Large N_c

- $N_c = 3$ nr of colors $\longmapsto$ N_c becomes a variable, which is sent to ∞ : $N_c \longmapsto \infty$

- $SU(N_c = 3)$ $\longmapsto$ $SU(N_c)$

- 'Large N_c' expansion : $A = a + \frac{b}{N_c} + \frac{c}{N_c^2} + \cdots$

  one may object: $N_c = 3$ is not large!

  However : $\rightarrow \frac{1}{N_c}$ in the amplitude $\longmapsto \frac{1}{N_c^2}$ in the width or cross section

  $\rightarrow e_{QED} \sim 1/3$

  $\rightarrow$ 'Way out' to study strongly coupled QCD where no expansion parameter existed

- Many phenomenological properties of QCD can be explained only in the large-N_c framework (OZI rule, succes of quark model, ...)

$SU_c(3)$

$|q\rangle = \begin{pmatrix} R \\ G \\ B \end{pmatrix}$; $|q\rangle \mapsto U|q\rangle$ $U \in SU_c(3)$

q_a $a = 1, 2, 3$

$|MESON\rangle = \sqrt{\frac{1}{3}}(\bar{R}R + \bar{G}G + \bar{B}B)$

invariant under $SU_c(3) \equiv$ 'white'

$|BARYON\rangle = N \cdot \varepsilon_{abc}\, q^a q^b q^c$

$= N \cdot \boxed{(RGB + BRG + GBR - GRB - BGR - RBG)}$

invariant under $SU_c(3)$

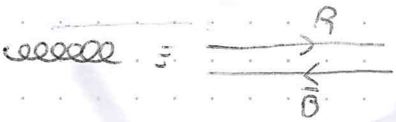
$|BARYON\rangle \mapsto |BARYON$

DIAGRAM:

$a = 1, 2, 3$

GLUON:

$A_\mu^\lambda \equiv \lambda = 1, 2, \ldots, 8$



9 combination (-1)
(double line notation)

$SU_c(N_c)$

$|q\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N_c} \end{pmatrix}$; $|q\rangle \mapsto U|q\rangle$ $U \in SU(N_c)$

q_a $a = 1, 2, \ldots, N_c$

$|MESON\rangle = \sqrt{\frac{1}{N_c}}(\bar{c}_1 c_1 + \ldots + \bar{c}_{N_c} c_{N_c})$

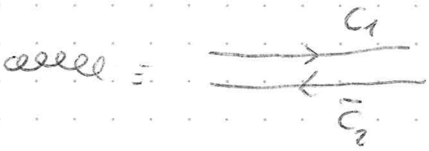
$|BARYON\rangle = N\, \varepsilon_{a_1 \ldots a_{N_c}}\, q^{a_1} q^{a_2} \ldots q^{a_{N_c}}$

$a_1, \ldots, a_{N_c} = 1, 2, \ldots, N_c$

invariant under $SU_c(N_c)$

$a = 1, 2, \ldots, N_c$

$A_\mu^\lambda \equiv \lambda = 1, 2, \ldots, N_c^2 - 1 \simeq N_c^2$



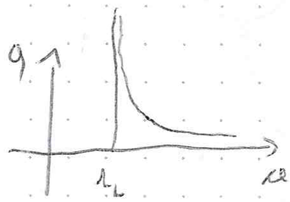
$A_\mu^{(12)}$

$N_c^2 (-1)$ combination

$A_\mu^{(a,b)}$ $a, b = 1, \ldots, N_c$

In QCD there is a coupling constant $g = g(\mu^2)$ (after renormalisation

$$g^2(\mu^2) = \frac{g_0^2}{1 + \frac{g_0^2}{8\pi^2}\left(\frac{11}{3} N_c - \frac{2}{3} N_f\right) \ln\left(\frac{\mu}{\mu_0}\right)}$$

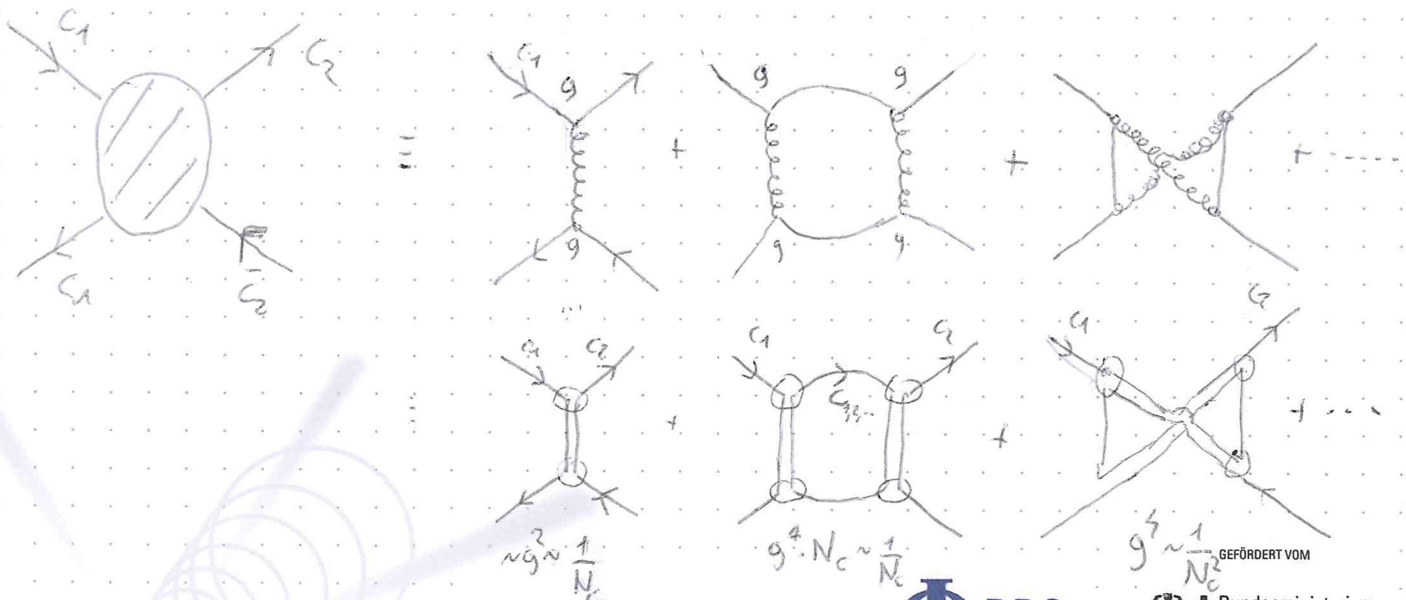


It is easy to perform the calculation in N_c, which amount to a simple change.

We thus see that $g \propto \frac{1}{\sqrt{N_c}}$

This is an important point, which allows how to calculate diagrams.

As an example: $c_1 \bar{c}_1 \mapsto c_2 \bar{c}_2$



PLANAR PLANAR NON-PLANAR

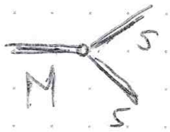
(1) In the large N_c limit only planar diagrams survive!
(2) " " " " " no virtual quarks are needed!

This is a great simplification: a large class of diagrams can be neglected!

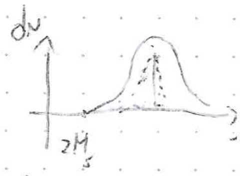
However: QCD cannot be solved in this limit. Also, confinement cannot be proven.

But certain properties of low-energy QCD can be indeed seen in the large N_c expansion:

- Be Q a $\bar{q}q$ state: $M_Q \propto N_c^0$ (constant in the large N_c limit
- Q, S $\bar{q}q$ states, $\mathcal{L} \sim QS^2$

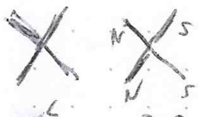


$A \propto \frac{1}{\sqrt{N_c}}$; $\Gamma_{Q \mapsto SS} \propto \frac{1}{N_c}$



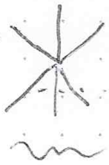
$d_Q \mapsto \delta(x - M_Q)$

- 4-th order interaction



Q^4 $\quad Q^2 S^2$ $\quad A \sim \frac{1}{N_c}$

- n-th leg interaction N_L



N_L $\quad A \sim \left(\frac{1}{\sqrt{N_c}}\right)^{N_L - 2}$

- Be G a glueball: $M_G \propto N_c^0$
- $G \mapsto SS$ $\quad \Gamma_{G \mapsto SS} \propto \frac{1}{N_c^2}$ "the glueball is very narrow in the large N_c limit"
- No tetraquark or multiquark states
- QCD is very simple in the large-N_c world: NON-INTERACTING RESONANCES $\bar{q}q$ and gg

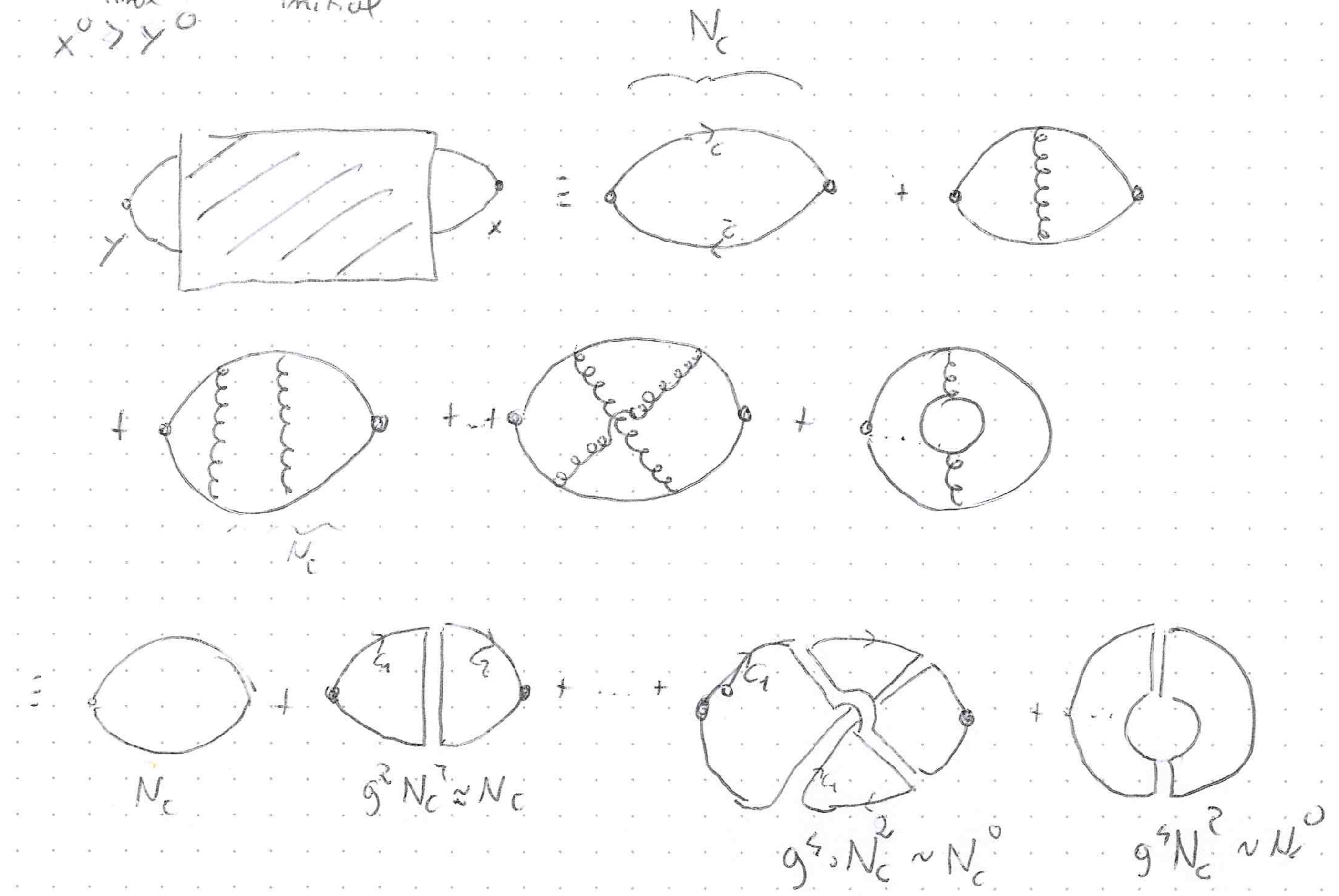
Let us consider a local quark current (of course 'white')

$J(x) = \bar{q}(x)\,\Gamma\, q(x)$ $\qquad$ $\Gamma = 1, i\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \ldots$

$J = (\bar{c}_1 c_1 + \bar{c}_2 c_2 + \cdots + \bar{c}_{N_c} c_{N_c})$

$\langle 0|T\underbrace{J(x)}_{\text{final}}\,\underbrace{J(y)}_{\text{initial}}|0\rangle = \langle 0|J(x)J(y)|0\rangle$

$x^0 > y^0$

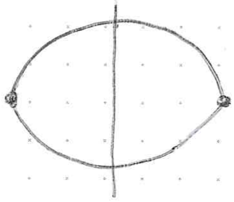


SUPPRESSED! $\qquad$ SUPPRESSED

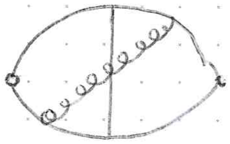
Thus: $\langle 0|J(x)J(y)|0\rangle \sim N_c$

Let us imagine to cut each of our diagrams in the "middle"!

For all planar diagram we always have a $\bar{q}q$ state as 'white' irreducible color state. it is not possible to have an intermediate state something like $(\bar{q}q)\cdot(\bar{q}q)$ state!



$= \bar{q}_a q_a$



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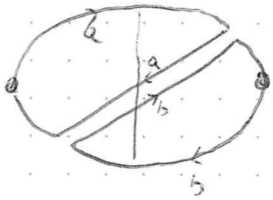
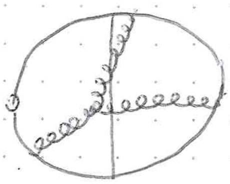


$= \bar{q}_a A_\mu^{(ab)} q_b = \bar{q} q$ meson "due to confinement"



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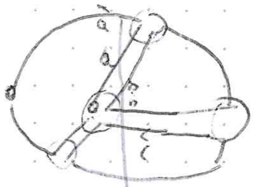


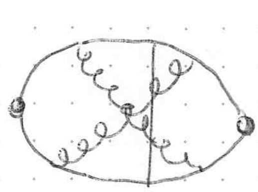
$= \bar{q}_a A_\mu^{(ab)} A_\nu^{(bc)} q_c = \bar{q} q$ meson

$g^4 \cdot N_c^3 \sim N_c$

IT IS IMPOSSIBLE TO OBTAIN SOMETHING LIKE $\bar{q}q \cdot gg$: $\underbrace{(\bar{q}_\ell q^\ell)}_{\text{white}} \cdot \underbrace{(A^{(m,j)} A^{(j,m)})}_{\text{also white}}$

I'd have meson glueball

We always have an irreducible $\bar{q} q$ state.



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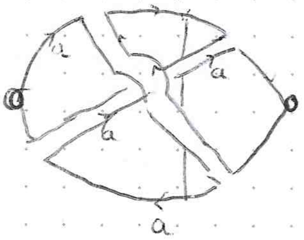
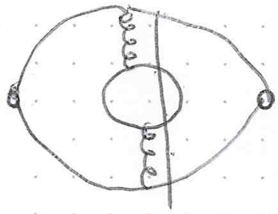
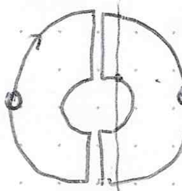


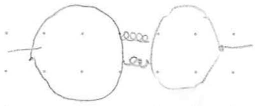


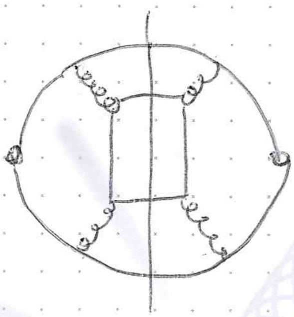


$(\bar{q}_a q)(\bar{q}_b ...)$

$g^4 N_c^2 = N_c^0$

SUPPRESSED





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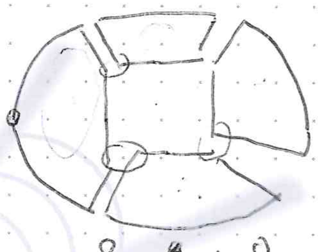
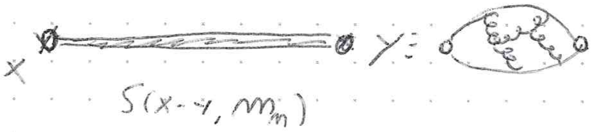


$g^8 N_c^4 = N_c^0$

SUPPRESSED!

$\langle 0|J(x)\,J(y)|0\rangle \sim N_c$ and only mesons are intermediate states!

There are, of course, many mesons with the correct quantum numbers!



In this limit $J(y)|0\rangle = \sum_n a_n b_n^\dagger(y)|0\rangle = \sum_n a_n |n, y\rangle$

$$\langle 0|J(x)J(y)|0\rangle = \sum_n |a_n|^2 \cdot S_n(x-y) = \sum_n |a_n|^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m_n^2} e^{ik(x-y)}$$

$$\sim \underline{N_c}$$

Only one solution is possible:

$a_n \propto \sqrt{N_c}$ $\quad a_n\,\delta(x-y) = \langle n, x| J(y) |0\rangle$; $\boxed{a_n = \langle n|J|0\rangle}$ $\sim f_\pi$

$m_n \propto N_c^0$

Moreover: $\langle 0|J(x)J(y)|0\rangle \sim \int \frac{d^4k}{(2\pi)^4} \left[\# \ln \frac{k^2}{\Lambda^2}\right] e^{ik(x-y)}$ in PQCD ($|k|^2 \gg \Lambda_{QCD}$)

then, an infinity number of mesons are needed $\sum_n = \sum_{n=1}^{\infty}$!

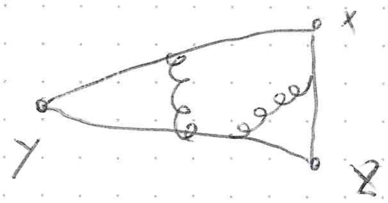
For each given quantum numbers J^{PC} there is an infinity number of mesons!

Calculating:

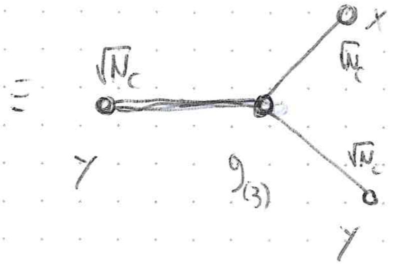
$m_n \sim N_c^0 \quad n = 0, \dots, \infty$

One can go further:

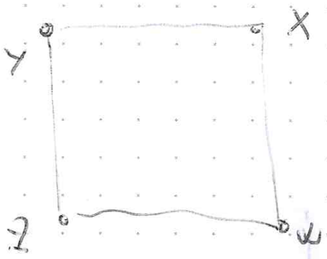
$\langle \alpha | J(y)\, J(x)\, J(z) | 0 \rangle$

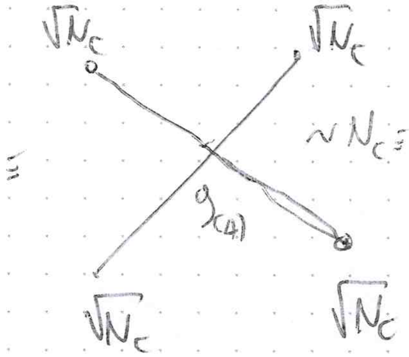


$\sim N_c$ At each intermediate

$\equiv$ 

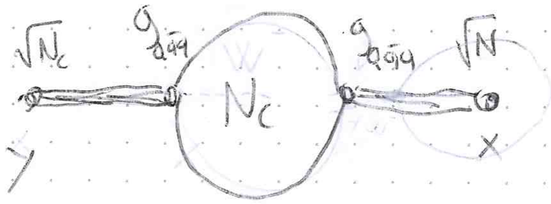
$(\sqrt{N_c})^3 \cdot g_{(3)} = N_c \longrightarrow g_{(3)} \sim \frac{1}{\sqrt{N_c}}$



$\equiv$ 

$\sim N_c = (\sqrt{N_c})$ $g_{(4)} \sim \frac{1}{N_c}$

- ρ_π



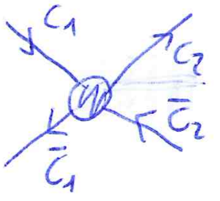
$\equiv g N_c \equiv g_{Q\bar{q}q}^2 \cdot N_c (\sqrt{N_c})^2$

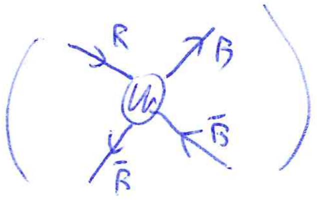
Y ≡ "we create a meson"

X ≡ W weak boson

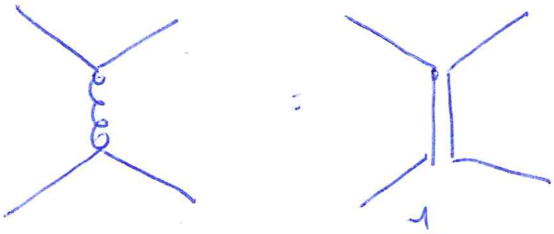
$g_{Q\bar{q}q} \propto \frac{1}{\sqrt{N_c}}$

Alternative derivation: $M_Q \sim N_c^0$

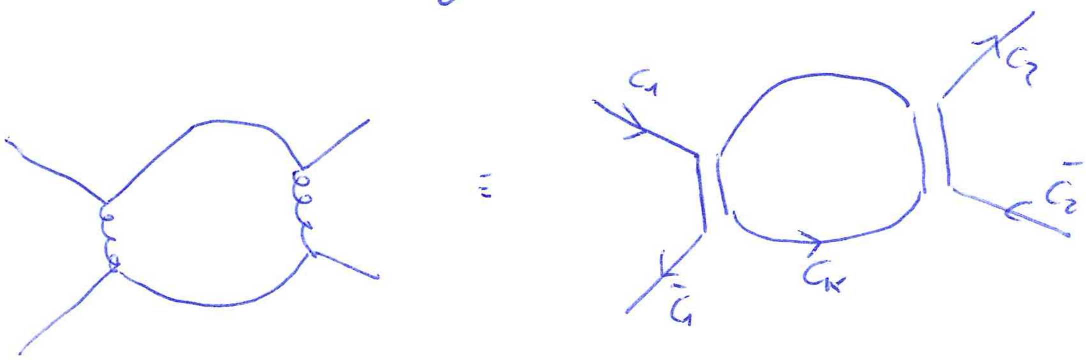




This goes as

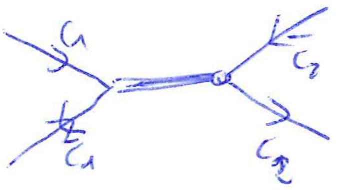


$\frac{1}{N_c}$



$$\frac{1}{N_c} \cdot N_c \cdot \frac{1}{N_c} = \frac{1}{N_c}$$

All that sum up to a bound state:



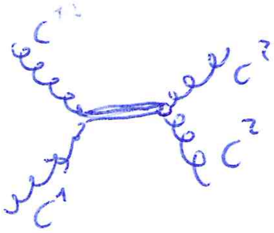
$$= g^2_{Q\bar{q}q} \frac{1}{p^2 - M^2} \propto \frac{1}{N_c}$$

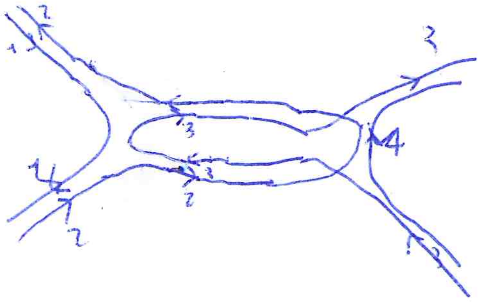
$$M \propto N_c^0$$

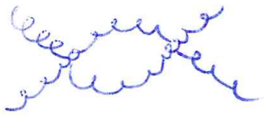
$$g_{Q\bar{q}q} \sim \frac{1}{\sqrt{N_c}}$$

The proof of glueballs is "longer" ... but goes in a similar way.

$C^a \quad a = 1 \dots N_c^2 - 1$

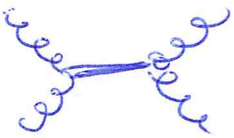






$$\frac{1}{N_c^2}$$

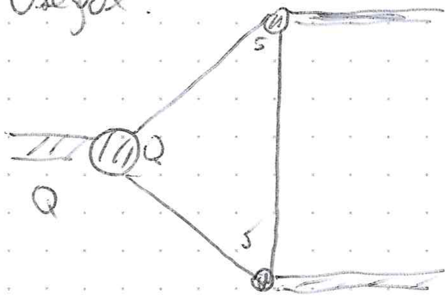
$$\frac{1}{N_c^2} + \frac{1}{N_c^2} \cdot N_c^2 \frac{1}{N_c^2} + \dots = \frac{1}{N_c^2} =$$



$$\propto \frac{1}{N_c} \cdot \frac{1}{p^2 - M_G^2} \frac{1}{N_c} = \frac{1}{N_c^2}$$

$$\boxed{g_{Ggg} \propto \frac{1}{N_c}}$$

Useful.

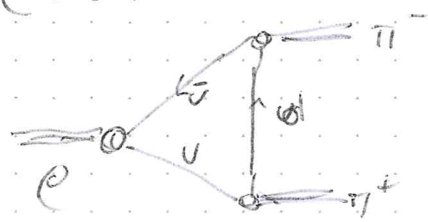


$Q \mapsto s\bar{s}$

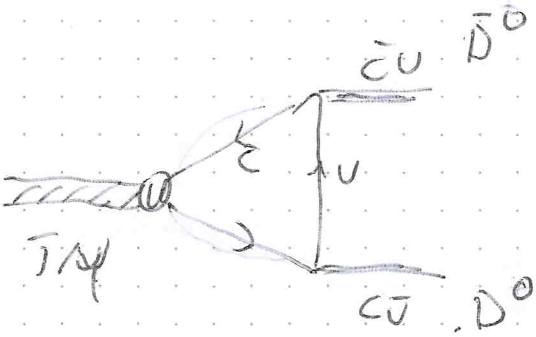
$$A \sim \left(\frac{1}{\sqrt{N_c}}\right)^3 \cdot N_c = \boxed{\frac{1}{\sqrt{N_c}}}$$

$$\Gamma \sim \frac{1}{N_c}$$

$\rho \mapsto \pi\pi$



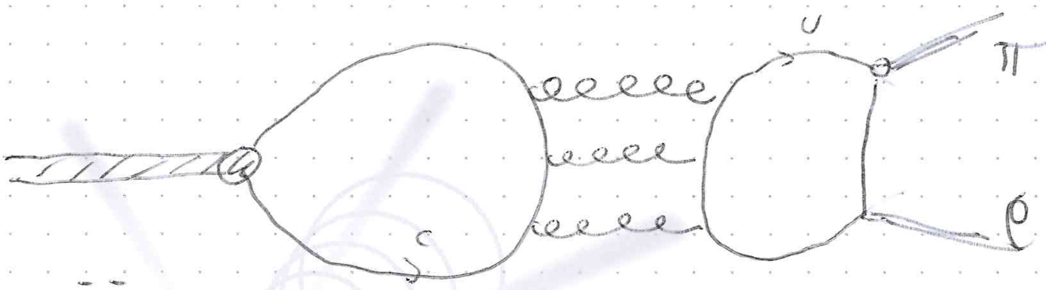
But what does it happen if such a decay is kynematically forbidden?



$2M_D > M_{J/\psi}$

forbidden!

How can J/ψ decays?



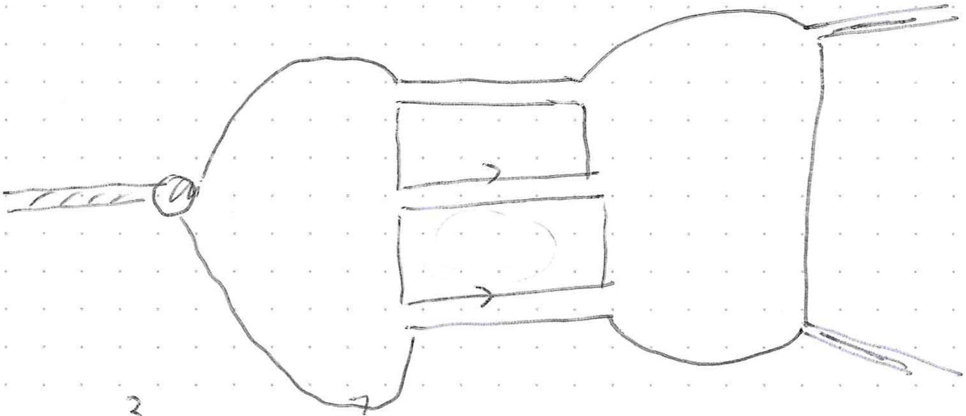
$\Gamma_{J/\psi} = 93.2$ keV

(87% in hadrons

13% with γ

1^{--}

OZI suppressed!!

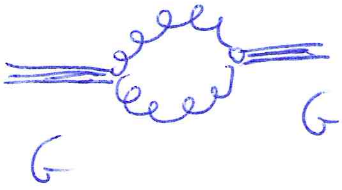


$$A \sim \left(\frac{1}{\sqrt{N_c}}\right)^3 \cdot g^6 \cdot N_c^3 = \frac{1}{N_c\sqrt{N_c}}$$

suppressed of a factor N_c with respect to the direct decay
without transverse gluons!

(that is, only of a factor $1/N_c$ ...)

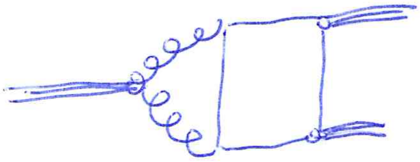
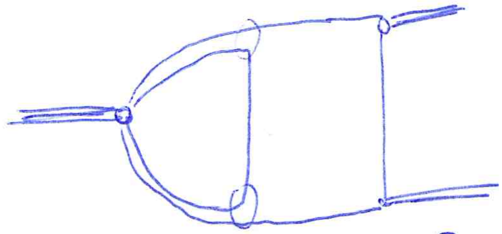
Glueball



$$M_G^2 \sim \frac{1}{N_c} \cdot N_c^2 \cdot \frac{1}{N_c} \simeq N_c^0 \quad !$$

$$g_{Gqq} \propto \frac{1}{N_c}$$

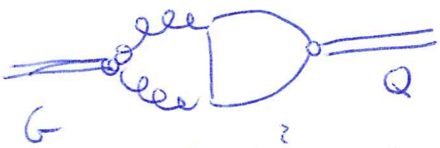
Glueball decay:

 = 

$$\frac{1}{N_c}\left(\frac{1}{\sqrt{N_c}}\right)^2\left(\frac{1}{\sqrt{N_c}}\right)^2 \cdot N_c^2 = \frac{1}{N_c}$$

$$\Gamma_{G \mapsto QQ} \propto \frac{1}{N_c^2}$$ (Here ↦ wave ... conflicting evidence ...).

Mixing



$$\frac{1}{N_c} N_c^2 \frac{1}{\sqrt{N_c}}\left(\frac{1}{\sqrt{N_c}}\right)^2 = \frac{1}{\sqrt{N_c}}$$

EXAMPLE 1:

σ-model:

$$V = \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2-F^2\right)^2 = \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2\right)^2 - 2F^2\frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2\right)$$

$$\lambda \propto \frac{1}{N_c}$$

$$F^2\lambda \propto 1 \mapsto F \propto \sqrt{N_c}$$

$$\boxed{\sigma_0 = F \propto \sqrt{N_c}}$$

$\sigma \mapsto \sigma + F$

$$V = \frac{\lambda}{4}\left(\cancel{F^2} + 2F\sigma + \sigma^2 + \vec{\pi}^2 \cancel{- F^2}\right)^2$$

$$= \frac{\lambda}{4}\left(4F^2\sigma^2\right) + \frac{\lambda}{4}\cdot 4F\sigma\,\vec{\pi}^2 + \dots$$

$$M_\sigma \simeq \lambda F^2 \sim N_c^0$$

$$A_{\sigma\to\pi\pi} \simeq \lambda F \sim \frac{1}{\sqrt{N_c}}$$ Just right for $\bar{q}q$ states!

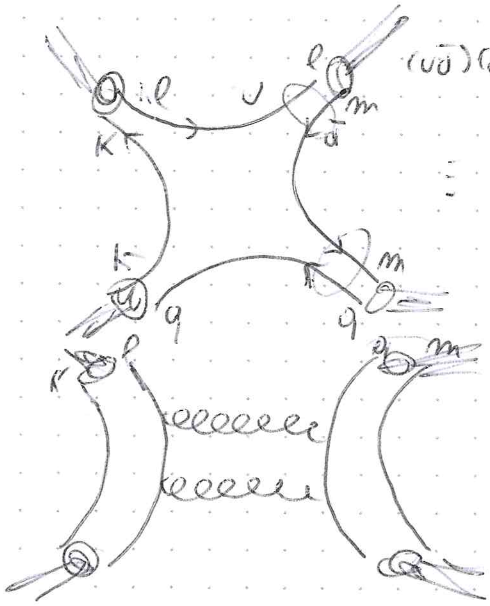
EXAMPLE 2

Our model:

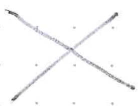
$\phi = S + iP$

$\lambda_2 \, \mathrm{Tr}\left[(\phi^\dagger\phi)^2\right]$

$\lambda_1 \left(\mathrm{Tr}\left[\phi^\dagger\phi\right]\right)^2$



$(u\bar{u})(d\bar{u})(u\bar{d})(d\bar{u})$

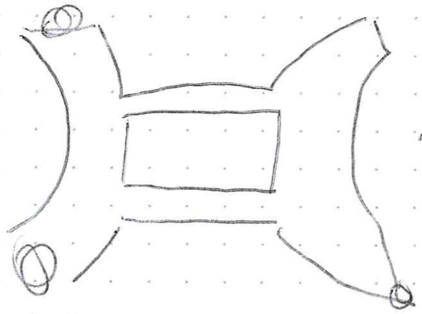


$\equiv \lambda_2 \sim \frac{1}{N_c}$

$g^4 \left(\frac{1}{\sqrt{N_c}}\right)^4 N_c^2 \sim \frac{1}{N_c^2}$

$\phi^\dagger_{kl}\,\phi_{lm}\,\phi^\dagger_{mq}\,\phi_{qk}$

$\phi^\dagger_{kl}\,\phi_{lk}\;\phi^\dagger_{mq}\,\phi_{qm}$



$\sim \lambda_1 \sim \frac{1}{N_c^2}$

The same holds for the h-terms!