Let us define the matrix $\underline{\Psi}$ $\Phi$ as:

$$\boxed{\Phi_{ij} = 2\bar{q}_{j,R}\, q_{i,L}}$$

$$\Phi = 2\begin{pmatrix} \bar{u}_R u_L & \bar{d}_R u_L & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}.$$

This is a matrix with specific combination of right-handed and left-handed quarks.

Why do we do this? The basic reason is that $\Phi$ has a very simple transformation under chiral transformation.

$$\boxed{\Phi \mapsto U_L \Phi U_R^{\dagger}}$$

from the left

from the right

What is $\Phi$?

$$\Phi = 2\bar{q}_j P_L P_L q_i = 2\bar{q}_j P_L q_i =$$

$$= 2\bar{q}_j \frac{1}{2}(1-\gamma^5) q_i =$$

$$= \bar{q}_j q_i + i\, \bar{q}_j (i\gamma^5) q_i$$

$$= S_{ij} + i P_{ij}$$

Ergo:

$$\boxed{\Phi = S + iP}$$

Parity:

$$\phi \mapsto \phi^\dagger(t, -\vec{x})$$

Charge conjugation:

$$\phi \mapsto \phi^T$$

## Flavor transformation:

$$\Phi \mapsto U_V \phi U_V^\dagger \qquad \left( \begin{array}{l} P \mapsto U_V P U_V^\dagger \\ S \mapsto U_V S U_V^\dagger \end{array} \right)$$

## Axial transformation

$$\Phi \mapsto U_A \phi U_A$$

[check → not a group]

How does $P$ transform under axial transf. ?

$$P = \frac{1}{2i}\left(\phi - \phi^{+}\right)$$

$$\mapsto \frac{1}{2i}\left(U_A \phi U_A^{+} - U_A^{+} \phi^{+} U_A^{+}\right)$$

$$= \frac{1}{2i}\left(U_A (S+iP) U_A^{+} - U_A^{+} (S-iP) U_A^{+}\right) =$$

$$P \mapsto \frac{1}{2} U_A P U_A^{+} + \frac{1}{2} U_A^{+} P U_A^{+} + \frac{1}{2i}\left(U_A S U_A - U_A^{+} S U_A^{+}\right).$$

$$P \mapsto \text{"}P\text{"} + \text{"}S\text{"}$$

The axial transf. mixes $P$ and $S$.

This is expected because the axial transf. changes parity (it adds a $\gamma^5$).

If we consider an infinitesimal transf:

$$U_A = e^{i\theta^a t^a} \simeq 1 + i\theta^a t^a$$

$$P \mapsto \frac{1}{2}\Big( (1+i\theta^a t^a) P (1+i\theta^a t^a) \Big) + \frac{1}{2}\Big( (1-i\theta^a t^a) P (1-i\theta^a t^a) \Big)$$

$$+ \frac{1}{2i}\Big( (1+i\theta^a t^a) S (1+i\theta^a t^a) - (1-i\theta^a t^a) S (1-i\theta^a t^a) \Big)$$

$$= P + \theta^a \{ t^a, S \}$$

$$\boxed{P \mapsto P + \theta^a \{ t^a, S \}}$$

Note, for $N_f = 2$ we have the following nice picture:

$$\begin{tikzcd} \sigma \arrow[d, leftrightarrow] \arrow[dr, dotted, leftrightarrow, "U_A(1)"] & \vec{a}_0 \arrow[d, leftrightarrow] \arrow[dl, dotted, leftrightarrow] \\ \vec{\pi} & \eta \end{tikzcd}$$

$SU_A(2) \updownarrow$ $\quad$ $U_A(1) \updownarrow$

(Because of this: I can construct a chiral model with $\sigma, \vec{\pi}$ only and without $\vec{a}_0$ and $\eta$... this fact has been

$$\begin{tikzcd} \vec{\rho} \arrow[d, leftrightarrow] & \omega \arrow[d, leftrightarrow] \\ \vec{a}_1 & f_1 \end{tikzcd}$$

Finally we are in the position to write down the $\sigma$-model (with glueball/dilaton) following:

- dilatation invariance (together with analiticity)
- chiral symmetry

$$\boxed{\mathcal{L} = \mathcal{L}_{dil} + \frac{1}{2} \mathrm{Tr}\left[ (\partial^\mu \Phi)^\dagger (\partial_\mu \Phi) - aG^2 \Phi^\dagger \Phi - \frac{\lambda_2}{2} (\Phi^\dagger \Phi) \right] - \frac{\lambda_1}{4} \left( \mathrm{Tr}\left[ \Phi^\dagger \Phi \right] \right)^2 + \mathcal{L}_{SB} + \mathcal{L}_{U_A(1)}}$$

- $\mathcal{L}_{SB} = \frac{1}{2} \mathrm{Tr}\left[ H (\Phi + \Phi^\dagger) \right]$

$H = \mathrm{diag}\{ h_0^1, h_0^2, \ldots, h_0^N \}$

- $h_0^i \propto m_i^2$.

$$\mathcal{L}_{U_A(1)} = c\,(\det \Phi - \det \Phi^\dagger)^2$$

Notice that the various terms are invariant under chiral transformation.

$$G^2 \mathrm{Tr}[\phi^\dagger \phi] \mapsto G^2 \mathrm{Tr}[U_R \phi^\dagger U_L^\dagger U_L \phi U_R^\dagger] = G^2 \mathrm{Tr}[\phi^\dagger \phi]$$

Similarly for:

$$\mathrm{Tr}[\phi^\dagger \phi \phi^\dagger \phi]$$

It is also easy to check that P and C are fulfilled.

- $\mathcal{L}_{SB}$ breaks chiral symmetry

$$\mathrm{Tr}[H(U_L \phi U_R^\dagger + U_R \phi^\dagger U_L^\dagger)] \neq \mathrm{Tr}[H(\phi + \phi^\dagger)]$$

Note:
if $H = h_0 \cdot 1_{N_f}$ $\mapsto$ we still have $U(N_f)_V$ - invariance.

The term $\mathcal{L}_{U_A(1)}$ is "peculiar".

Notice that

$$\det\phi \mapsto \det(U_L \phi U_R^\dagger) = \det U_L \det\phi \det U_R^\dagger$$

- Thus is $= \det\phi$ if we consider $SU(N_f)_R \times SU(N_f)_L$ but not for $U(N_f)_R \times U(N_f)_L$.

- In particular, $\det\phi$ is not invariant under $U_A(1)$:

$$U_L^\dagger = U_R = e^{i\alpha} \cdot 1_{N_f}.$$

$$\det U_R^\dagger = e^{-i\alpha \cdot N_f}$$

Ergo:

$$\det\phi \mapsto e^{-2i\alpha N_f} \cdot \det\phi$$

(This is indeed a general transf.)

(One gets a mass for the $\eta$ field... $\eta_N$ and $\eta_S$).

Dilatation invariance and analiticity prohibite terms of the type

$$\kappa\left(\mathrm{Tr}[\phi^{+}\phi]\right)^{6}$$

$\Downarrow$

dimension $[\mathrm{Energy}]^{-2}$.

Namely, in order to have dil. inv. one has

$$\frac{\alpha}{G^{2}}\left(\mathrm{Tr}[\phi^{+}\phi]\right)^{2} \to$$ but it explodes for $G \to 0$. Analiticity is broken.

No further freedom is possible in the Lagrangian as long as only $\phi$ is involved.

Let us study the field condensation. $N_f = 2$.

$$\phi = \sigma \left(\sqrt{2}\, t^0\right)$$

$$V(G,\sigma) = V_{dil}(G) + a\, G^2 \sigma^2 + (\lambda_1 + \lambda_2)\, \sigma^4$$

$$V_{dil}(G) = \frac{1}{2} \frac{m_G^2}{\Lambda^2} G^4 \left( \ln \frac{G}{\Lambda_G} - \frac{1}{4} \right)$$

In principle, full minimisation of $G$ and $\sigma$...

but let us take the easy way

(Enough for our purposes)

$G$ condenses in view of $V_{dil}(G)$.

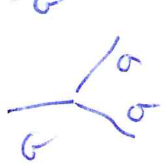
$$G = \Lambda_G + G$$

$$V_{\rm eff}(G) = \frac{1}{2} m_G^2 G^2 + \# G^4 + \dots$$

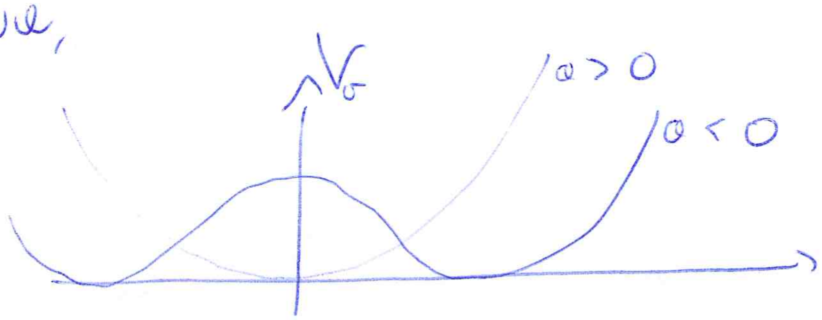
But then the $\sigma$-part becomes:

$$V = a(\Lambda_G + G)^2 \sigma^2 + (\lambda_1 + \lambda_2)\sigma^4$$

$$= \underbrace{a \Lambda_G^2 \sigma^2 + (\lambda_1 + \lambda_2)\sigma^4}_{V_\sigma} + 2a\Lambda_G G \sigma^2 + \dots$$




Now,




$a < 0$ is realised in nature.

$$\partial_\sigma V_\sigma = +2a\Lambda_G^2\sigma + 4(\lambda_1+\lambda_2)\sigma^3 = 0$$

$\sigma = 0$ is always a sol.

$$\boxed{\sigma^2 = \frac{-2a\Lambda_G^2}{4(\lambda_1+\lambda_2)} > 0 \text{ for } a < 0.}$$

$$\boxed{\sigma_0 \approx f_\pi = \sqrt{\frac{-a}{2(\lambda_1+\lambda_2)}}\,\Lambda_G^2}$$

$$\boxed{\sigma_0 = \phi \sim G_0}$$

If $G_0 = 0 \rightarrow$ no "condensate".

Due to $aG^2\sigma^2 \mapsto$ the condensate of $\sigma$ is only possible if "first" the dilaton has condensed "already".

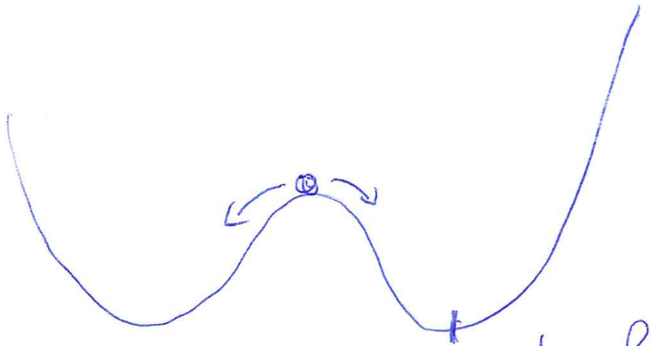
(of course, all takes place simultaneously... Full minimization... But for $G_0 = 0 \rightarrow \sigma = 0$, ~~but~~ not viceversa).

Breaking of symmetry → $\sigma$ gets a vev!

$\sigma$ is scalar, that is why ... (indeed, care is needed... see later on

$J^{PC}=0^{++}$

(Also $\pi^0$ could condense....)



$\sigma_0 = \phi \sim f_\pi \neq 0$.

Buridan, the donkey and the usual problem of philosophy: good questions, wrong answers.

Second example: the elegant dinner (A. Salam) and the realization with Fowler and the Pope!!!!!

(Axial-) Vector sector

Similarly, in the vector sector one defines

$$R^{\mu}_{ij} = 2\,\bar{q}_{j,R}\,\gamma^{\mu} q_{i,R}$$

$$L^{\mu}_{ij} = 2\,\bar{q}_{j,L}\,\gamma^{\mu} q_{i,L}$$

The transformations are:

$$\begin{cases} R^{\mu} \mapsto U_R R^{\mu} U_R^{+} \\ L^{\mu} \mapsto U_L L^{\mu} U_L^{+} \end{cases}$$

→ extension to vector mesons

- IMPORTANCE OF THIS OPERATION FOR PHENOMENOLOGY.
- "COMPLETE AS POSSIBLE" DESCRIPTION.

$$R^{\mu}_{ij} = 2\bar{q}_j P_L \gamma^{\mu} P_R q_i = 2\bar{q}_j P_L \gamma^{\mu} q_i$$

$$= \bar{q}_j \gamma^{\mu} q_i - \bar{q}_j \gamma^5 \gamma^{\mu} q_i$$

$$= V_{ij} - A_{ij}$$

$$R = V - A$$

$$\begin{cases} L^{\mu}_{ij} = 2\bar{q}_j P_R q_i = V_{ij} - A_{ij} \\ L = V + A \end{cases}$$

→ see file.