INTRODUCING QUARKS

$$\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}\left[G_{\mu\nu} G^{\mu\nu} \right]$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_0 [A_\mu, A_\nu]$$

$$A_\mu = A_\mu^a t^a \qquad a = 1, \dots, N_c^2 - 1 \qquad (N_c = 3)$$

$$\boxed{\mathcal{L}_{QCD} = \sum_{i=1}^{N_f} \mathrm{Tr}\left[\bar{q}_i (i \gamma^\mu D_\mu - m_i) q_i \right] + \mathcal{L}_{YM}}$$

$$D_\mu = \partial_\mu - i g_0 A_\mu$$

$$\mathrm{Tr}[t^a t^b] = \frac{1}{2} \delta^{ab}$$

The Lagrangian is invariant under local gauge transformation of the color group:

$$\begin{cases} q_i \mapsto U(x) q_i \\ A_\mu \mapsto U A_\mu U^\dagger - \frac{i}{g_0} U(x) \partial_\mu U^\dagger(x) \end{cases}$$

$q_i(x)$ is the quark field.

What does it mean to perform a parity transformation?

$$\boxed{\begin{gathered} \vec{x} \mapsto -\vec{x} \\ q_i'(t,\vec{x}) = \gamma^0 q_i(t,-\vec{x}) \end{gathered}}$$

Dirac

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Particle $(+1)$

Antiparticle (-1)

This is the origin of the intrinsically $\neq$ parity of a particle and antiparticle.

Charge Conjugation

$$q(t,\vec{x}) \mapsto q'(t,\vec{x}) = C\bar{q}^{t}(t,\vec{x})$$

$$\boxed{q' = C\bar{q}^{t}}$$

C is such that

$$C^{-1}\gamma^{\mu}C = (-\gamma^{\mu})^{t}$$

In the Dirac notation $C = -i\gamma^{2}\gamma^{0}$

$$C^{+} = C^{-1} = -C$$

There is however a tremendous subtlety...

For a "classical" Dirac field (also in first quantisation)
(in the Dirac rep.)

$$\bar{\psi}^t = (\psi^\dagger \gamma^0)^t = \gamma^{0\dagger} \psi^{\dagger t} = \gamma^0 \psi^*$$

But this is not true when the field is quantised.

For a quantised field we should use the expression

$$\psi^c = C \bar{\psi}^t$$

and "t applies only to the spinors u and v but not to the operators b_k and d_k.

Nonet of pseudoscalar fields

$$P_{ij} = \bar{q}_j \, i\gamma^5 q_i \qquad i, j = u, d, s$$

$$P =' \begin{pmatrix} \bar{u}\, i\gamma^5 u & \bar{d}\, i\gamma^5 u & \bar{s}\, i\gamma^5 u \\ \bar{u}\, i\gamma^5 d & \bar{d}\, i\gamma^5 d & \bar{s}\, i\gamma^5 d \\ \bar{u}\, i\gamma^5 s & \bar{d}\, i\gamma^5 s & \bar{s}\, i\gamma^5 s \end{pmatrix}$$

$$\equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_u}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_u}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \eta_s \end{pmatrix}$$

$\hookrightarrow$ Achtung: same "symmetries and transformations" ... but not actually the same thing.

$J = 0$

Namely, P_{ij} is a scalar object. There is "no transformation" under Lorentz transformation. It is invariant.

Note:
The matrix P appears in linear and nonlinear chiral representations.
(In ChPT the matrix P is the basic ingredient).

Parity:

$$\bar{q}_j i\gamma^5 q_i \longmapsto \bar{q}_j \gamma^0 i\gamma^5 \gamma^0 q_i = - \bar{q}_j \gamma^0 q_i$$

$P = -1.$

The parity is negative... we have a pseudoscalar

Charge conjugation

$$q_i \mapsto C\bar{q}^t = C(q^+\gamma^0)^t = C\gamma^{0t} q^{+t}$$

$$q_i^+ \mapsto (C\bar{q}^t)^+ = \bar{q}^{t+} C^+ = [(q^+\gamma^0)^t]^+ C^+ = [\gamma^{0t} q^{+t}]^+ C^+$$

$$= q^t \gamma^{0*} C^+ =$$

$$\bar{q}_i = q_i^+\gamma^0 \mapsto q_i^t \gamma^0 C^+ \gamma^0 = - q_i^t C^+$$

$$\bar{q}_j i\gamma^5 q_i \mapsto - q_j^t C^+ i\gamma^5 C \bar{q}_i^t = - q_j^t i\gamma^{5t} \bar{q}_i^t = \bar{q}_i i\gamma^5 q_j$$

Ergo:

$$P_{ij} \mapsto P_{ji} \Rightarrow \boxed{P \mapsto P^t}$$

Ergo, the elements on the diagonal are eigenstates with eigenvalue + 1!

Those which are not on the diagonal are interchanged

(but still with +1).

Ergo:

$$\boxed{J^{PC} = 0^{-+}}$$

(remember all the discussion of yesterday).

Note: the matrix P is hermitian:

For fixed i and j

$$P_{ij}^{+} = (\bar{q}_j i\gamma^5 q_i)^{+} = q_i^{+}(-i\gamma^{5+})(q_j^{+}\gamma^0)^{+} =$$

$$= q_i^{+}(-i)\gamma^5\gamma^0 q_j = q_i^{+}\gamma^0 i\gamma^5 q_j =$$

$$= \bar{q}_i i\gamma^5 q_j = P_{ji}$$

If now we consider

$$P^{+} = \begin{pmatrix} P_{11} & P_{12} & \cdots \\ P_{21} & & \\ & & \end{pmatrix}^{+} = \begin{pmatrix} P_{11}^{+} & P_{21}^{+} & P_{31}^{+} \\ P_{12}^{+} & & \\ & & \end{pmatrix} = P$$

+ of a matrix of elements: ACHTUNG!!!!

Ergo: $\boxed{P^{+} = P}$

Being the matrix P hermitian it can be expressed as:

$$P = \sum_{i=0}^{8} X^i \overbrace{(\sqrt{2}\, t^i)}^{\text{convention}} = \sum_{i=0}^{8} X^i \frac{\lambda^i}{\sqrt{2}}$$

$\hookrightarrow$ generators of $U(3)$ and basis of Hermitian matrices....

How to determine P^i? Very simple. Multiply by t^j and take the trace.

$$P \cdot t^j = \sum_{i=0}^{8} X^i (\sqrt{2}\, t^i)\, t^j$$

$$\mathrm{Tr}\left[P t^j\right] = \sum_{i=0} X^i \sqrt{2}\, \frac{1}{2}\, \delta^{ij} = X^j$$

$$\boxed{X^j = \sqrt{2}\, \mathrm{Tr}\left[P t^j\right]}$$

X^5 can be also written as:

$$X^5 = \bar{q}\sqrt{2}t^5 q = (\bar{u}, \bar{d}, \bar{s}) \cdot \sqrt{2}t^5 \cdot \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

Namely:

$$X^5 = \sqrt{2}\,\mathrm{Tr}[P t^5] = \sqrt{2} P_{kr} t^5_{rk} = \sqrt{2}\,\bar{q}_r (i\gamma^5) q_k t^5_{rk} =$$

$$= \bar{q}_r \sqrt{2} t^5_{rk} (i\gamma^5) q_k = \mathrm{Tr}[\bar{q}\sqrt{2}t^5 (i\gamma^5) q]$$

qed.

Flavour transformation

$$\boxed{q_i \mapsto U_{ik}\, q_k}$$

$\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ are mixed ....
Norm is preserved!

This is a symmetry of $\mathcal{L}_{QCD}$ if
$m_1 = m_2 \ldots = m_{N_f}$ (see later!)
$m_u \approx m_d$ ✓ ; m_s is already larger; m_c is "be[illegible]

$U \in U(N_f)$

($N_f = 3$ in the examples that we study...)

$$q_1 \mapsto U_{11} q_1 + U_{12} q_2 + U_{13} q_3 \mapsto \begin{cases} q_1^+ \mapsto q_1^+ U_{11}^* + q_2^+ U_{12}^* + q_3^+ U_{13}^* \\ q_1^+ \mapsto q_1^+ U_{11}^+ + q_2^+ U_{21}^+ + q_3^+ q_{31}^+ \end{cases}$$

$$q_1^+ \mapsto q_r^+ U_{r1}^+$$

$$\boxed{q_j^+ \mapsto q_r^+ U_{rj}^+}$$

$$P_{ij} = \bar{q}_j\, i\gamma^5 q_i \mapsto \bar{q}_r^{\,+} U_{rj}^+\, i\gamma^5\, U_{ik} q_k =$$

$$= U_{ik} \left(q_r^+\, i\gamma^5 q_k \right) U_{rj}^+$$

$$= U_{ik}\, P_{kr}\, U_{rj}^+$$

Proof of the invariance of $\mathcal{L}_{QCD}$ under flavour transformation

if $m_i = 0$

$$\sum_i \mathrm{Tr}\left[\bar{q}_i \, i\not{D} \, q_i\right]$$

$$\mapsto \sum_i \left[U_{ik} \, \mathrm{Tr}\left(\bar{q}_r \, i\not{D} \, q_k\right) U^{+}_{ri} \right]$$

$$= \sum_i U_{ik} U^{+}_{ri} \, \mathrm{Tr}\left[\bar{q}_r \, i\not{D} \, q_k\right]$$

$$= \sum_{r,k} \delta_{kr} \, \mathrm{Tr}\left[\bar{q}_r \, i\not{D} \, q_k\right] = \sum_r \mathrm{Tr}\left[\bar{q}_r \, i\not{D} \, q_r\right] \quad \checkmark$$

=

The mass term is in general not invariant:

$$\sum_i m_i \mathrm{Tr}\left[\bar{q}_i q_i\right] \mapsto \sum_i m_i U_{ik} U^{+}_{ri} \, \mathrm{Tr}\left[\bar{q}_r q_k\right]$$

only if $m_1 = m_2 = \dots = m_{N_f}$ it can be factorized out.

Ergo:

$$\boxed{P \mapsto UPU^\dagger}$$

This is also called "adjoint representation".
(In contrast to the fundamental transf. of quarks!)

is the flavor transformation of the matrix P.

Note, if you take $N_f = 2$ one has a isospin transformation

Even more generally, if you set

$$U_I = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

→ This is an isospin transformation because we just rotates u and d.

Flavor invariant object.

$$X^i = \bar{q}\,(\sqrt{2}\,t^i)\,q$$

Under a flavor transformation this object transforms as

$$X^i \mapsto \bar{q}\,U^\dagger(\sqrt{2}\,t^i)\,U\,q$$

In general $U^\dagger \sqrt{2}\,t^i U$ is a combination of t^j..

but for $i=0$ we have that $t^0 = \frac{1}{\sqrt{2N_f}} 1_N \rightarrow$

$$X^0 \mapsto X^0$$

$$X^0 = \bar{q}\,\sqrt{2}\,t^0 q = \bar{q}\,\frac{\sqrt{2}}{\sqrt{2\cdot 3}}\,1\,q = \frac{1}{\sqrt{3}}\left(\bar{u}u + \bar{d}d + \bar{s}s\right)$$

Other nonets : scalar, vector, axial-vector.

Now, there is nothing special in the $i\gamma^5$ ... It affects parity and C, but not flavor transformation.

We can repeat the considerations for $\bar{q}_j \Gamma q_i$.

In particular, we have:

- scalar nonet

$$S_{ij} = \bar{q}_j q_i \qquad J^{PC} = 0^{++}$$

($L = S = 1$ coupled to $J = 0$).

$\to f_0(1370)$, $f_0(1710)$, $K_0^*(1430)$, $a_0(1450)$

- vector nonet

$$V_{ij} = \bar{q}_j \gamma^\mu q_i \qquad J^{PC} = 1^{--}$$

($L = 0$, $S = 1$, coupled to $J = 0$)

ρ, K^*, ω, ϕ

- Axial-vector nonet

$$A_{ij} = \bar{q}_j \gamma^5 \gamma^\mu q \qquad J^{PC} = 1^{++}$$

($L = 1$, $S = 1$ coupled to $J = 1$)

Here we finished the possibilities for the ~~mo~~ matrices... going next means

Example beyond

Pseudo-vector mesons:

$$B_{ij} = \frac{1}{\sqrt{2}}\left(\bar{q}_j \gamma^5 (\partial_\mu q_i) - (\partial^\mu \bar{q}_j) \gamma^5 q_i\right)$$

The ∂_μ [illegible] $L = 1$

$$\overset{L=S=0}{\bar{q}\gamma^5 q} \mapsto \overset{S=0,\, L=1}{\bar{q}\gamma^5 \partial_\mu q}$$