Let us now consider the famous chiral symmetry and analyze its spontaneous and explicit breakings.

$$\begin{cases} P_R = \frac{1}{2}(1+\gamma^5) \\ P_L = \frac{1}{2}(1-\gamma^5) \end{cases}$$

$$P_R P_L = 0$$

$$P_R^2 = P_R$$

$$P_L^2 = P_L$$

The quark field can be decomposed as:

$$q_i = q_{i,R} + q_{i,L}$$

Then let us define the transformation $U(N_f)_R$ and $U(N_f)_L$

$$U(N_f)_R \mapsto \ldots$$

$U(N_f)_R$:

$$q_{i,R} \mapsto U^{(R)}_{ij}\, q_{j,R} \qquad U_R \in U(N_f)$$

$$q_i = q_{i,R} \mapsto \underbrace{U^{(R)}_{ij}\, q_{j,R}} + q_{i,L}$$

we rotate only this piece

Easy to check: $U(N_f)_R$ is a group.

$U(N_f)_L$:

$$\boxed{q_{i,L} \mapsto U^{(L)}_{ij}\, q_{j,L}}$$

$$q_i = q_{i,R} + q_{i,L} \mapsto q_{i,R} + U^{(L)}_{ij}\, q_{j,L}$$

This is also a group.

$$U(N_f)_R \times U(N_f)_L$$

$$q_i = q_{i,R} + q_{i,L} \mapsto U^{(R)}_{ij}\, q_{j,R} + U^{(L)}_{ij}\, q_{j,L}$$

We separate the two components...

Is there a symmetry of $\mathcal{L}_{QCD}$?

PARTLY.

$$\sum_i \mathrm{Tr}\left[\bar{q}_i (i\gamma^\mu D_\mu) q_i\right]$$

$$q_i = q_{i,R} + q_{i,L}$$

$$q_i^\dagger = q_{i,R}^\dagger + q_{i,L}^\dagger = [P_R q_i]^\dagger + [P_L q_i]^\dagger$$

$$= q_i^\dagger P_R + q_i^\dagger P_L$$

$$\bar{q}_i = q_i^\dagger \gamma^0 = \bar{q}_i P_L + \bar{q}_i P_R$$

$$\boxed{\bar{q}_{i,R} = \bar{q}_i P_L}$$

$$\boxed{\bar{q}_{i,L} = \bar{q}_i P_L}$$

$$\bar{q}_i i\gamma^\mu D_\mu q_i = (\bar{q}_{i,R} + \bar{q}_{i,L}) i\gamma^\mu D_\mu (q_{i,R} + q_{i,L})$$

Now, the mixed terms vanish:

$$\bar{q}_{i,R} i\gamma^\mu D_\mu q_{i,L} = \bar{q}_i P_L i\gamma^\mu D_\mu P_L q_{i,L} =$$

$$\{\gamma^5, \gamma^\mu\} = 0$$

$$\bar{q}_i \, i\gamma^\mu P_R P_L \, q_i = 0$$

Ergo we are left with:

$$\mathcal{L}_{int} = \sum_i \mathrm{Tr}\left[\bar{q}_{i,R} (i\gamma^\mu D_\mu) q_{i,R} + \bar{q}_{i,L} (i\gamma^\mu D_\mu) q_{i,L}\right]$$

This term is invariant under the separate action of $U(N_f)_R$ and $U(N_f)_L$.

This is chiral symmetry of the QCD Lagrangian.

...but... the mass term delivers an explicit breaking of it.

$$\mathcal{L}_{mass} = \sum_i m_i \bar{q}_i q_i \quad \text{(Tr. over color omitted!)}$$

$$= \sum_i m_i (\bar{q}_{i,R} + \bar{q}_{i,L})(q_{i,R} + q_{i,L})$$

$$= \sum_i m_i (\bar{q}_{i,R} q_{i,L} + \bar{q}_{i,L} q_{i,R})$$

which is <u>not</u> invariant under chiral symmetry.

(Namely: $\bar{q}_{i,R} q_{i,R} = \bar{q}_i \underline{P_L} P_R q_i = 0$!!!!

Only if $m_i = 0$ chiral symmetry is preserved.

$U(N_f)_V \times U(N_f)_A$

$U(N_f)_R \times U(N_f)_L$

$$q_i = q_{i,R} + q_{i,L} \mapsto U^{(R)}_{ij} q_{j,R} + U^{(L)}_{ij} q_{j,L}$$

arbitrary unitary matrices

Still, I can choose them to be equal (nothing forbids me).

$$U^{(R)} = U^{(L)} = U^{(V)}$$

$$q_i \mapsto U^{(V)}_{ij} (q_{j,R} + q_{j,L}) = U^{(V)}_{ij} q_j$$

$$\boxed{\text{Matrix rotation} \quad q \mapsto Uq}$$

We recover the usual "flavor" transformations.

So, if we transform the right-handed and the left-handed fermions in the same way $\longmapsto$ ... we get the flavor transformation.

But there is another possibility:

$$U^{(R)} = U^{(L)^\dagger} = U^{(A)}$$

Then:

$$q_i = q_{i,R} + q_{i,L} \mapsto U^{(A)}_{ij} q_{j,R} + U^{(A)^\dagger}_{ij} q_{j,L}$$

$$= U^{(A)}_{ij} P_R q_j + U^{(A)^\dagger}_{ij} P_L q_{j,L}$$

It is better to use here a matrix notation:

$$q = \begin{pmatrix} q_u \\ q_d \\ q_s \end{pmatrix} !$$

$$q = q_R + q_L \mapsto U^{(A)} q_R + U^{(A)^\dagger} q_L =$$

$$= U^{(A)} P_R q + U^{(A)^\dagger} P_L q$$

$$U^{(A)} = e^{i\alpha^a t^a} \simeq 1 + i\alpha^a t^a$$

$$U^{(A)\dagger} = e^{-i\alpha^a t^a} \simeq 1 - i\alpha^a t^a$$

Ergo:

$$q \xmapsto{A} [1 + i\alpha^a t^a]\,\tfrac{1}{2}(1+\gamma^5)q + [1 - i\alpha^a t^a]\,\tfrac{1}{2}(1-\gamma^5)q$$

$$= q + i\alpha^a t^a \gamma^5 q + \dots$$

$$= e^{i\alpha^a t^a \gamma^5} q$$

$$\boxed{q \xmapsto{A} e^{i\alpha^a t^a \gamma^5} q}$$

is the axial transformation!

- Achtung: $U_A(N_f)$ is not a group...

$$e^{i\alpha^a t^a \gamma^5} \cdot e^{i\varphi^b t^b \gamma^5} \neq e^{i\delta^c t^c \gamma^5}$$

The operation is not internal

- $U_V(N_f) \times U_A(N_f)$ is a symmetry of the Lagrangian.

The conserved Noether currents are:

$$V^{\mu,a} = \bar{q}\gamma^\mu t^a q$$

$$A^{\mu,a} = \bar{q}\gamma^5\gamma^\mu t^a q$$

$$\begin{cases} \partial_\mu V^{\mu,a} = 0 \\ \partial_\mu A^{\mu,a} = 0 \end{cases}$$

IMPORTANT CONSIDERATIONS:

• $\partial_\mu V^{\mu,a} = 0$

is conserved also for $m_i \neq 0$ (but equals!)

$$\partial_\mu V^{\mu,a} = i\bar{q}\,[\hat{m}, t^a]\,q \qquad \hat{m} = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_{N_f} \end{pmatrix}$$

• $\partial_\mu A^{\mu,a} = 0$

is valid only for $\hat{m} = 0$, otherwise:

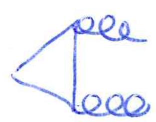
$$\partial_\mu A^{\mu,a} = i\bar{q}\,\gamma^5 \{\hat{m}, t^a\}\,q \neq 0$$

Moreover, as we shall see, $A^{\mu,a}$ is spontaneously <u>broken</u>! (Very important property of QCD!!!!)
($\mapsto$ Goldstone bosons: π, K, η)

• $a = 0$

$$\partial_\mu A^0_\mu = 2\bar{q}\hat{m}\gamma^5 q - \frac{g^2 N_f}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$$

with $\tilde{G}^{a,\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$

This is the so-called axial anomaly .

It is then explicitly broken by $\hat{m}$ and by the anomaly.
η' gets a high mass $\mapsto$ it is not a Goldstone boson (~ 1 GeV).