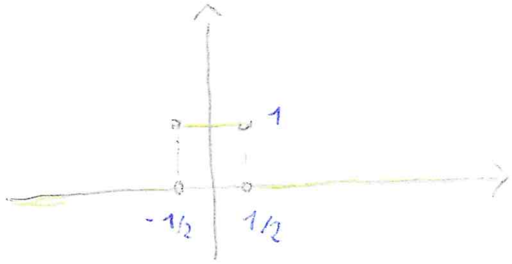
DIRAC δ → A SIMPLE WAY TO PRESENT IT

$$\delta_\varepsilon(x) = \begin{cases} 0 & |x| > \varepsilon \\ \dfrac{1}{2\varepsilon} & |x| \le \varepsilon \end{cases}$$

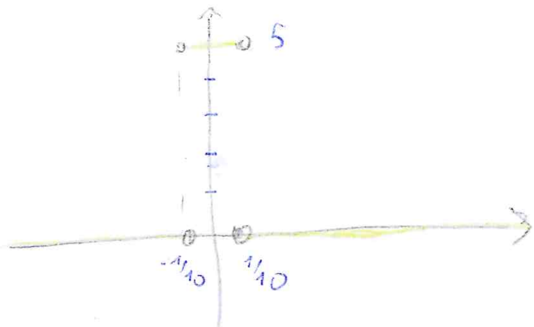
$\forall \varepsilon$ one has a different function.

$\varepsilon = \frac{1}{2}$

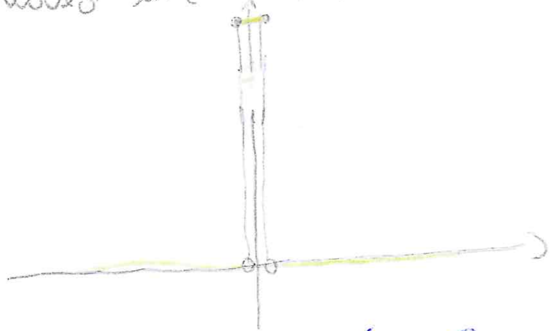


We are interested to small ε:

$\varepsilon = \frac{1}{10}$



In particular, we would like to take the limiting process $\varepsilon \mapsto 0$



$$\delta(x) \equiv \lim_{\varepsilon \mapsto 0^+} \delta_\varepsilon(x) = \begin{cases} 0 \text{ for } x \neq 0 \\ \infty \text{ for } x = 0 \end{cases}$$

ACHTUNG:

The def. $\delta(x) = \lim_{\varepsilon \to 0^+} \delta_\varepsilon(x)$ is "pathological"...

what we get is not a function, but a rather strange construct

which is zero everywhere and is ∞ for $x = 0$...

Indeed, $\delta(x)$ is a "distribution" (generalisation of the concept of function).

The definition of the properties of distributions is not possible with the developed background.

However, even with an "intuitive approach" one can derive and describe almost all relevant features.

From a "physical" point of view:

The Dirac-δ appears everywhere.. mass and charge distribution of point-like particles.

- Fourier-Transform of "a pure sound": Dirac-delta in the frequency.
- Solution of so-called Green functions in E-Dyn and Field Theories.
- QM (in vector spaces) (as an important 1-dim. potential)

the analogon of the "1"...

Property: $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \delta_\varepsilon(x)\, dx = 1$$

In fact: the function $\delta_\varepsilon(x)$ is constructed in such a way that the area reads $A = 2\varepsilon \cdot \frac{1}{2\varepsilon} = 1.$

Formally:

$$\int_{-\infty}^{\infty} \delta_\varepsilon(x) dx = \int_{-\infty}^{-\varepsilon} \delta_\varepsilon(x)\, dx + \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(x) dx + \int_{\varepsilon}^{\infty} \delta_\varepsilon(x)\, dx =$$

$$= 0 + \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\, dx + 0 =$$

$$= \frac{1}{2\varepsilon} \cdot \left(x \right)_{-\varepsilon}^{+\varepsilon} = \frac{1}{2\varepsilon} \left(\varepsilon - (-\varepsilon) \right) = \frac{2\varepsilon}{2\varepsilon} = 1 \; !$$

Note, the result is indipendent on ε. Ergo also for $\varepsilon \to 0$:

$$\boxed{\int_{-\infty}^{\infty} \delta(x)\, dx = 1}$$

The 'function' $\delta(x)$ is such that is zero everywhere except for $x=0$, and for $x=0$ is $\delta(0) = \infty$ in such a way that the area is 1 :)

$$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x) = f(0)}$$

Let us now consider

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \qquad \text{where } f(x): R \mapsto R.$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} f(x) \lim_{\varepsilon \mapsto 0} \delta_\varepsilon(x)\,dx = \lim_{\varepsilon \mapsto 0} \int_{-\infty}^{\infty} f(x)\,\delta_\varepsilon(x)\,dx =$$

$$= \lim_{\varepsilon \mapsto 0} \left[\int_{-\varepsilon}^{\varepsilon} f(x)\,\delta_\varepsilon(x)\,dx \right] = \lim_{\varepsilon \mapsto 0} f(0)\,\frac{1}{2\varepsilon}\,2\varepsilon = \lim_{\varepsilon \mapsto 0} f(0) = f(0).$$

Ergo:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

(one writes also $f(x)\,\delta(x) = f(0)\,\delta(x)$) (but care is needed....)

Examples:

$$\int_{-\infty}^{\infty} e^{x}\,\delta(x)\,dx = e^{0} = 1$$

$$\int_{-\infty}^{\infty} (x+2)^{2}\,\delta(x)\,dx = 2^{2} = 4$$

$$\int_{-\infty}^{\infty} x\,\delta(x)\,dx = 0$$

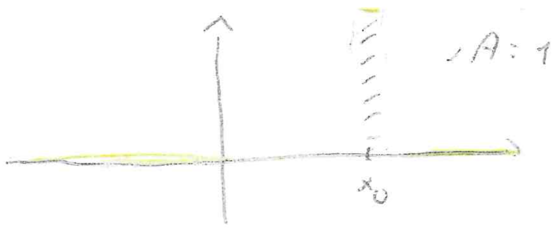
Note also that:

$$\int_{-\infty}^{\infty} \delta(x)\,f(x)\,dx = \int_{-a}^{a} \delta(x)\,f(x)\,dx \qquad \forall a > 0$$

Shift $\to x_0$

We can shift the δ-function in a straitforward manner.

$\delta(x - x_0)$



Peak in x_0

The discussion is the same.

$$\int_{-\infty}^{\infty} \delta(x - x_0)\, dx = 1$$

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0).$$

Examples:

$$\int_{-\infty}^{\infty} e^{x} \delta(x-1)\, dx = e$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1} \left[\delta(x-1) + \delta(x+1) \right] dx = \frac{1}{2} + \frac{1}{2} = 1$$

$$\int_{-\infty}^{\infty} x \left[\delta(x-3) - \delta(x+3) \right] dx = 3 - (-3) = 6.$$

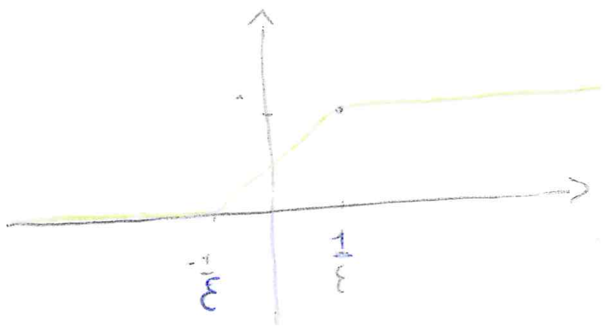
Relation to $\vartheta(x)$

Remind the function $\vartheta(x)$?

$$\vartheta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

$\vartheta(x)$ is strongly related to the $\delta(x)$. In order to see it let us express $\vartheta(x)$ also as a limiting process.

$$\vartheta_\varepsilon(x) = \begin{cases} 0 & x < -\varepsilon \\ \frac{1}{2\varepsilon}(x+\varepsilon) & |x| \leq \varepsilon \\ 1 & x > \varepsilon \end{cases}$$



Note that $\vartheta_\varepsilon(0) = \frac{1}{2}$ $\forall x$ (indipendently on ε!)

$$\vartheta(x) \equiv \lim_{\varepsilon \to 0} \vartheta_\varepsilon(x) = \begin{cases} 0 & x < 0 \\ 1/2 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$\frac{d\vartheta_\varepsilon}{dx} = \begin{cases} 0 & x \leq -\varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \\ 0 & x > \varepsilon \end{cases} = \begin{cases} 0 & |x| > \varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \end{cases} = \delta_\varepsilon(x)$$

Ergo we have that

$$\frac{d\vartheta_\varepsilon(x)}{dx} = \delta_\varepsilon(x)$$

By taking the limit $\varepsilon \mapsto 0$ we get

$$\boxed{\delta(x) = \frac{d\vartheta(x)}{dx}}$$

Conversely, one can also express $\vartheta(x)$ as

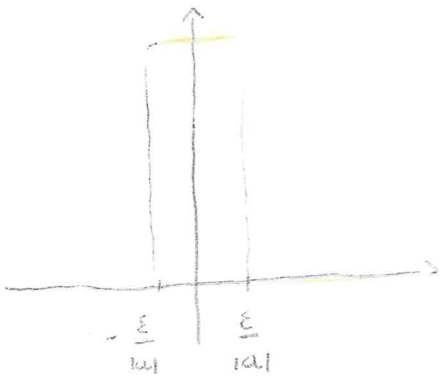
$$\vartheta(x) = \int_{-\infty}^{x} \delta(z)\,dz\,.$$

important property

$$\boxed{\delta(ax) = \frac{1}{|a|}\,\delta(x) \qquad \forall a \neq 0 .}$$

$$\delta(ax) = \lim_{\varepsilon \to 0} \delta_\varepsilon(ax)$$

$$\delta_\varepsilon(ax) = \begin{cases} 0 & \text{for } |ax| > \varepsilon \\ \frac{1}{2\varepsilon} & \text{for } |ax| < \varepsilon \end{cases} = \begin{cases} 0 & \text{for } |x| > \frac{\varepsilon}{|a|} \\ \frac{1}{2\varepsilon} & \text{for } |x| \leq \frac{\varepsilon}{|a|} \end{cases}$$



$$A = \frac{1}{2\varepsilon} \cdot \frac{2\varepsilon}{|a|} = \frac{1}{|a|}$$

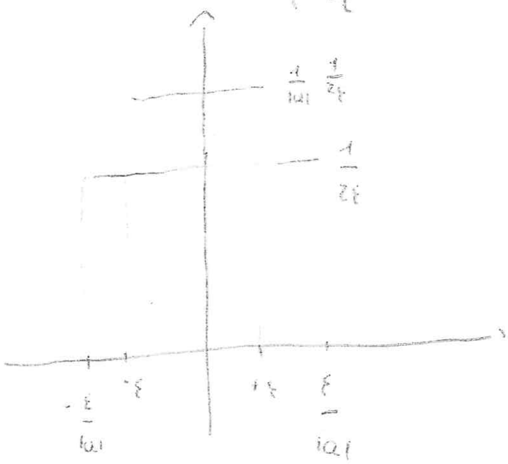
$$\int_{-\infty}^{\infty} \delta(ax)\,dx = \frac{1}{|a|} \qquad \int_{-\infty}^{\infty} \delta(ax)\, f(x)\,dx = \frac{1}{|a|} f(0)$$

We therefore can write

$$\delta(ax) = \frac{1}{|a|}\,\delta(x) .$$

Note that this is true when the limit is taken, but not for a finite ε.

$$\delta_\varepsilon(ax) \neq \frac{1}{|a|}\delta_\varepsilon(x) = \frac{1}{|a|} \cdot \begin{cases} 0 & |x| > \varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \end{cases}$$



Same A, but different shape... but for $\varepsilon \to 0$ you do not see any difference. This shows clearly the peculiarity of a limiting process.

Simple generalization:

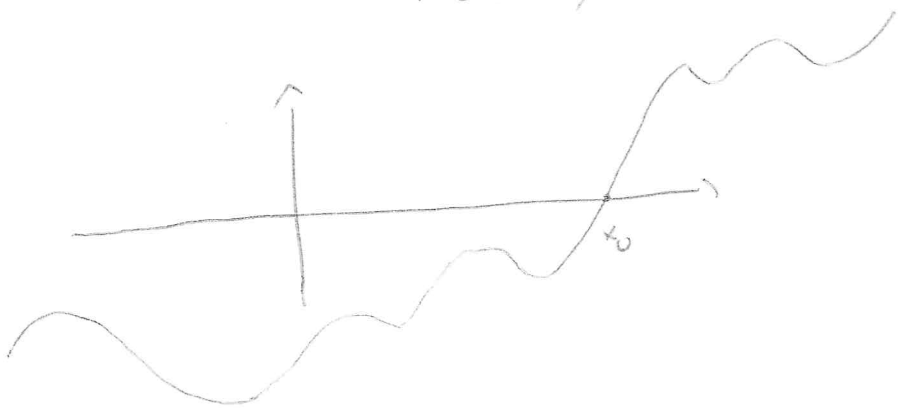
$$\boxed{\delta(a(x-x_0)) = \frac{1}{|a|}\delta(x-x_0)}$$

$\delta(f(x))$

$f(x): \mathbb{R} \mapsto \mathbb{R}$

For x_0 : $f(x_0) = 0$. Let us assume that : x_0 is the only point for which $f(x)$ vanishes

Moreover, we also assume that $f'(x_0) \neq 0$.



What is $\delta(f(x))$ in this case?

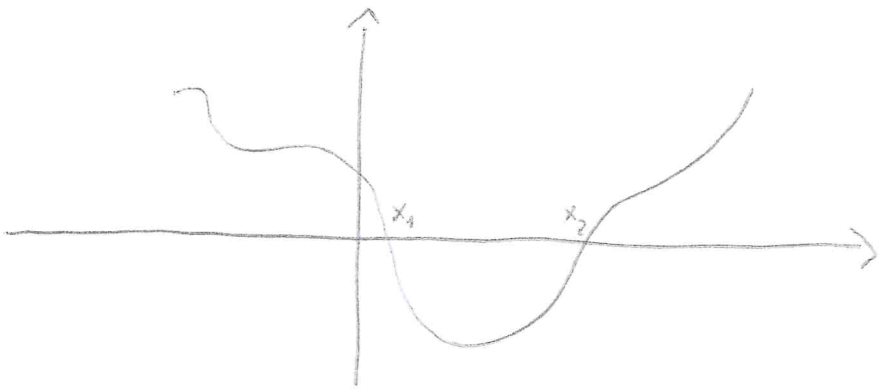
$\forall x \neq x_0$ $f(x) \neq 0$; ergo: $\delta(f(x)) = 0$ $\forall x \neq x_0$.

For $x = x_0$ we must be careful, for $x \simeq x_0$ we can approximate $f(x)$ as a Taylor expansion:

$$f(x) = \underbrace{f(x_0)}_{\substack{=0 \\ \text{per hp}}} + \underbrace{f'(x_0)}_{\substack{\neq 0 \\ \text{per hp}}} \cdot (x - x_0) + \dots = f'(x_0)(x - x_0)$$

Ergo:

$$\delta(f(x)) = \delta(f'(x_0)(x - x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$



if $f(x)$ / $f(x_1)=0$, $f(x_2)=0$ and $f'(x_1)\neq 0$, $f'(x_2)\neq 0$

we can repeat the same reasoning:

$$\delta(f(x)) = 0 \quad \forall x \neq x_1, x \neq x_2 .$$

For $x \simeq x_1$ write $f(x) \simeq f'(x_1)(x-x_1)$

$$\delta(f(x)) \underset{x\simeq x_1}{\simeq} \frac{1}{|f'(x_1)|}\,\delta(x-x_1)$$

For $x \simeq x_2$ write $f(x) \simeq f'(x_2)(x-x_2)$

$$\delta(f(x)) \underset{x\simeq x_2}{\simeq} \frac{1}{|f'(x_2)|}\,\delta(x-x_2)$$

Put alltogether:

$$\delta(f(x)) = \frac{1}{|f'(x_1)|}\,\delta(x-x_1) + \frac{1}{|f'(x_2)|}\,\delta(x-x_2)$$

We can generalise this procedure:

$f(x): \mathbb{R} \mapsto \mathbb{R}$

with $\begin{cases} f(x_i) = 0 & (i = 1, \ldots, N) \\ \text{and} \\ f'(x_i) \neq 0 \end{cases}$.

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{1}{|f'(x_i)|} \delta(x - x_i).$$

Example:

$\delta(x^2 - a^2) \qquad (a > 0).$

$f(x) = x^2 - a^2$; $f(x) = 0 \rightarrow x^2 - a^2 = 0 \qquad x = \pm a.$ $\begin{pmatrix} x_1 = a \\ x_2 = -a \end{pmatrix}$.

$f'(x) = 2x$: $\begin{cases} f'(x_1) = 2a \neq 0 \\ f'(x_2) = -2a \neq 0 \end{cases}$

Ergo:

$$\delta(x^2 - a^2) = \frac{1}{|2a|} \delta(x - a) + \frac{1}{|-2a|} \delta(x + a) =$$

$$= \frac{1}{2a} \delta(x - a) + \frac{1}{2a} \delta(x + a).$$

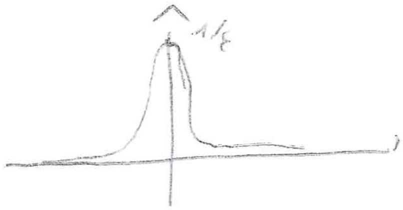
For instance:

$$\int_{-\infty}^{\infty} x^2 \delta(x^2 - a^2) dx = \frac{a^2}{2a} + \frac{1}{2a}(-a)^2 = a.$$

The chosen way to determine the $\delta(x)$ as a limit $\varepsilon \mapsto 0$ is <u>not</u> unique! Actually, there is an "∞" of way to reach the same result.

For instance, let us consider

$$L_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2+\varepsilon^2}$$



(For $\varepsilon \mapsto 0$ we get also in this case a very peaked function for $x=0$ and a vanishing function elsewhere).

$$\int_{-\infty}^{\infty} L_\varepsilon(x)dx = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty} \frac{dx}{x^2+\varepsilon^2} = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty} \frac{dx}{\varepsilon^2\left(\frac{x^2}{\varepsilon^2}+1\right)} \qquad z = \frac{x}{\varepsilon} \quad dx = \varepsilon dz$$

$$= \frac{\varepsilon}{\pi}\cdot\frac{1}{\varepsilon^2}\int_{-\infty}^{\infty}\frac{\varepsilon\, dz}{z^2+1} = \frac{1}{\pi}\Big[\arctan z\Big]_{-\infty}^{\infty} = \frac{1}{\pi}\Big[\underbrace{\arctan \infty}_{\frac{\pi}{2}} - \underbrace{\arctan(-\infty)}_{\left(-\frac{\pi}{2}\right)}$$

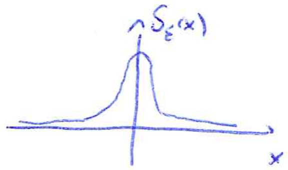
$$= 1 \; ! \quad (\forall \varepsilon).$$

$$\boxed{\delta(x) = \lim_{\varepsilon \mapsto 0^+} L_\varepsilon(x)}$$

All the properties studied before are "still" fulfilled!

One can go further; for instance:

$$\delta(x) = \lim_{\varepsilon \mapsto 0} \frac{1}{\sqrt{\pi}} \frac{1}{\varepsilon} e^{-x^2/\varepsilon^2}$$

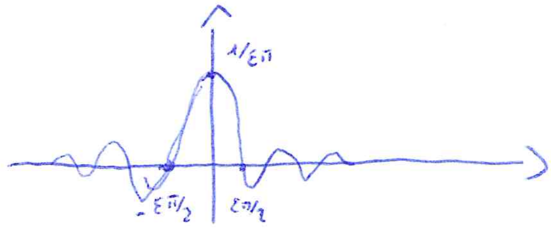


is also OK... the idea is always the same. Same discussion.

There is however a rather different representation (an integral representation):

To see it we start from:

$$\delta(x) = \lim_{\varepsilon \mapsto 0} \underbrace{\frac{1}{\pi} \frac{\sin(x/\varepsilon)}{x}}^{\delta_\varepsilon(x)}$$



$\frac{x_0}{\varepsilon} = \frac{\pi}{2} \rightarrow$

This is a 'bit' different... it can be also negative.

$$\delta_\varepsilon(x) = \frac{1}{\pi} \frac{\sin(x/\varepsilon)}{x} = \frac{1}{2\pi} \int_{-1/\varepsilon}^{1/\varepsilon} e^{ikx} dk =$$

$$= \frac{1}{2\pi} \frac{1}{ix} \left(e^{ikx} \right)_{-1/\varepsilon}^{1/\varepsilon} = \frac{1}{2\pi} \frac{1}{ix} \left(e^{ix/\varepsilon} - e^{-ix/\varepsilon} \right) =$$

$$= \frac{1}{\pi} \frac{\sin\left(\frac{x}{\varepsilon}\right)}{x}$$

$\checkmark$

$$\delta(x) = \lim_{\varepsilon \mapsto 0} \frac{1}{2\pi} \int_{-1/\varepsilon}^{1/\varepsilon} e^{ikx} dk \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

$$\boxed{\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk}$$

This expression naturally enters also when doing Fourier transformation (next week)

Note, in this representation one can write $\underline{\underline{\theta(x)}}$ /

$$\frac{d\,\theta(x)}{dx} = \delta(x)$$

as

$$\theta(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k-i\varepsilon}\,dk \;\hat{=}\; \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k}\,dk$$

Namely:

$$\frac{d\theta(x)}{dx} = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\underbrace{\left(\frac{ik}{k-i\varepsilon}\right)}_{1} e^{ikx} = \delta(x).$$

(This is fully understandable only in the context of function analysis)

Derivative of the $\delta(x)$:

$$\frac{d\delta(x)}{dx}$$

$$\int_{-\infty}^{\infty} \frac{d\delta(x)}{dx} f(x)\, dx = -\int_{-\infty}^{\infty} \delta(x) \frac{df(x)}{dx}\, dx + \underbrace{\left[\delta(x) f(x)\right]_{-\infty}^{\infty}}_{=0} = -\left(\frac{df(x)}{dx}\right)_0 = -f'(0).$$

But formally:

$$f(x) = x\,\varphi(x)$$

$$f'(x) = \varphi(x) + x\varphi'(x)$$

$$f'(0) = \varphi(0)$$

$$\int_{-\infty}^{\infty} \frac{d\delta(x)}{dx}\, x\,\varphi(x)\, dx = -\varphi(0)$$

Ergo

$$\boxed{x \frac{d\delta(x)}{dx} = -\delta(x)}$$

Generalization:

$$\boxed{x^n \frac{d^{(n)}\delta(x)}{dx} = (-1)^n n!\, \delta(x)}$$