U(N) and SU(N)

Groups

"AT THE BEGINNING THERE IS THE GROUP"

Group $G = (X, \odot)$

where X is a set of elements $\begin{cases} \{\text{banana, apple}, \dots\} \\ \{1, 2, 3, \dots\} \quad , \quad \{\text{matrices}\} \\ \text{or whatever you like} \dots \end{cases}$:)

In general:

$X : \{a, b, c, \dots\}$

and $\odot$ is an internal operation:

$\forall a, b \in X$

$a \odot b \in X$

Then, the 3 following properties must hold:

$$a \odot (b \odot c) = (a \odot b) \odot c$$

$$\exists\, 1_G \,/\, 1_G \odot a = a \odot 1_G = a$$

$$\forall a \in X \; \exists\, a^{-1} \in X \,/\, a \odot a^{-1} = a^{-1} \odot a = 1_G .$$

Examples:

1) $\{ Z = \{0, \pm 1, \pm 2, \ldots\}, + \}$

is a group.

$a + b \in Z$

- $(a+b)+c = a+(b+c)$
- $1_G = 0$

$0 + a = a + 0 = a$

- $a^{-1} = -a$

$+a^{-1} + a = a + a^{-1} = a$

2) $\{R, +\}$ is also such.

$\{N = \{1, 2, 3, \ldots\}, + \}$

is <u>not</u> such.

The internal operation is <u>ok</u>, but the "unity" is not there and also the inverse is not included.

3) $\{ N + \{0\} = \{0, 1, 2, \ldots\}, + \}$

No inverse.

4) $\{\mathbb{R}, \cdot\}$

Is this a group?

- internal operation: ok.

$(a \cdot b) \cdot c = a (b \cdot c)$

$\exists\, 1_G = 1$:

$a \cdot 1 = 1 \cdot a = 1$

But if $a = 0$ this is not the case !!!

Similarly: $0\, 1 = 1 \cdot 0 = 0$!!!

However, for $a = 0 \Rightarrow$ NO INVERSE.

Ergo: $\{\mathbb{R}, \cdot\}$ is not a group.

5) $\{\mathbb{R} - \{0\}, \cdot\}$ is a group.

All the problems above disappear.

U(N)

Let us consider $N\times N$ complex matrices.

$$A = \begin{pmatrix} 1 & 2+i & \cdots & 5-3i \\ \vdots & & & \vdots \\ i & 7 & \cdots & 21-\frac{4}{3}i \end{pmatrix}$$

Out of the set of matrices we restrict our attention to unitary matrices.

def: A unitary $N\times N$ ~~matrix~~ U is a $N\times N$ complex matrix such that

$$\boxed{U^{+}\cdot U = U\cdot U^{+} = 1_N \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}}$$

$$U^{+} = U^{t\,*}$$

Example:

$$U = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

is a unitary 2×2 matrix (actually even $\perp$).

$X = \{$ set of unitary $N \times N$ matrices $\}$

$\cdot$: usual product of matrices

is $U(N) = \{X, \cdot\}$ a group? Yes.

$U_1, U_2 \to U_1 \cdot U_2 \in X$? Yes.

$$(U_1 U_2)^+ U_1 U_2 = U_2^+ \underbrace{U_1^+ U_1}_{1} U_2 = U_2^+ U_2 = 1$$

- $U_1(U_2 U_3) = (U_1 U_2) U_3$
- $1_G = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$
- $U^{-1} \in X$:

$$(U^{-1})^+ U^{-1} = (U^{+t})^+ U^{+t} = U^t U^{+t} = (U^+ U)^t = 1_G^t = 1_G \quad \text{qed.}$$

Ergo, yes $U(N)$ is a group.

Notation: one writes simply $U \in U(N)$...

Importance in Physics:
- whole SM
- QCD $\rightarrow$ color (exact!) ; $\rightarrow$ flavor (approximate $\mapsto$ chiral and flavor symmetries, sym. breaking
- why unitary? QM! Preservation of the norm!

X

$\underline{U = e^{iT}}$

We want to find a clever way to express a unitary matrix U.
For our purposes it is very useful to express U in an exponential form.

$$U = e^{iT} = 1 + iT + \frac{(iT)^2}{2!} + \frac{(iT)^3}{3!} + \cdots$$

(This is always possible

$$U^\dagger = e^{-iT^\dagger}$$

Now, if $T^\dagger = T$

Achtung: $e^A e^B \neq e^{A+B}$
$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]] - [B,[A,B]])} \cdots$

$$U^\dagger U = e^{-iT} e^{iT} \overset{\text{Achtung}}{=} e^{-iT+iT} = e^{0_N} = 1_N. \quad \checkmark$$

T: set of $N\times N$ matrices such that $\{T^\dagger = T\}$.

These are hermitian matrices.

Take a basis of Hermitian matrices: $\{t_0, \ldots, t_m\}$ with $m = N^2 - 1$

where by convention $t_0 = \frac{1}{\sqrt{2N}} 1_N$

$\boxed{\text{Why?}}$

$$T = \vartheta_a t_a$$

$$U = e^{i\vartheta_a t_a}$$

The basis

$$\{t_0, t_1, \dots, t_{N^2-1}\}$$

is chosen such that

$$\mathrm{Tr}[t^a t^b] = \frac{1}{2}\delta^{ab}$$

For $N=3$

$$\boxed{t^a = \frac{\lambda^a}{2}}$$

λ^a Gell-Mann-matrices for $N=3$.

$a = 0, 1, \dots, 8$

$$t^0 = \frac{1}{\sqrt{6}} 1_3$$

For $N=2$

$$\begin{cases} t^a = \frac{1}{2}\sigma^a \\ t^0 = \frac{1}{\sqrt{4}} 1_2 \end{cases}$$

σ^a = Pauli matrices

SU(N):

is analogous to U(N) but there is an additional requirement: $\det U = +1$.

Note that if U is unitary, $U^\dagger U = 1$, it follows that

$$\det(U^\dagger U) = \det 1_N = 1$$

$$\det U^\dagger \det U = 1$$

$$|\det U|^2 = 1$$

$$|\det U| = 1$$

$$\det U = e^{i\phi}$$

Then, SU(N) is the further requirement that the phase is zero.

$$SU(N) = \{X, \cdot\}$$

$$X = \{U \,/\, U^\dagger U = UU^\dagger = 1_N \text{ and } \det U = 1\}$$

Exponential form of SU(N):

$$U = e^{iT}$$

$T \,/\, T^{+} = T$ as before

$$\text{Det}\, U = e^{i\,\text{Tr}[T]} = 1 \quad \text{if } \text{Tr}[T] = 0 .$$

If the basis of Hermitian matrices of the Algebra of $U(N)$ was

$\{t_0, t_1, \ldots, t_m\}$ with $\text{Tr}[t_a t_b] = \frac{1}{2} \delta_{ab}$

$t_0 = \frac{1}{\sqrt{2N}} 1_N \mapsto \text{Tr}[t_0] = \frac{N}{\sqrt{2N}} \neq 0$, $\text{Tr}[t_a] = 0 \quad \forall a = 1, \ldots, m = N^2 - 1$

Then:

$\{t_1, \ldots t_m\}$ with $\text{Tr}[t_a t_b] = \frac{1}{2} \delta_{ab}$

$\text{Tr}[t_a] = 0$

$$U = e^{i\theta_a t_a} \qquad a = 1,$$

i.e. just as $U(N)$ but without the identity matrix $t_0 = \frac{1}{\sqrt{2N}} 1_N$.

Algebra of $SU(N)$:

$$[t_a, t_b] = i f_{abc} t_c$$

Namely, $[t_a, t_b]$ is traceless and anti-hermitian, therefore the validity of the previous equation.

(Lie Algebra: concept which goes beyond our mathematics...

but intuitively it is as follows:

- top. space: a set in which you define neighbourhood and limits.
- Manifold: a space which is locally isomorphic to R^N
- Lie Group: a group and a manifold at the same time.
- Lie Algebra: the "exponential" piece of the Lie Group elements.

The algebra $(A,+)$ is a vect. space on $(R,+,\cdot)$
with the operation $[\cdot,\cdot]: A\times \longrightarrow A$
with

$$\begin{cases} [a,b] = -[b,a] \\ ad[a,[b,c]] = 0. \end{cases}$$

$$A = e^{B} \rightarrow \quad \det A = e^{\ln B}$$

If A is diagonal $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix}$ then $\det A = a_1 a_2 \cdots a_N$

B is also diagonal with

$$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{pmatrix}$$

$$a_K = e^{b_K} \qquad (b_K = \ln a_K)$$

Ergo:

$$\det A = a_1 \cdots a_N = e^{b_1} \cdots e^{b_n} = e^{b_1 + \cdots + b_n} = e^{\mathrm{Tr}\, B}$$

[if A is not diagonal we proceed as follows.

$$U \,/\, U A U^{-1} = D_A = U e^{B} U^{-1} = e^{D_B}$$

$$\det A = \det[UAU^{-1}] = \lambda_1 \cdots \lambda_N = e^{\mathrm{Tr}[D_B]} = e^{\mathrm{Tr}[B]}$$
]

Center Z_N of $SU(N)$

As a last step we study the center of $SU(N)$ because it is in the "center" of many discussions about QCD.

$$C: \{ Z \;/\; Z \text{ is diagonal but belongs to } SU(N) \}$$

A diagonal matrix

$$Z = e^{i\alpha_0 t_0} \in U(N) \text{ but not to } SU(N) \text{ because } \det Z \neq 1 \text{ in general.}$$

Still, there are some exceptions:

$$Z = e^{i\alpha \frac{1}{\sqrt{2N}} \sqrt{2N}\, 1_N} = e^{i\alpha}\, 1_N$$

$$\det Z = e^{iN\alpha} = 1$$

$$\boxed{\alpha = \frac{2\pi n}{N} \qquad n = 0, 1, 2, \ldots, N-1}$$

$$\boxed{Z_n = e^{i\frac{2\pi n}{N}} \cdot 1_N \qquad n = 0, 1, \ldots, N-1}$$

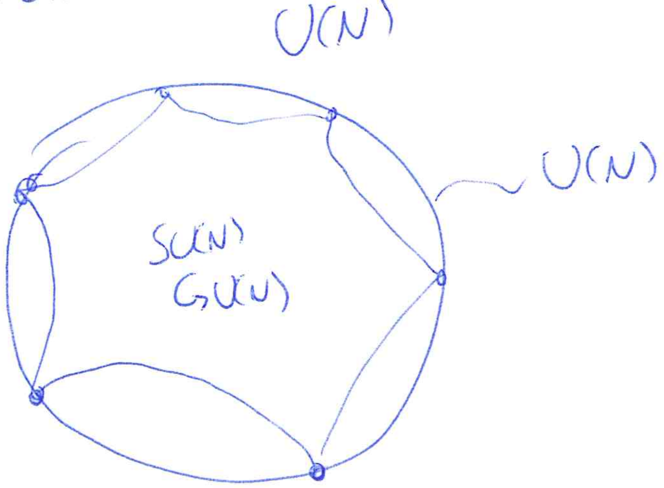
Note

$C = \{ z_{m_i} \}$

is a group.

$z_{m_1} \cdot z_{m_2}$ is an element of C, the identity and the inverse are part of it.

(Prove it as an exercise).



The center elements are special elements which belong to $SU(N)$, although they can be expressed also as

Why relevant in QCD? Indeed, the center is relevant at nonzero temperature. At present, it does only mathematics.

We come back in due time to this point.

Explicitely:

$$\mathbb{Z}_3 = C_{SU(3)} = \{ \mathbb{1}_3 , e^{i 2\pi/3} \mathbb{1}_3 , e^{i 4\pi/3} \mathbb{1}_3 \}$$