AT THE BEGINNING THERE IS THE LAGRANGIAN....

AND THE ACTION.

$\varphi_i(x)$

$x = (t, \vec{x})$ $\quad\quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_D \end{pmatrix}$

D=3 in our world.

$$\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} = \frac{\partial \mathcal{L}}{\partial \varphi_i}$$

$$S = \int d^4x \, \mathcal{L}$$

is the action.

→ Follow page 1-2-3-4

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - V$$

$$\Box \varphi = -\frac{\partial V}{\partial \varphi}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{4} A_\mu^2$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = -m^2 A^\mu$$

$m \neq 0 \to \partial_\nu A^\nu = 0$

$m = 0$ g.i. under $A^\nu \mapsto A^\nu + \partial^\nu \alpha(x)$

Consider the transformation

$$\begin{cases} \varphi_i(x) \longmapsto \varphi_i'(x) = \varphi_i + \delta\varphi_i + \dots \\ x \longmapsto x' = x + \delta x \end{cases}$$

One has the following inequality:

$$dS' - dS = \mathcal{L}\left(\varphi_i'(x'), \partial_\mu' \varphi_i'(x')\right) d^4x' - \mathcal{L}\left(\varphi_i(x), \partial_\mu \varphi_i(x)\right) d^4x$$

$$= \left(\partial_\mu J^\mu\right) d^4x$$

with $\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta\varphi_i + \delta x^\mu \cdot \mathcal{L}}$

This is the Noether current.

If the transformation is a symmetry of the action; $dS' = dS$,

it follows that $\boxed{\partial_\mu J^\mu = 0.}$

This is a conserved current and $Q = \int d^3x \, J^0$ is a conserved charge.

- TRANSLATION → this is a symmetry of all "known" fundamental theories

$$x^\mu \mapsto x'^\mu = x^\mu + a^\mu$$

$$\varphi_i'(x') = \varphi_i(x)$$

There are four conserved currents:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial^\nu \varphi_i - g^{\mu\nu}\mathcal{L}$$

$$\partial_\mu T^{\mu\nu} = 0 \qquad \nu = 0,1,2,3$$

$$P^\nu = \int d^3x \, T^{0\nu}$$

are the four conserved charges (for φ_i solving the eom).

$P^0 = E$

P^k is the three-momentum of the field configuration (solution of the eom).

$P^k = \int d^3x \, \rho^k \qquad \rho^k$ momentum density

$P^0 = E = \int d^3x \, \mathcal{E} \qquad \mathcal{E} = T^{00}$ energy density

- Lorentz transf. : this is also a symmetry of all known fundam. theories

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

with $\quad g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \qquad (x'^2 = x^2)$

The conserved currents are given by:

$$J_\mu^{\alpha\beta} = - T^\beta_\mu x^\alpha + T^\alpha_\mu x^\beta$$

There are six of them (because $J_\mu^{\alpha\beta} = - J_\mu^{\beta\alpha}$).

The charges are:

$\vec{J} = \left(\int d^3x \, J_0^{12}, \int d^3x \, J_0^{23}, \int d^3x \, J_0^{31} \right)$ is the angular momentum

$$= \int d^3x \, \vec{x} \wedge \vec{P}$$

$$\vec{K} = \left(\int d^3x \, J_0^{01}, \int d^3x \, J_0^{02}, \int d^3x \, J_0^{03} \right)$$

is the boost ...

Dilatation: this is the "third" important spacetime transf.

It is not a symmetry of the SM.

It plays a crucial role in QCD.

Indeed, a Lag. is dilatation invariant only if there are dimensionless parameters!!!!

$$x^\mu \mapsto x'^\mu = \lambda^{-1} x^\mu$$

$$\varphi_i(x) \mapsto \varphi'_i(x') = \lambda^{d} \varphi(x)$$

$$\left[\begin{array}{l} q'_i(x') = \lambda^{3/2} q(x) \\ A'^a_\mu(x') = \lambda A^a_\mu(x) \end{array}\right]$$

The associated current is:

$$\boxed{J^\mu = x_\nu T^{\mu\nu}}$$

if the dilatation is a symmetry of the Lagrangian then - at least at the classical level:

$$\partial_\mu J^\mu = 0$$

$$\partial_\mu J^\mu = g_{\mu\nu} T^{\mu\nu} + x_\nu \partial_\mu T^{\mu\nu} = g_{\mu\nu} T^{\mu\nu} = 0$$

$$(= T^{00} - T^{11} - T^{22} - T^{33} = 0)$$!!!! in a few seconds more on this!!!!!

and dualism fondo

Internal symmetries $\varphi_i(x) \mapsto \varphi_i'(x)$
$x'^\mu = x^\mu$

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta\varphi_i}$$

→ These are the chiral symmetry....

They change from problem to problem and depends on the details of the Lagrangian.

Gluon field

$$A_\mu = A_\mu^a t^a \qquad a = 1, \dots, N_c^2 - 1$$

N_c nr of colors.

Let us start from the transformation.

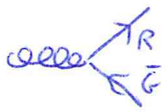
A_μ is in the so-called "adjoint representation".

$$A_\mu \mapsto U A_\mu U^\dagger$$

(Namely, a gluon is like a color-anticolor state...

$R\,G\,B$

$\bar{R}\,\bar{G}\,\bar{B}$



there are $9 - 1 = 8$ combination.

↳ this is the 'singlet'.)

If $U = U(x) = e^{i\alpha^a_{(x)} t^a}$ we have a local gauge transf.

$$\boxed{A_\mu \mapsto U(x) A_\mu U^\dagger(x) - \frac{i}{g_0} U(x) \partial_\mu U^\dagger(x)}$$

Note, for $N_c = 1$ $U = e^{i\alpha(x)}$, $A_\mu \mapsto A_\mu + \frac{\partial_\mu \alpha(x)}{g_0}$

This is the gauge transf. of the em field!

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_0 [A_\mu, A_\nu]$$

$$\boxed{G_{\mu\nu} \mapsto U(x) G_{\mu\nu} U^\dagger(x)}$$

(The $\neq$ pieces eat each other... and for field tensor we get an easy transformation!!!)

Proof (Just put in and verify it).

$$G_{\mu\nu} \mapsto \partial_\mu \left(U A_\nu U^+ - \frac{i}{g_0} U \partial_\nu U^+ \right) - \partial_\nu \left(U A_\mu U^+ - \frac{i}{g_0} U \partial_\mu U^+ \right)$$

$$- i g_0 \left[U A_\mu U^+ - \frac{i}{g_0} U \partial_\mu U^+ , U A_\nu U^+ - \frac{i}{g_0} U \partial_\nu U^+ \right]$$

$$= U (\partial_\mu A_\nu - \partial_\nu A_\mu) U^+ + \partial_\mu U A_\nu U^+ + U A_\nu \partial_\mu U^+ - \partial_\nu U A_\mu U^+ - U A_\mu \partial_\nu U^+$$

$$- \frac{i}{g_0} \partial_\mu (U \partial_\nu U^+) + \frac{i}{g_0} \partial_\nu (U \partial_\mu U^+)$$

$$- i g_0 U [A_\mu, A_\nu] U^+ - i g_0 \left(\left[-\frac{i}{g_0} U \partial_\mu U^+ , U A_\nu U^+ \right] + \left[U A_\mu U^+ , -\frac{i}{g_0} U \partial_\nu U^+ \right] \right.$$

$$- i g_0 \left[-\frac{i}{g_0} U \partial_\mu U^+ , -\frac{i}{g_0} U \partial_\nu U^+ \right] =$$

$$= U G_{\mu\nu} U^+ + \partial_\mu U A_\nu U^+ + U A_\nu \partial_\mu U^+ - \partial_\nu U A_\mu U^+ - U A_\mu \partial_\nu U^+$$

$$- \frac{i}{g_0} \partial_\mu U \partial_\nu U^+ + \frac{i}{g_0} \partial_\nu U \partial_\mu U^+$$

$$- \left(U \partial_\mu U^+ U A_\nu U^+ - U A_\nu U^+ U \partial_\mu U^+ + \dots \right)$$

$$+ \frac{i}{g_0} \left(U \partial_\mu U^+ U \partial_\nu U^+ - U \partial_\nu U^+ U \partial_\mu U^+ \right) = U G_{\mu\nu} U^+$$

Use $\partial_\mu (U U^+) = \partial_\mu U \cdot U^+ + U \partial_\mu U^+ = 0$] $\to$ use this trick.

The YM-Lagrangian is given by:

$$\mathcal{L}_{YM} = -\frac{1}{2} \operatorname{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$$

- By construction it is invariant under local color transformation:

$$\mathcal{L}_{YM} = -\frac{1}{2} \operatorname{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right] \longmapsto -\frac{1}{2} \operatorname{Tr}\left[U G_{\mu\nu} U^{\dagger} U G^{\mu\nu} U^{\dagger}\right]$$
$$= -\frac{1}{2} \operatorname{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right] = \mathcal{L}_{YM}.$$

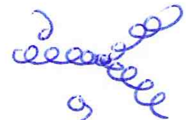
- Translation and Lorentz invariant: ✓

- interaction:

$$\mathcal{L}'_{YM} = -\frac{1}{2} \operatorname{Tr}\left[\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i g_0 [A_\mu, A_\nu]\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu - i g_0 [A^\mu, A^\nu]\right)\right]$$

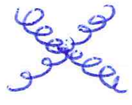
$a = 1, \ldots, N_c^2 - 1$

$$= -\frac{1}{2} \operatorname{Tr}\left[\left(\partial_\mu A_\nu - \partial_\nu A^\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\right]$$



$$- \frac{1}{2} \operatorname{Tr}\left[-i g_0 [A_\mu, A_\nu]\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\right]$$

g_0



$$- \frac{1}{2} \operatorname{Tr}\left[-g_0^2 [A_\mu, A_\nu][A^\mu, A^\nu]\right]$$

g_0^2

Thus, gluons shine in their own light.

This is one of the crucial differences from QED.

(Example of lamp vs lamp).

→ Beginning of QCD: do we have glueballs? → Answer: yes. (I come back in a moment to this).

Now, the only parameter entering in $\mathcal{L}_{YM}$ is g_0.

g_0 is a "pure number".

There is no parameter with a dimension...

(classical)
→ invariance under scale transformation.

$$A_\mu(x) \mapsto A'_\mu(x') = \lambda x'$$

$$x' = \lambda^{-1} x$$

$$\partial'_\mu = \lambda \partial_\mu$$

it follows that:

$$\mathcal{L}_{YM} \mapsto \lambda^4 \mathcal{L}_{YM}$$

But then the action is invariant:

$$\mathcal{L}(A'_\mu(x'))\,d^4x' = \mathcal{L}(A_\mu(x))\,d^4x$$

Then, the dilation current

$$J^{\mu} = x_{\nu} T^{\mu\nu} \qquad \text{with } T^{\mu\nu} = \frac{\partial \mathcal{L}_{YM}}{\partial(\partial_{\mu} A_{\rho})} \partial^{\nu} A_{\rho} - g^{\mu\nu} \mathcal{L}_{YM}$$

$$\partial_{\mu} J^{\mu} = 0 \,!!! \qquad + \text{"symm"}.$$

is a classical symmetry of the YM-Lag.

Now we realize the following points:

- this is paradoxical!!!
  If there is no scale, where does M_{proton} come from?
  (or M_{ρ}, or f_{π}, or whatever scale in QCD)
  Namely, we know that the quarks ($m_u \simeq m_d \simeq 5$ MeV $\ll M_p \sim 1$ GeV)
  can't do that.

Note: as a consequence it is not the Higgs which gives us mass!!! This is a misunderstanding due to marketing of few people....

The origin of mass is in QCD, or better is in the gluons, also in the YM part of QCD!!

Moreover, there is <u>confinement</u>.

There is <u>no</u> free floating gluon moving around.

The spectrum of this theory consists <u>only</u> of glueballs.

Namely, as we shall see very soon, the interaction among gluons becomes very strong at small energies.

Gluons are always confined into glueballs.

- Even without quarks, $\mathcal{L}_{YM}$ generates glueballs.

  There is a full spectrum of them.

  The lightest one is a scalar state $J^{PC} = 0^{++}$ with a mass of about 1.5 GeV.

  The currents are given by:

J^{PC}

0^{++} $\quad G = \mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$ (local color invariant ... 2+3+4 gluons)

$\tilde{G}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$

0^{-+} $\quad \tilde{G} = \mathrm{Tr}\left[G_{\mu\nu} \tilde{G}^{\mu\nu}\right]$

2^{++} $\quad H^{\mu\nu} = \mathrm{Tr}\left[G^{\mu\rho} G_{\rho}{}^{\nu}\right]$

2^{-+} $\quad \tilde{H}^{\mu\nu} = \mathrm{Tr}\left[\tilde{G}^{\mu\rho} G_{\rho}{}^{\nu}\right]$

1^{--} $\quad J_{\alpha} = d^{abc}\, \partial^{\alpha}\, \mathrm{Tr}\left[G^{a}_{\mu\nu} G^{b,\nu\rho} G^{c}_{\rho\alpha}\right]$ → more complicated...

...

...

→ Show the spectrum of lattice QCD (in the quenched approx: gluons only). Comment on the lightest state: it is extreemely important for us!!!!

- Then, how does a dimensionful nr come into the YM theory?

- This is one of the deepest issue in YM, QCD, and without exaggerating, in the whole theoretical physics.

- First, in the classical version of the theory there is NO way how this could happen.

  A scale-invariant theory cannot generate a dimension in its classical VS.

- But in the quantum VS of a theory everything is $\neq$!

  When you renormalize a theory you actually introduce a cutoff Λ which is the maximal energy scale of that theory

  (that is - also - a minimal length $l_{min} \sim 1/\Lambda$).

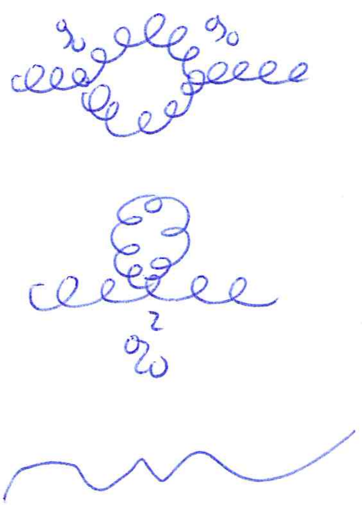
  Beyond Λ new physics is valid (whatever it is ...

  (Merging of quantum physics and general relativity)

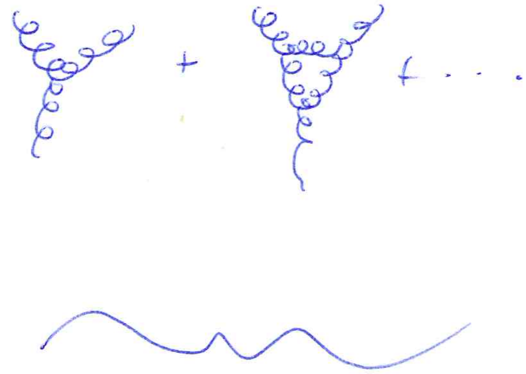
Now, when doing renormalisation one obtains that

$$g_0 \overset{\text{Ren.}}{\longmapsto} g(\mu)$$

↳ energy scale



gluon propagator



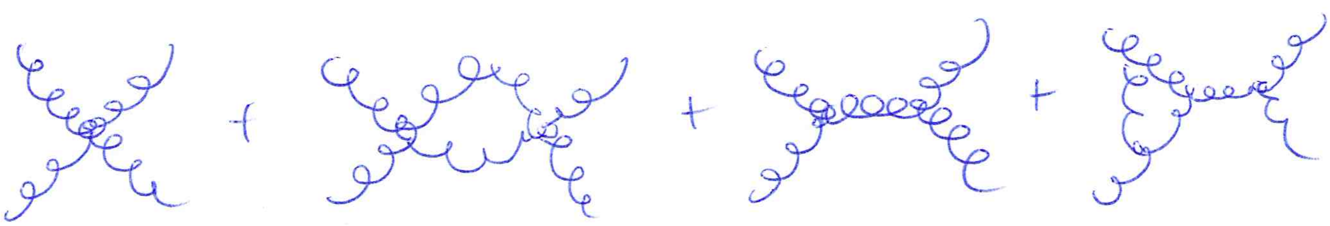
vertex

We are not concerned with these diagrams, that is we will not calculate them.

μ → energy at which you probe the interaction.

Another possibility is "scattering".

$g(\mu)$



+ ...

μ can be the energy in the c.o.m.

One has thus a $\neq$ coupling constant in dependence of the energy at which the gluons scatter.

The β function is defined that:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} \qquad (\text{variation of } g).$$

(Note: for $g = g_0 = \text{const} \longmapsto \beta(g) = 0$).

In the case of a pure YM theory we have (perturbative calc.):

$$\beta(g) = \mu \frac{\partial g}{\partial \mu} = -b g^3 < 0 \qquad \boxed{b = \frac{11 N_c}{48 \pi^2}}$$

Ergo, $g(\mu)$ decreases with increasing μ.

$\longrightarrow$ Asymptotic freedom..

$\longrightarrow$ QED is different... $\beta(g) > 0$ ..

$\partial_\mu J^\mu_D \neq 0$ in YM theory. Because of $\beta(g) \neq 0$

one gets:

$$\boxed{\partial_\mu J^\mu_D = \frac{\beta(g)}{4g} G^a_{\mu\nu} G^{a,\mu\nu} \neq 0}$$

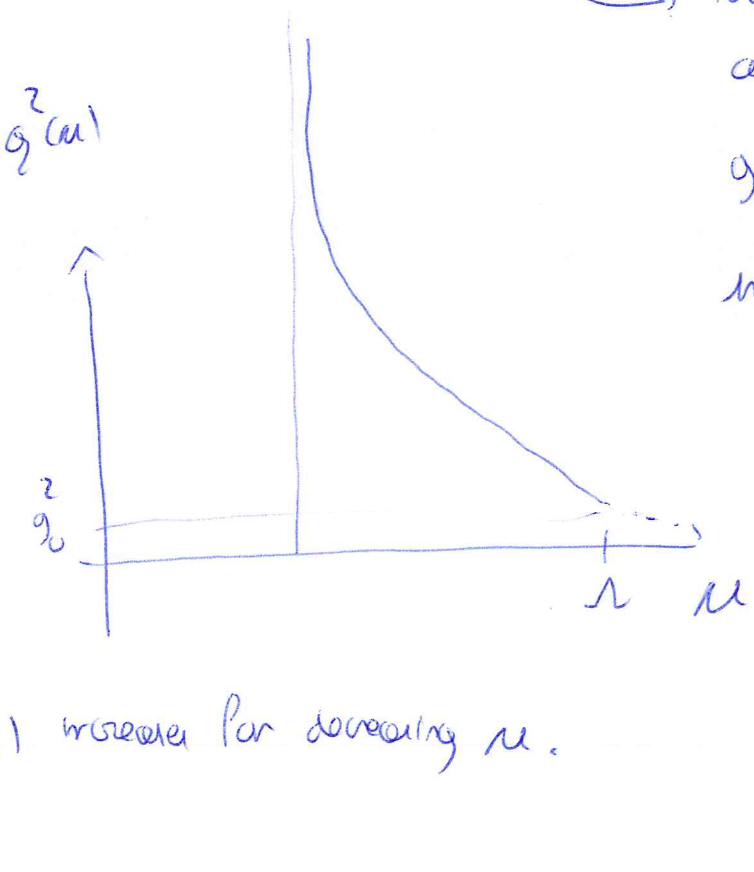
Trace anomaly.

Moreover, let us solve the eq:

$$g^2(\mu) = \frac{g_0^2}{1 + 2bg_0^2 \ln\left(\frac{\mu}{\Lambda}\right)}$$

this is the very large Λ cited before

g_0 is small (a very small number).



$g(\mu)$ increases for decreasing μ.

$g^2(\mu)$ has a pole for Λ_{YM} : $g^2(\Lambda_{YM}) = \infty$.

$$1 + 2 b g_0^2 \ln \frac{\mu}{\Lambda} = 0$$

$$\Lambda_{YM} = \text{"Landau Pole"} = \underset{\substack{\downarrow \\ \text{large num}}}{\Lambda}\, e^{\underbrace{-\frac{1}{2 b g_0^2}}_{\text{small num}}}$$

Note: we cannot calculate Λ_{YM} because we don't know Λ and g_0.

But we can see that and have such a scale energy.

All this goes under the name of <u>dimensional transmutation</u>.

$\Lambda \rightarrow$ large energy

$\Lambda_{YM} \mapsto$ "small" energy.

In Nature:

$\Lambda_{YM} \simeq 250$ MeV $\rightarrow$ scale on which everything depends.

$f_\pi \simeq 100$ MeV $\quad M_p \simeq 1$ GeV $\simeq 3 \Lambda_{YM}$

$m^* \simeq \Lambda_{YM}$

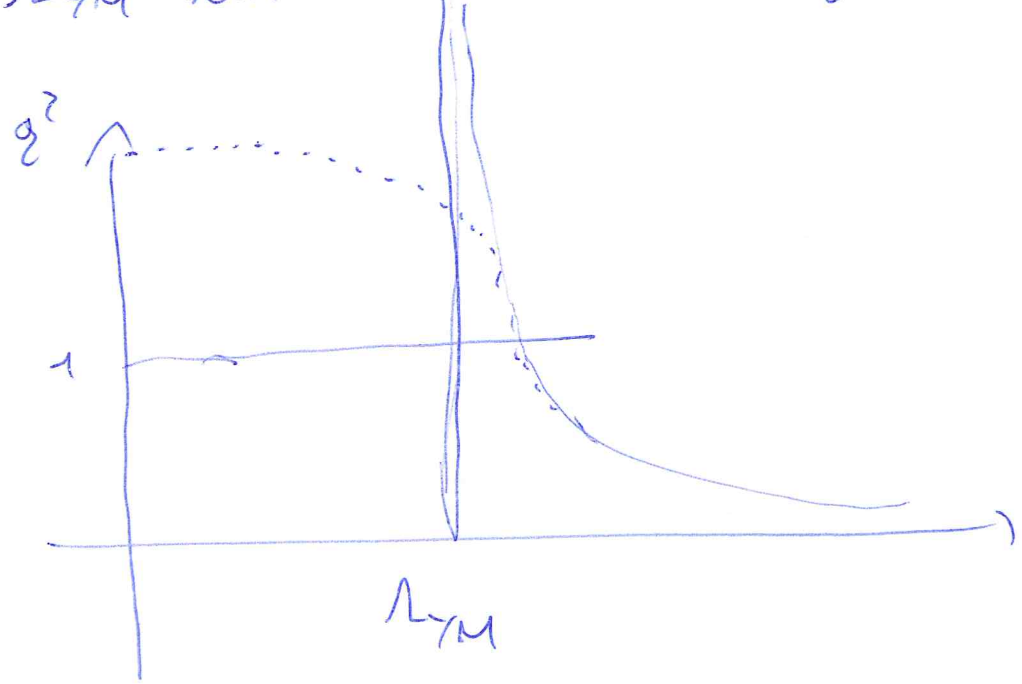
Note:

In reality $g^2(\mu)$ doesn't explode at Λ_{YM}...

Namely, this is a calculation based on one-loop...

valid only when g is small.

But Λ_{YM} sets the scale at which g^2 becomes large.



Theoretically, solvable (FRG approach).

There is no way we could overestimate the importance of this. It is "crucial".

VEV : vacuum's expectation value

4

$$\langle T_\mu^{\ \mu} \rangle = -\langle \frac{11 N_c \alpha_s}{48\pi} G^a_{\mu\nu} G^{a,\mu\nu} \rangle \sim -\frac{11 N_c}{48} \Lambda M$$

Note that:

$$\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} = \frac{1}{2} (\vec{E}^{a^2} - \vec{B}^{a^2})$$

In the Weyl gauge:

$$E_i^a = \dot{A}_i^a$$

$$B_i^a = \frac{1}{2} f_{ijk} (\partial_j A_k^a - \partial_k A_j^a + f^{abc} A_j^b A_k^c)$$

At a perturbative level: $\langle \vec{E}^{a^2} - \vec{B}^{a^2} \rangle = 0$

Non-perturbative field configurations (such as instantons) can generate $\langle \vec{E}^{a^2} - \vec{B}^{a^2} \rangle > 0$

Effective description of the YM theory in its confined part

$\longrightarrow$ Dilaton Lagrangian

Confinement: only glueballs... let us take the lightest one: G.

It is a "scalar", it has the quantum nrs of the vacuum, then it can condense.

and it does condense.

$$\boxed{\mathcal{L}_{dil} = \frac{1}{2}(\partial_\mu G) - V(G)}$$

Which is the correct potential?

eom:

$$\partial_\mu \partial^\mu G = -\frac{\partial V}{\partial G}$$

Under a dilatation transf:

$$G'(x') = \lambda G(x)$$

$$x' = \lambda^{-1} x$$

$$J^{\mu}_{D} = x_{\nu} T^{\mu\nu}$$

$$\partial_{\mu} J^{\mu}_{D} = g_{\mu\nu} T^{\mu\nu}$$

$$\left[T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} G)} \partial^{\nu} G - g^{\mu\nu} \mathcal{L} = \partial^{\mu} G \partial^{\nu} G - g^{\mu\nu} \mathcal{L} \right.$$

$$\partial_{\mu} J^{\mu}_{D} = \partial_{\mu} G \partial^{\mu} G - 4 \mathcal{L} =$$

$$= (\partial_{\mu} G)^2 - 4 \left(\frac{1}{2} (\partial_{\mu} G)^2 - V \right) =$$

$$= - \left((\partial_{\mu} G)^2 - 4 V \right)$$

Use the e.o.m:

$$\partial_\mu G \partial^\mu G = \partial_\mu (G \partial^\mu G) - G \Box G =$$

$$= \partial_\mu (G \partial^\mu G) + G \frac{\partial V}{\partial G}$$

Ergo:

$$\partial_\mu J^\mu_D = -(G \partial_G V - 4V)$$

=

When is it zero?

$$V = \lambda G^4$$

$$G \partial_G V - 4V = G \cdot 4\lambda G^3 - 4\lambda G^4 = 0!$$

This is expected!!! Infact: λ is dimensionless.

And this is the only case in which $\partial_\mu J^\mu_D$ is zero for a scalar field theory (at a classical level at least)

In QCD:

$$\partial_\mu J^\mu_{YM D} = \frac{\beta(g)}{4g} \underbrace{G^a_{\mu\nu} G^{a,\mu\nu}}_{\sim G^4}$$

We thus have for our theory with one scalar field:

$$\boxed{\partial_\mu J^\mu_D = -c \cdot G^4}$$

$$(G \partial_G V - 4V) = c\, G^4$$

The solution is "unique":

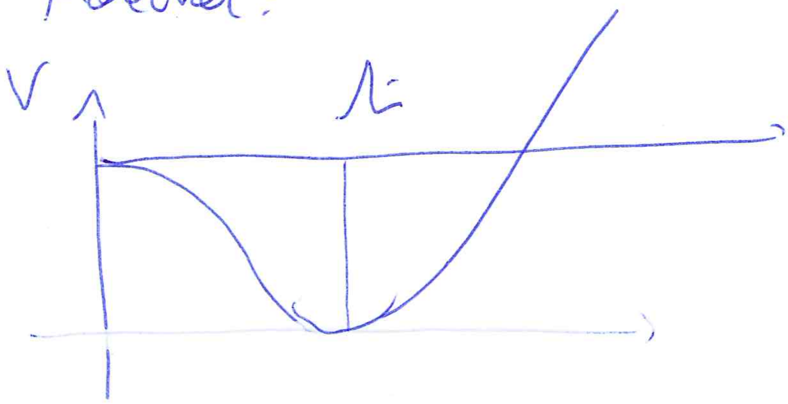
$$V(G) = \frac{1}{4} \frac{m_G^2}{\Lambda^2} \left(G^4 \ln \frac{G}{\Lambda} - \frac{G^4}{4} \right)$$

↳ explicit breaking (at the "classical level of the effective Lagrangian) of the trace anomaly.

Λ has the dimension of energy. It is $\sim \Lambda_{YM}$. It sets the energy of the composite field.

$$\partial_\mu J^a_D = -\frac{1}{4}\frac{m_G^2}{\Lambda^2} G^4 \quad \text{as desired.}$$

Moreover:



$V(G)$ has a minimum per $G = G_0 = \Lambda$:

A "condensate" of G is generated.

$$\boxed{\langle \partial_\mu J^a_D \rangle = -\frac{1}{4} m_G^2 \Lambda^2 \rightarrow \text{condensate} \sim \langle G^a_{\mu\nu} G^{a,\mu\nu} \rangle}$$

Comments:

- why only "one" G? This is obviously a simplification.
- The "logarithm" already describes the loop of the underlying gluonic theory (Coleman effective potential!)

Up to now: we are in the vacuum...

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu}\right]$$

$$A_\mu \mapsto U A_\mu U^\dagger - \frac{i}{g_0} U \partial_\mu U^\dagger$$

obviously, the Lag. $\mathcal{L}_{YM}$ is invariant under transf. of the center...

indeed the transformation is very simple, trivial indeed:

$$U = z_n \mathbb{1}_{N_c} \qquad z_n = e^{i\frac{2\pi n}{N_c}} \qquad n = 0, 1, \dots, N_c - 1$$

$$A_\mu \mapsto A_\mu$$

So, why one often speaks about this center symmetry?

It appears to be very trivial...

and, in fact, it is such if we are only in the vacuum!

Finite temperature: we expect $\to$ generates in the vacuum / $\to$ gluons at high T — phase transition in the middle

$$Z_{YM}(T) = \int_{PBC} DA_\mu \, e^{-S^E_{YM}}$$

$$S^E_{YM} = \int_0^\beta d\tau \int d^3x \, \mathcal{L}^E_{YM} \quad \text{with } \beta = \frac{1}{T}$$

PBC means:

$$A_\mu(0,\vec{x}) = A_\mu(\beta,\vec{x})$$

One considers $U(\tau,\vec{x})$ /

$U(\beta,\vec{x}) \neq U(0,\vec{x})$ but belongs to another element of the center.

For instance)

$$\begin{cases} U(0,\vec{x}) = 1_{N_c} \\ U(\beta,\vec{x}) = z_n = e^{i2\pi n/N_c} \quad n = 1,\dots,N_c-1 \end{cases}$$

with smooth derivative...
$$\begin{cases} A'_\mu(0) = A_\mu(0) \\ A'_\mu(\beta) = z_n A_\mu(\beta) z_n^{-1} = A_\mu(\beta) \end{cases}$$

$Z_{YM}(T)$ is invariant under this generalised transf.

⌈Note, when introducing the quarks this symmetry is broken:

Namely, for a fermion field $(q \mapsto Uq)$

$$q(0,\vec{x}) = -q(\beta,\vec{x})$$

$$q(\tau,\vec{x}) \to q'(\tau,\vec{x}) \quad / \quad \begin{aligned} q'(0,\vec{x}) &= q(0,\vec{x}) \\ q'(\beta,\vec{x}) &= z_n\, q(\beta,\vec{x}) = \\ &= -z_n\, q(0,\vec{x}) \\ &\neq -q'(0,\vec{x}) \end{aligned}$$

(The antiperiodicity is explic. broken ...).

This is why the 'center symmetry' is broken by quarks.

Polyakov line

$$L(\vec{x}) = P\, e^{iq\int_0^\beta d\tau\, A_4(\tau,\vec{x})} \qquad \left[\begin{array}{l} t = -i\tau \\ A^0 = -iA^4 \end{array} \right]$$

Under a gauge transf. it transforms as:

$$L(\vec{x}) \mapsto U(0,\vec{x})\, L(\vec{x})\, U^+(\beta,\vec{x}) = z_m^+ L(\vec{x})$$
$$= e^{-i2\pi m/N} L(\vec{x})$$

Polyakov loop: trace over the Polyakov line:

$$l = \frac{1}{N} \operatorname{Tr} L$$

$\langle l \rangle$ = expectation value of the Polyakov loops.

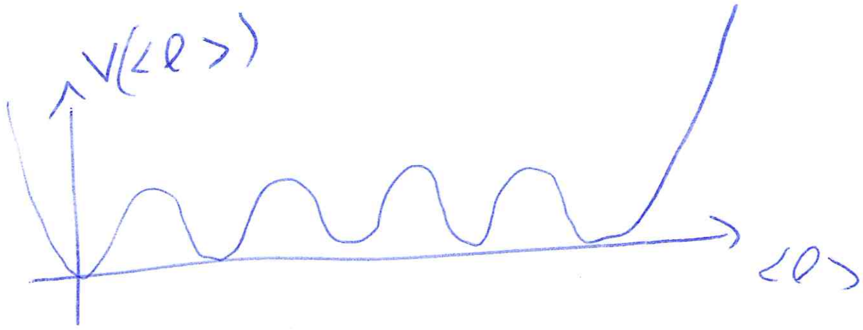
For very high T $q \mapsto 0 \to \langle l \rangle = 1 \ldots$

However, every other value $\langle l \rangle = e^{2\pi i m/N} = z_m$

is OK for $T \mapsto \infty$.

one has a degeneracy... all of them are ok.

The realised one follows from the phen. of spont. symm. breaking.



For high T

N distinct minima.

indeed, this is also understandable in the following way:

$\langle \ell \rangle = z_n \ell_0$

$$\ell_0 \sim e^{-F_{\text{pair}}/T}$$

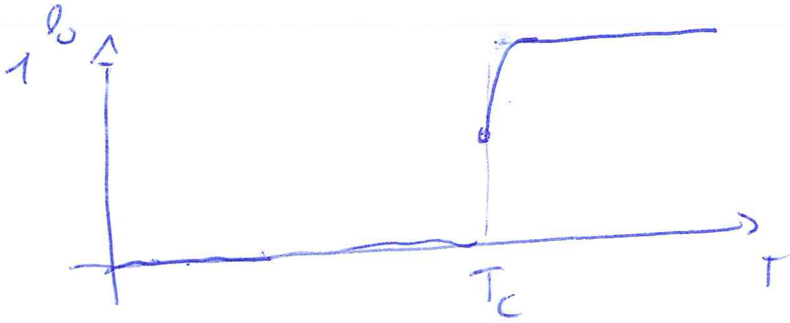
$\hookrightarrow$ energy of a probe $q\bar{q}$ pair.

For $\ell_0 = 1$

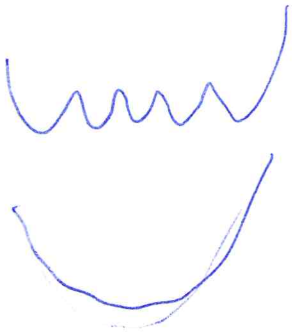
$F_{\text{pair}} = 0 \mapsto$ deconfinement of quarks (and indeed of gluons)

On the contrary, for $T=0$ we have $\ell_0 = 0 \to F_{\text{pair}} = \infty$.

Then, we expect that:



1° order



Spont. symm. breaking at high T !!!

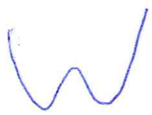
"Average" over all configurations of the order for small T.

$N = 2$

$N = 3$

$\{1_2, -1_2\}$

$\langle \ell \rangle$ is either 1 or -1


high T


small T

• When introducing quarks → cross-over !!!!