma/2)^2} ,}$$

where N is chosen in way that $\int_{-\infty}^{\infty} dM\, d(M) = 1$ and where Γ is the full width at half maximum and m is the maximum of the distribution.

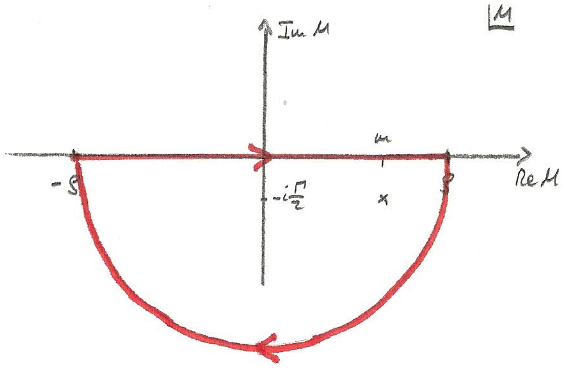
- Now we can calculate the survival probability for the unstable system in the Breit-Wigner-limit:

$$\boxed{P_{BW}(t) = |a_{BW}(t)|^2 , \qquad a_{BW}(t) = \int_{-\infty}^{\infty} dM \frac{N}{(M-m)^2 + (\Gamma/2)^2} e^{-iMt}}$$

- For the evaluation of survival probability amplitude, one has to use some complex analysis, namely the residue theorem:

$\Rightarrow$ The denominator has a first order pole at : $M = m - i\frac{\Gamma}{2}$

$\Rightarrow$ The parameter t is larger than zero, therefore we close the contour in the upper half-plane :



- Then, using the residue theorem, one obtains

$$\boxed{\begin{aligned}
a_{BW}(t) &= \lim_{\rho\to\infty} \int_{-\rho}^{\rho} dM \frac{N}{(M-m)^2 + (\Gamma/2)^2} e^{-iMt} \\
&= -2\pi i \; Res\left(I, m - i\tfrac{\Gamma}{2}\right) \\
&= -2\pi i \left\{ \left(M - m + i\tfrac{\Gamma}{2}\right) \cdot \frac{N}{(M-m)^2 + (\Gamma/2)^2} e^{-iMt} \right\}_{M = m - i\frac{\Gamma}{2}} \\
&= -2\pi i \left\{ \left(M - m + i\tfrac{\Gamma}{2}\right) \frac{N}{(M-m+i\frac{\Gamma}{2})(M-m-i\frac{\Gamma}{2})} e^{-iMt} \right\}_{M = m - i\frac{\Gamma}{2}} \\
&= -2\pi i \frac{N}{M - m - i\frac{\Gamma}{2}} e^{-iMt} \Big|_{M = m - i\frac{\Gamma}{2}} = -2\pi i \cdot \frac{N}{-i\Gamma} e^{-i(m - i\frac{\Gamma}{2})t} \\
&= \frac{2\pi N}{\Gamma} e^{-imt} e^{-\frac{\Gamma}{2}t} ,
\end{aligned}}$$

where we used, that the residue of a function for a p-order pole is given by

$$\boxed{\mathrm{Res}[f; z_0] = \frac{1}{(p-1)!} \frac{d^{p-1}}{dz^{p-1}} \left\{ (z-z_0)^p f(z) \right\}_{z=z_0}.}$$

- Now from the normalization integral one obtains the normalization constant N:

$$\begin{aligned}
1 &= N \int_{-\infty}^{\infty} dM \frac{1}{(M-m)^2 + (\Gamma/2)^2} \\
&= N \int_{-\infty}^{\infty} dM \frac{4}{\Gamma^2} \frac{1}{\left(\frac{M-m}{\Gamma/2}\right)^2 + 1} \\
&= N \frac{4}{\Gamma^2} \frac{\Gamma}{2} \int_{-\infty}^{\infty} d\xi \frac{1}{1+\xi^2} \\
&= N \frac{2}{\Gamma} \left\{ \underbrace{\arctan(\infty)}_{} - \underbrace{\arctan(-\infty)}_{} \right\} \\
&= N \frac{2}{\Gamma} \left\{ \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right\} \\
&= N \frac{2\pi}{\Gamma} \qquad \Rightarrow \quad N = \frac{\Gamma}{2\pi}
\end{aligned}$$

- Combining all results, we obtain the following expression for the survival probability amplitude in the Breit-Wigner-limit:

$$\boxed{a_{BW}(t) = e^{-imt} e^{-\frac{\Gamma}{2}t}.}$$

- Therefore the survival probability $p(t)$ is identical to the estimation (from the exponential decay law):

$$\boxed{p(t) = |a(t)|^2 = e^{-\Gamma t}}$$

$\Rightarrow$ This result can also be obtained by perturbative calculations!

- Now we come to the deviations from the exponential decay law.

  To this end, we'll show, that the survival probability cannot be an exponential for short times and we'll also show, that $p(t)$ will decrease slower than an exponential function for very long times.

- First of all, we show that for $t=0$, the survival probability cannot be an exponential:

  ① The survival probability must hold the condition

$$\boxed{a^*(t) = a(-t)}$$

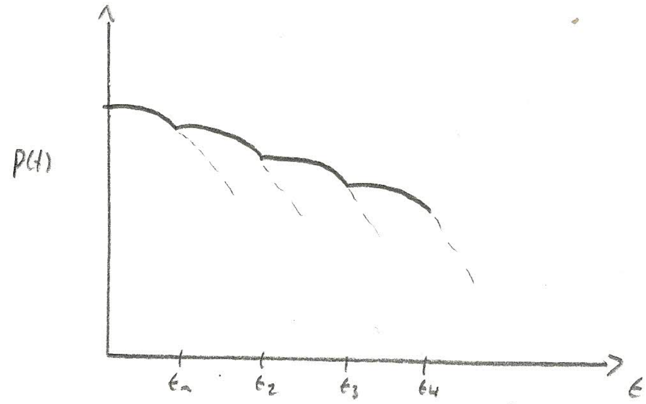
for $\varrho(M)$ to be real.

②
$$\left.\frac{dp(t)}{dt}\right|_{t=0} = \left\{\frac{da^*(t)}{dt}\,a(t) + a^*(t)\,\frac{da(t)}{dt}\right\}\Big|_{t=0}$$
$$= \left\{\frac{da(-t)}{dt}\,a(t) + a(-t)\,\frac{da(t)}{dt}\right\}\Big|_{t=0}$$
$$= \left\{\int dM\,(+iM)\,\varrho(M)\,e^{+iMt}\cdot a(t) + a(-t)\int dM\,(-iM)\,\varrho(M)\,e^{-iMt}\right\}_{t=0}$$
$$= \int dM\, iM\,\varrho(M) - \int dM\, iM\,\varrho(M)$$
$$= 0$$

$\Rightarrow$ This shows, that for $t=0$ $p(t)$ cannot be an exponential!

- But $t=0$ was the time at which our state was prepared. This involves, that the last observation was at $t=0$!

- The condition $\frac{dp(t)}{dt}=0$ must also hold for further observations at $t=t_i$.



$\Rightarrow$ observations, which don't destroy the state, lead to the Zeno-effect (in this case)

- Finally, we show, that for large times $p(t)$ will decrease more slowly than an exponential:

- First of all one needs a mathematical theorem from complex analysis:

  "If $a(t)$ and $\rho(M)$ are Fourier transforms of each other, so that $\rho(M) = 0$ for $M < M_0$ and $\int_{-\infty}^{\infty} dt\, |a(t)|^2$ is finite, then $\int_{-\infty}^{\infty} dt \frac{|\ln|a(t)||}{1+t^2}$ is also finite."

- In our case:

  ① $a(t)$, $\rho(M)$ are Fourier transforms of each other

  ② $\rho(M) = 0$ has to be 0 for $M < M_0$, because for each decay, there has to be a threshold. This corresponds to a ground state energy M_0!

  ③ $\int_{-\infty}^{\infty} dt\, |a(t)|^2$ is finite.

- Now if $|a(t)|$ would be an exponential, the expression $|\ln|a(t)||$ would be linear in t. But, if $|\ln|a(t)||$ is linear in t, the expression

$$\int_{-\infty}^{\infty} dt \frac{|\ln|a(t)||}{1+t^2}$$

is not finite!

$\Rightarrow$ therefore $|\ln|a(t)||$ has to increase slower than linear, which excludes, that $|a(t)|$ is an exponential!