$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2} M_0^2 S^2$$

Free particle with mass M_0.... this is obviously stable.

The propagator takes the form

$$\Delta_S(p^2) = \frac{1}{p^2 - M_0^2 + i\varepsilon} \qquad p^2 = p^{0\,2} - \vec{p}^{\,2};$$

$$p^2 = p^{0\,2} - \vec{p}^{\,2}.$$

Imagine to be in the rest frame of the state S...

$$\vec{p} = 0$$

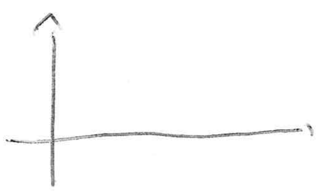
$$p^0 = x$$

$$\Delta_S(x) = \frac{1}{x^2 - M_0^2 + i\varepsilon}$$

spectral function

$$d_S(x) = \frac{2x}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im}\left[\Delta_S(x)\right] \vartheta(x) = \frac{2x}{\pi} \frac{\varepsilon}{(x^2 - M_0^2)^2 + \varepsilon^2} \vartheta(x)$$

$$= \frac{2x}{\pi} \pi\, \delta(x^2 - M_0^2)\, \vartheta(x) = \delta(x - M_0)$$



$$\int_0^\infty d_S(x)\, dx = 1,$$

Amplitude for survival probability:

$$a(t) = \int_{-\infty}^{\infty} d_S(x)\, e^{-ixt}$$

In the present case:

$$a(t) = e^{-iM_0 t}$$

$$p(t) = 1 \qquad \forall t \qquad \text{(stable)}.$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S_1)^2 - \frac{1}{2} M_1^2 S_1^2 + \frac{1}{2}(\partial_\mu S_2)^2 - \frac{1}{2} M_2^2 S_2^2 + g S_1 S_2$$

Mixing Phenom.

$$= \frac{1}{2}(\partial_\mu S_1)^2 - \frac{1}{2}(\partial_\mu S_2)^2 - \frac{1}{2}(S_1, S_2)\begin{pmatrix} M_1^2 & g \\ g & M_2^2 \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$$

We have then:

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \quad ; \quad \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{pmatrix}$$

$\tilde{M}_1^2, \tilde{M}_2^2$ are the solutions of

$$(M_1^2 - \lambda)(M_2^2 - \lambda) - g^2 = 0.$$

Note that this makes sense as long as $\lambda_i > 0$.

$$\lambda^2 - (M_1^2 + M_2^2)\lambda + (M_1^2 M_2^2 - g^2) = 0;$$

$$\lambda_{1,2} = \frac{M_1^2 + M_2^2 \pm \sqrt{(M_1^2+M_2^2)^2 - 4(M_1^2 M_2^2 - g^2)}}{2} = \frac{M_1^2 + M_2^2 \pm \sqrt{(M_1^2 - M_2^2)^2 + 4g^2}}{2}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{S}_1)^2 + \frac{1}{2}(\partial_\mu \tilde{S}_2)^2 - \frac{1}{2}\tilde{M}_1^2 \tilde{S}_1^2 - \frac{1}{2}\tilde{M}_2^2 S_2^2$$

Ergo, für [illegible]

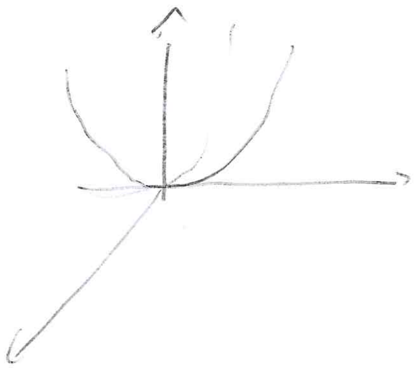
$$\lambda_1 = \frac{M_1^2 + M_2^2 - \sqrt{(M_1^2 - M_2^2)^2 + 4\varrho^2}}{2} > 0$$

$$M_1^2 + M_2^2 > \sqrt{(M_1^2 - M_2^2)^2 + 4\varrho^2}$$

$$(M_1^2 + M_2^2)^2 > (M_1^2 - M_2^2)^2 + 4\varrho^2$$

$$4 M_1^2 M_2^2 > 4\varrho^2$$

$$\boxed{M_1^2 M_2^2 > \varrho^2}$$



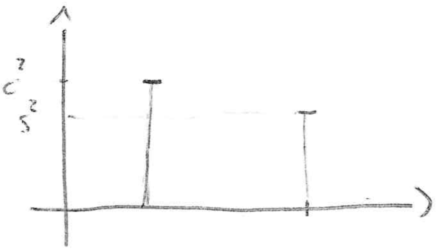
$$\tilde{S}_1 \mapsto \Delta_{\tilde{S}_1} = \frac{1}{p^2 - \tilde{M}_1^2} \quad , \quad \tilde{S}_2 \mapsto \Delta_{\tilde{S}_2} = \frac{1}{p^2 - M_2^2}$$

What about S_1? (or S_2) This is more tricky... $\overline{}_{S_1} + \overline{}_{S_1} \cdots \overline{}_{S_2} \cdots \overline{}_{S_1} + \cdots$

The result for the spectral function is:

One could do the same thing in the day...

$$d_{S_1}(x) = c^2\,\delta(x - \tilde{M}_1) + s^2\,\delta(x - \tilde{M}_2)$$



$$\int_0^\infty d_{S_1}(x)\,dx = c^2 + s^2 = 1,$$

$$a(t) = \int_0^\infty d_{S_1}(x)\, e^{-ixt}\, dx = c^2 e^{-i\tilde{M}_1 t} + s^2 e^{-i\tilde{M}_2 t}$$

$$P(t) = |a(t)|^2 = c^4 + s^4 + c^2 s^2 e^{i(\tilde{M}_2 - \tilde{M}_1)t} + s^2 c^2 e^{-i(\tilde{M}_2 - \tilde{M}_1)t}.$$

$$= c^4 + s^4 + 2c^2 s^2 \cos\left[(\tilde{M}_2 - \tilde{M}_1)t\right].$$

Indeed, the propagator of S_1 is:

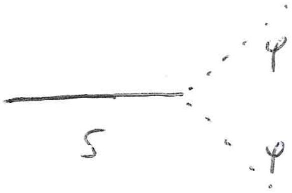
$$\boxed{D_{S_1} = c^2 \Delta_{\tilde{S}_1} + s^2 \Delta_{\tilde{S}_2}}$$

out of which it follows

$$d_{S_1} = c^2 \delta(x - \tilde{\mu}_1) + s^2 \delta(x - \tilde{\mu}_2).$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_0^2 S^2 + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 + g S \varphi^2$$

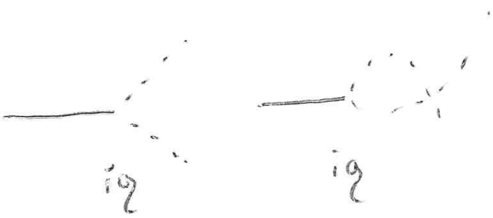
$$[g] = [\text{Energy}]$$



This is a decay of S into $\varphi\varphi$. The decay width can be calculated following the usual strategy (S-matrix, amplitude, ...)

$$\boxed{\Gamma_{S\mapsto\varphi\varphi} = \eta_f \cdot \frac{|\vec{K}|}{8\pi M_0^2} |-iM|^2 \, \theta(M_0 - 2m}$$

In the present case:



$$\Rightarrow -iM = 2ig \qquad |-iM| = 2g$$

$$\eta_f = \frac{1}{2}$$

$$\Gamma_{S\mapsto\varphi\varphi} = \frac{|\vec{K}|}{8\pi M_0^2} (\sqrt{2}\, g)^2$$

What is $|\vec{K}|$?

Rest frame of S:

$$(M_0, \vec{0}) = (E_1, \vec{K}_1) + (E_2, \vec{K}_2)$$

$$\vec{K}_1 + \vec{K}_2 = 0 \qquad \vec{K}_1 = \vec{K} = -\vec{K}_2$$

$$E_1 = E_2 = \sqrt{m^2 + \vec{K}^2}$$

$$M_0 = 2\sqrt{m^2 + \vec{K}^2}$$

$$\boxed{|\vec{K}| = \sqrt{\frac{M_0^2}{4} - m^2}}$$

$$\Gamma_{S\mapsto\varphi\varphi} = \frac{\sqrt{\frac{M_0^2}{4} - m^2}}{8\pi M_0^2} \left(\sqrt{2}\, g\right)^2 \theta(M_0 - 2m) \quad ; \quad P(t) = e^{-\Gamma_{S\mapsto\varphi\varphi} t}$$

Note, for $S \mapsto \varphi_1 \varphi_2$ with masses m_1, m_2 we get

$$\Gamma_{S\mapsto\varphi_1\varphi_2} = \frac{|\vec{K}_1|}{8\pi M_0^2} |-i\mathcal{M}|^2 \theta(M_0 - m_1 - m_2).$$

If one has other forms such as derivatives:

$$\mathcal{L} = g\, S\, (\partial_\mu \varphi)(\partial^\mu \varphi)$$

Use the new Feyn. rules to evaluate the amplitude.

$$-i\mathcal{M} = -(k_1 \cdot k_2) \cdot g$$

$$\langle f | S | in \rangle$$

$$\varphi = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega}} \left(a_{\vec{k}}\, e^{-ikx} + a^{+}_{\vec{k}}\, e^{ikx} \right)$$

$$\partial_\mu \varphi = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega}} \left(a_{\vec{k}} (-ik_\mu)\, e^{-ikx} + a^{+}_{\vec{k}} (ik_\mu)\, e^{ikx} \right)$$

What is k^2? $k^2 = m^2$.

=

$$\mathcal{L} = g\, S\, (\partial_\mu \varphi_1)(\partial^\mu \varphi_2)$$

$$-iM = (-i k_{1,\mu})(-i k_2^\mu)\, i g = i g (k_1 \cdot k_2)$$

$$P = k_1 + k_2$$

$$P^2 = (k_1 + k_2)^2 = k_1^2 + k_2^2 + 2 k_1 k_2 = 2m^2 + 2 k_1 k_2$$

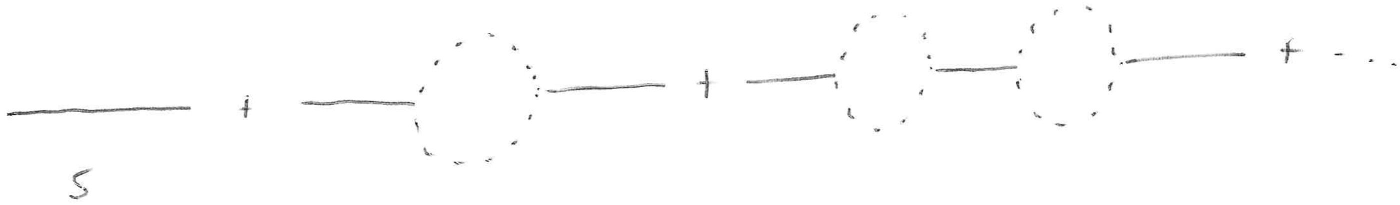
Ergo:

$$k_1 k_2 = \frac{M_0^2 - 2m^2}{2}$$

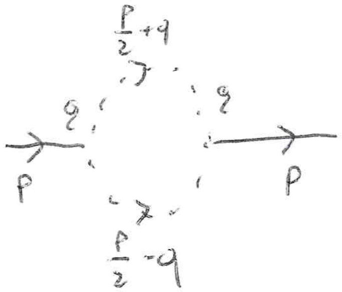
$$\boxed{|-iM| = g^2 \left(\frac{M_0^2 - 2m^2}{2} \right)^2 .}$$

It is always possible to express Γ as function of masses...

Back to $\mathcal{L}_1 = g S \varphi^2$. What about the propagator of S?



Bubble contribution



$$\Sigma(p) \propto \int \frac{d^4q}{(2\pi)^4} \frac{1}{\left[\left(\frac{p}{2}+q\right)^2 - m^2 + i\varepsilon\right]} \frac{1}{\left[\left(\frac{p}{2}-q\right)^2 - m^2 + i\varepsilon\right]}$$

it is logarithm. divergent ... $\int q^3 dq \cdot \frac{1}{q^4}$

We can modify it by putting a cutoff.

The propagator takes the form:

$$\Delta_S(p^2) = \frac{1}{p^2 - M_0^2 + (\sqrt{2}g)^2 \Sigma(p^2) + i\varepsilon};$$

All the things said before should ok here:

mass:

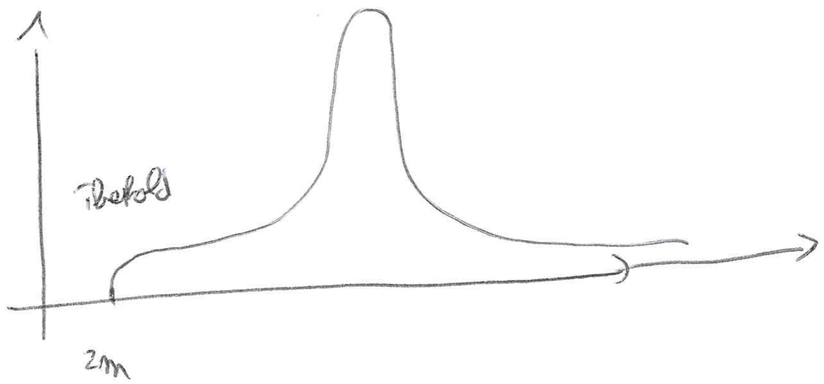
$$\to M^2 - M_0^2 + (\sqrt{2} g)^2 \, Re \, \Sigma(M^2) = 0$$

$M < M_0$

actually, while true...
the tree-level decay width should actually be evaluated at M!!
(and not at M_0)

spectral function:

$$d_S(x) = \frac{2x}{\pi} \lim_{\varepsilon \to 0} \operatorname{Im}\left[\Delta_S(x)\right]$$



$$\boxed{a_S(t) = \int_0^\infty d_S(x)\, e^{-ixt}\, dx}$$

$d_S(x)$ is not a Breit-Wigner... still, BW works extremely good.

$$\Delta_S(p^2) = \frac{1}{p^2 - M_0^2 + g^2\Sigma(p^2) + i\varepsilon} = \underbrace{\int_0^\infty dx\, \frac{d_S(x)}{p^2 - x^2 + i\varepsilon}}$$

sum of many free propagators, weighted by $d_S(x)dx$.

=

$$d_S(x) = \frac{2x}{\pi} \frac{(\sqrt{2}g)^2 \operatorname{Im}\Sigma(x)}{(x^2 - M_0^2 + (\sqrt{2}g)^2 \operatorname{Re}\Sigma)^2 + ((\sqrt{2}g)^2 \operatorname{Im}\Sigma)^2}$$

optical theorem:

$$\boxed{(\sqrt{2}g)^2 \operatorname{Im}\Sigma(x) = x\,\Gamma(x)}$$

$$(\sqrt{2}g\; \Gamma(x) = \underbrace{\frac{\sqrt{\frac{x^2}{4} - m^2}}{8\pi x^2}}\,[\sqrt{2}g]^2\, \theta(x-2m)$$

i.e. decay with a 'running mass'.

How do I get the usual B-W limit?

$$d_S(x) = \frac{2x}{\pi} \frac{x\,\Gamma(x)}{(x^2 - M_0^2 + (\sqrt{2}g)^2 \operatorname{Re}\Sigma)^2 + (x\,\Gamma(x))^2}$$

$$\simeq N \frac{2M}{\pi} \frac{M\,\Gamma(M)}{(x^2 - M^2)^2 + (M\,\Gamma(M))^2}\, \theta(x-2m)$$

I need an extra-normalisation ...

$$d_5^{BW,rel}(x) = N \frac{\vartheta(x-2m)}{(x^2-M^2)^2 + (M\Gamma)^2} = N \frac{\vartheta(x-2m)}{(x-M)^2(x+M)^2 + (M\Gamma)^2} =$$

$$\simeq \frac{N}{(2M)^2} \frac{\vartheta(x-2m)}{(x-M)^2 + (\Gamma/2)^2} \simeq \tilde{N} \frac{1}{(x-M)^2 + (\Gamma/2)^2} = d_5^{BW,nr}(x) .$$

The usual trick with the cutoff is:

* perform the integral over q^0

* Put a 'cutoff' on the tree-momentum $\vec{k}$.

Using a hard cutoff:

$$\Sigma(p^2 = x^2) = \frac{-\sqrt{4m^2-x^2}}{8\pi^2 x} \arctan\left(\frac{\Lambda x}{\sqrt{\Lambda^2+m^2}\sqrt{4m^2-x^2}}\right)$$

$$-\frac{1}{8\pi^2}\ln\left(\frac{m}{\Lambda+\sqrt{\Lambda^2+m^2}}\right).$$

Note that for $\Lambda \mapsto \infty$ we get

$$\Sigma(x) = \underbrace{\frac{-\sqrt{4m^2-x^2}}{8\pi^2 x}\arctan\left(\frac{x}{\sqrt{4m^2-x^2}}\right)}_{\text{let us neglect it ....}} \underbrace{-\frac{1}{8\pi^2}\ln\left(\frac{m}{\Lambda}\right)}$$

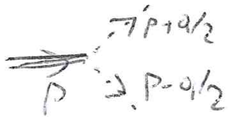
$$M^2 - M_0^2 + \frac{g^2}{8\pi^2}\ln\left(\frac{\Lambda}{m}\right) + \# = 0$$

$$M^2 \simeq M_0^2 - \frac{g^2}{8\pi^2}\ln\left(\frac{\Lambda}{m}\right) \quad \to \text{ren. in front of our eyes...}$$

cutoff:

$\Lambda \to$ in hadrons 1-2 GeV... in a fundamental theory: $M_{Planck} = 10^{19}$

It means that, when $|\vec{q}| \gtrsim \Lambda \to$ the interaction is 'smaller'...



There is a limit on how fast the virtual particles circulate in the loop.

The present theory is renormalizable... but this kind of log. dependence on the cutoff takes indeed place for all the fermionic particles...
the bosons (gauge bosons) have a mass which is radiatively protected.

But the Higgs is not... for the Higgs one has:

$$\boxed{M_H^2 - M_0^2 - \#\Lambda^2 + \underbrace{\#}_{\text{finite or log. terms}} = 0}$$

$\begin{cases} \text{if } \Lambda \sim 10^{19}\ \text{GeV} \\ M_0 \sim 10^{19}\ \text{GeV} \end{cases}$ in such a way that M is 125 GeV.

Krass, oder?

$a(t)$ and $p(t)$ depend only slightly on the cutoff.