Rabi oscillations

$$H = H_0 + H_1$$

$$\begin{cases} H_0 = M_0 |S\rangle\langle S| + \omega |\omega\rangle\langle\omega| \\ H_1 = g\left(|S\rangle\langle\omega| + |\omega\rangle\langle S|\right) \end{cases}$$

Note:

$\langle S|S\rangle = \langle\omega|\omega\rangle = 1$

$\langle S|\omega\rangle = 0$

$1 = |S\rangle\langle S| + |\omega\rangle\langle\omega|$

One can introduces the eigenstates $|E_1\rangle$, $|E_2\rangle$ as:

$$\begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} |\omega\rangle \\ |S\rangle \end{pmatrix}$$

$c = \cos\vartheta$

$s = \sin\vartheta$

$$\vartheta = \frac{1}{2}\arctan\left(\frac{-2g}{M_0 - \omega}\right)$$

$$H = E_1 |E_1\rangle\langle E_1| + E_2 |E_2\rangle\langle E_2|$$

$$\begin{cases} E_1 = \omega c^2 + M_0 s^2 + 2gcs = \dfrac{\omega + M_0 - \sqrt{(\omega+M_0)^2 + 4(g^2 - \omega M_0)}}{2}; \\ E_2 = M_0 c^2 + \omega s^2 - 2gcs = \dfrac{\omega + M_0 + \sqrt{(\omega+M_0)^2 + 4(g^2 - \omega M_0)}}{2}. \end{cases}$$

Eigenvalues of the matrix

$$\begin{pmatrix} \omega & g \\ g & M_0 \end{pmatrix}$$

Check: $\vartheta = 0$ (that is, $g = 0$):

$$\begin{cases} E_1 = \omega \\ E_2 = M_0 \end{cases}$$

survival probability amplitude

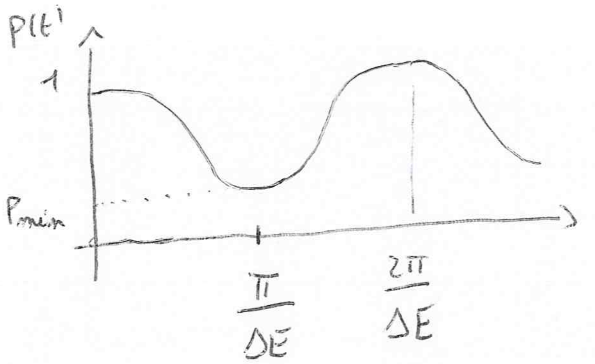
$$a(t) = \langle S | e^{-iHt} | S \rangle = s^2 e^{-iE_1 t} + c^2 e^{-iE_2 t}$$

$$|S\rangle = s|E_1\rangle + c|E_2\rangle$$
$$\langle S| = \langle E_1| s + \langle E_2| c$$

$P(t) = |a(t)|^2$ is the survival probability.

$$P(t) = |a(t)|^2 = s^4 + c^4 + s^2 c^2 e^{i(E_1 - E_2)t} + s^2 c^2 e^{i(E_2 - E_1)t}$$
$$= s^4 + c^4 + 2 s^2 c^2 \cos\left[(E_2 - E_1)t\right]$$

$$t = 0 \to P(0) = s^4 + c^4 + 2 s^2 c^2 = (s^2 + c^2)^2 = 1$$



Rabi oscillations

The time scale of the oscillation is $\sim \frac{1}{\Delta E}$.

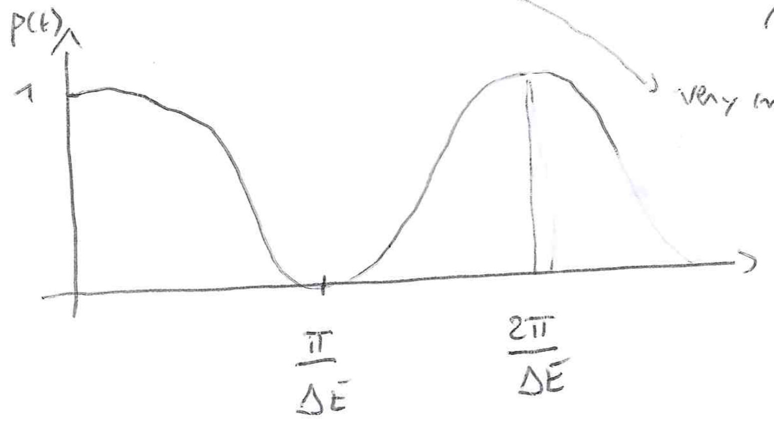
The period is $\frac{2\pi}{\Delta E}$.

$P(t)$ is minimal if $t = \frac{\pi}{\Delta E} \mapsto P(t = \frac{\pi}{\Delta E}) = P_{min} = c^4 + s^4 - 2 s^2 c^2$
$$= (c^2 - s^2)^2$$
$$= (\cos^2\theta - \sin^2\theta)^2$$

You see that for $\theta = \frac{\pi}{4}$ $\rightarrow P_{min} = 0$.

$\theta = \pi/2$

very important for us



$P_{min} = 0$.

That is, for $t = \frac{\pi}{\Delta E}$ $P(t = \frac{\pi}{\Delta E}) = 0$! The system is with probability

100% described by $|w\rangle$.

That is, the transition $|s\rangle \mapsto |w\rangle$ occurred with prob = 100% at $t = \frac{\pi}{\Delta E}$

N.b: for $\theta = \frac{\pi}{4}$

$$a(t) = \frac{1}{2}\left(e^{-iE_1 t} + e^{-iE_2 t}\right) =$$

$$P(t) = |a(t)|^2 = \frac{1}{2} + \frac{1}{2}\cos(\Delta E t) = \cos^2\left(\frac{\Delta E t}{2}\right)$$

Recall:

$$\begin{cases} \cos^2 x + \sin^2 x = 1 \\ \cos^2 x - \sin^2 x = \cos(2x) \end{cases}$$

$$2\cos^2 x = 1 + \cos(2x)$$

$$\boxed{\cos^2 x = \frac{1 + \cos(2x)}{2}}$$

$$2\sin^2 x = 1 - \cos(2x)$$

$$\boxed{\sin^2 x = \frac{1 - \cos(2x)}{2}}$$

In general, one can solve the system exactly:

$$|\nu(t)\rangle = e^{-iHt}|S\rangle$$

$$|\nu(0)\rangle = S$$

$$|\nu(t)\rangle = a(t)|S\rangle + r(t)|\omega\rangle$$

$\Rightarrow$ Determine $a(t)$ and $r(t)$.

$$P(t) = |a(t)|^2 = \langle S|\nu(t)\rangle = \langle S|e^{-iHt}|S\rangle$$

$$\begin{cases} P(t) = |a(t)|^2 = \text{survival prob.} \\ \bar{P}(t) = |r(t)|^2 = 1 - |a(t)|^2 = \text{transition probability} \end{cases}$$

For $\theta = \pi/4$

$$\begin{cases} P(t) = \cos^2(2\Delta E t/2) \\ \bar{P}(t) = \sin^2(2\Delta E t/2) \end{cases}$$

Check: $t = \pi/\Delta E \rightarrow P(\pi/\Delta E) = 0$

$$\bar{P}(\pi/\Delta E) = 1$$

Short time behaviour:

In general:

$$P(t) = c^4 + s^4 + 2 s^2 c^2 \cos\left((E_2 - E_1) t \right) =$$

$$= c^2 + s^4 + 2 s^2 c^2 \left[1 - \frac{1}{2} (E_2 - E_1)^2 t^2 + \dots \right] =$$

$$= (c^2 + s^2)^2 - s^2 c^2 \Delta E^2 t^2 + \dots = 1 - s^2 c^2 \Delta E^2 t^2 + \dots$$

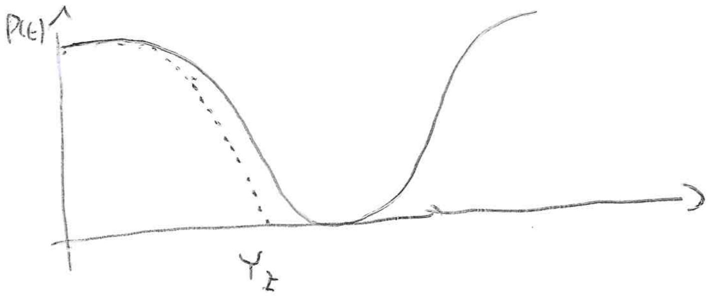
$$= 1 - \frac{t^2}{\tau_Z^2}$$

τ_Z "zero time" $\qquad \tau_Z^2 = \dfrac{1}{s^2 c^2 \Delta E^2} \longrightarrow \tau_Z = \dfrac{1}{|s c \Delta E|}$

For $\theta = \frac{\pi}{4}$ we get: $\qquad \rightarrow \tau_Z = \dfrac{2}{\Delta E}$

$$\begin{cases} P(t) \simeq 1 - \dfrac{\Delta E^2 t^2}{4} \\ \bar{P}(t) = 1 - P(t) \simeq \dfrac{\Delta E^2 t^2}{4} \end{cases}$$

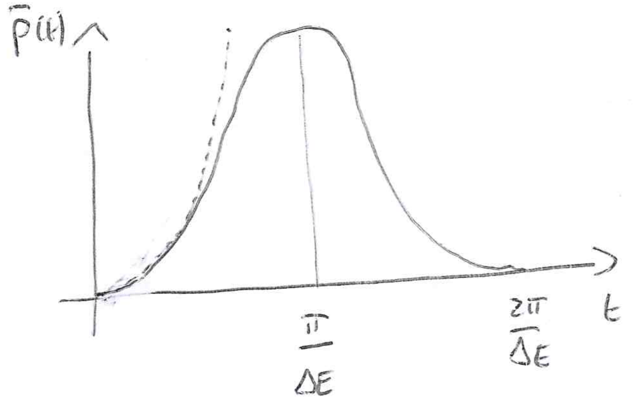
[obviously, we could have gotten it by expanding directly $\cos^2\left(\frac{\Delta E t}{2}\right)$ as
$\left(1 - \frac{1}{2} \frac{\Delta E^2}{4} t^2 \dots\right)^2 \simeq 1 - \frac{\Delta E^2}{4} t^2 + \dots$]



obviously, for $t > \tau_Z$ the approx. is meaningless...

→ indeed, it is strictly valid for $t \ll \tau_Z$ ... but practically for $t \lesssim \tau_Z$

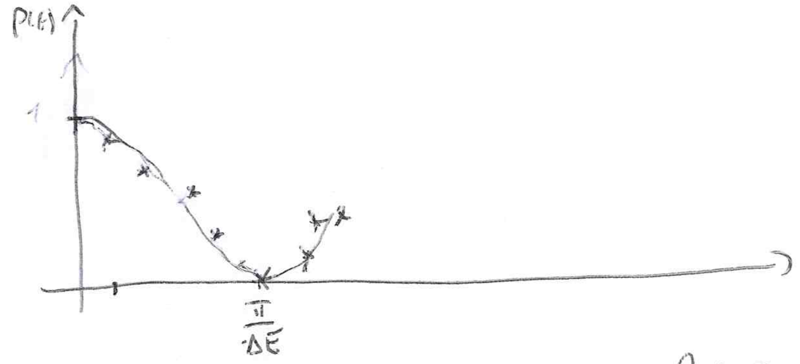
Similarly, for $\bar{p}(t)$ we get:



We can think of the following possibilities:

* The apparatus destroys our state... then, for one state we can perform only one measurement.

Still, we can repeat the experiment many times and/for different values of τ... then, we can measure $p(t)$



$\tau = 0$, N measurement... N times we find S... then is, probability of $p(t) = \frac{\text{Nr of times}}{N}$

$\tau = \frac{\pi}{10 \cdot \Delta E}$, $p(\tau) = 1 - \frac{\Delta E^2}{4} \frac{\pi^2}{10} \frac{1}{\Delta E}$ ..

We can verify $p(t)$ and $\bar{p}(t)$... interesting, but not spectacular!...

* The apparatus tells us if the state is in $|S\rangle$ or $|W\rangle$, but leaves $|S\rangle$ untouched but destroys the state if $|W\rangle$ is found....

$$|s(t)\rangle = e^{-iHt} |S\rangle$$

Now, at time τ we do the measurement.

$$p(\tau) = \cos^2\left(\frac{\Delta E \, \tau}{2}\right)$$

$$\bar{p}(\tau) = \sin^2\left(\frac{\Delta E}{2} \tau\right)$$

Measuring such a system

~~OK,~~

We can now perform a measurement of the system. We start with $|S\rangle$ for $t=0$.

At time "Υ" we ascertain if I still have $|S\rangle$ or not.

$$\begin{cases} \text{The probability to "still" find the system in } |S\rangle \text{ is } P(t=\Upsilon) \\ \text{The " " " " " " } |W\rangle \text{ is } \bar{P}(t=\Upsilon). \end{cases}$$

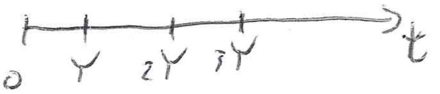
Now, if Υ is very small ($\Upsilon << \Upsilon_2$) we will have ($\alpha = \pi/4$)

$$\begin{cases} P(t=\Upsilon) = 1 - \dfrac{\Delta E^2 \Upsilon^2}{4} \\ \bar{P}(t=\Upsilon) = \dfrac{\Delta E^2 \Upsilon^2}{4} \end{cases}$$

Now, we have to "ask" what the measurement actually does to the system.

Namely, a "measurement" can destroy or modify the quantum system, but in some cases measurement can also occur without "interaction".

Now, let us repeat the measurement ....



After the first meas. at time τ, the state is again $|s\rangle$ with probability $p(\tau)$ ... then, everything starts from the very beginning.

After 2τ ... the probability to still have $|s\rangle$ is:

$$p(\tau)^2$$ (and not $p(2\tau)$ ... this would be the case without measuring at τ ...)

$$|s(t)\rangle = e^{-iHt}|s\rangle \quad \text{for } t \in (0,\tau) \quad \longrightarrow \quad |s(t)\rangle = e^{-iH(t-\tau)}|s\rangle \quad \text{for } \in (\tau, 2\tau) \quad ? \ldots$$

After N steps:

Prob. to find still $|s\rangle$ after N steps: $p(\tau)^N$.

Now, a very interesting case is when τ is very small ...

then:

$$p(\tau)^N \simeq \left(1 - \frac{\Delta E^2 \tau^2}{4}\right)^N$$

Now, recall that

$$\left(1+\frac{a}{n}\right)^n \sim e^a$$ for n very large...

Then:

$$P(Y)^N = \left(1 - \frac{\Delta E^2}{4}\left(\frac{T}{N}\right)^2\right)^N = \left(1 - \frac{\Delta E^2}{4}\frac{T^2}{N}\cdot\frac{1}{N}\right)^N$$

$$= e^{-\frac{\Delta E^2}{4}\frac{T^2}{N}}$$

if we keep T fixed and send $N \mapsto \infty$, we find:

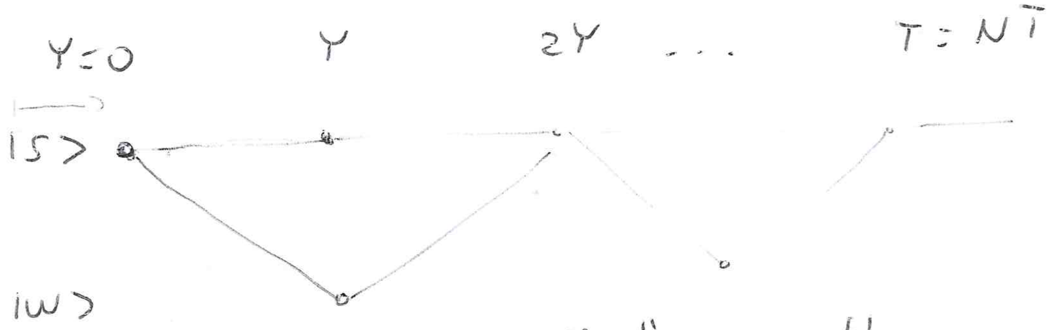
$$P(Y)^N \longmapsto 1 \;!!!$$

The system stays in its "undecayed" original state

because of frequent measurements!

* There is however a third possibility: the measurement preserves both $|S\rangle$ and $|W\rangle$...

Then the situation is more complicated:



"recombination effects" are present!

There are now many possibilities now... the calculation is lengthy (and not that easy):

$$\left(\text{Prob. to find } |S\rangle \text{ after } N \text{ steps}\right) = \frac{1}{2}\left[1 + \left(2P(\tau) - 1\right)^N\right]$$

$$P^{N\ \text{steps}}_{|S\rangle \mapsto |S\rangle}$$

For $N = 1$: $P(\tau) = \cos^2(\Delta E \tau) \Rightarrow P(\tau) = 0$!!!

Let us consider now

$$\boxed{T = \frac{\pi}{2\Delta E} = N\tau \mapsto \tau = \frac{\pi}{2\Delta E} \cdot \frac{1}{N}}$$

$$\boxed{P^{\text{after } N \text{ steps}}_{|S\rangle \mapsto |S\rangle} = \frac{1}{2}\left[1 + \cos^N\left(\frac{\pi}{N}\right)\right]}$$

Again... for N very large one gets "1"!

$P_{|S\rangle \mapsto |S\rangle} \simeq 1$!!!

Also here → zero!

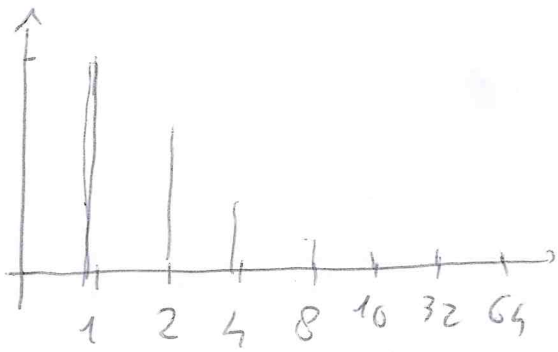
$$P_{|s\rangle \mapsto |w\rangle}^{\text{after } N \text{ steps}} = \frac{1}{2}\left[1 - \cos^N\left(\frac{\pi}{N}\right)\right]$$

is now the probability that the transition has occurred ...

This has been measured!

$N = 1$

$$P_{|s\rangle \mapsto |w\rangle} = 1!$$



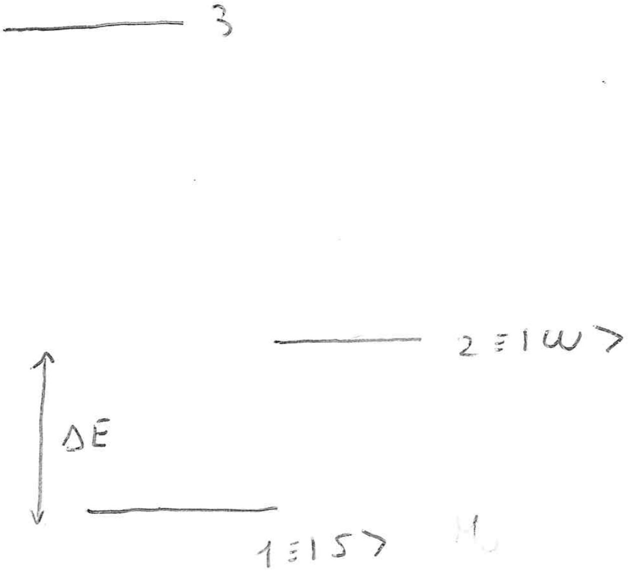
$N = 64$

$$P_{|s\rangle \mapsto |w\rangle}^{\text{after 64 steps}} \simeq 0 \qquad \left(P_{|s\rangle \mapsto |s\rangle}^{\text{after 64 steps}} = 1!\right)$$

Quantum-Zeno-effect!

Haro et al, Phys Rev A:

This is exactly what it has been done...



$|s\rangle$, $|w\rangle$ are "stable". (Transition $|w\rangle \mapsto |s\rangle$ is supposed to be forbidden).

Hamiltonian: H_0.

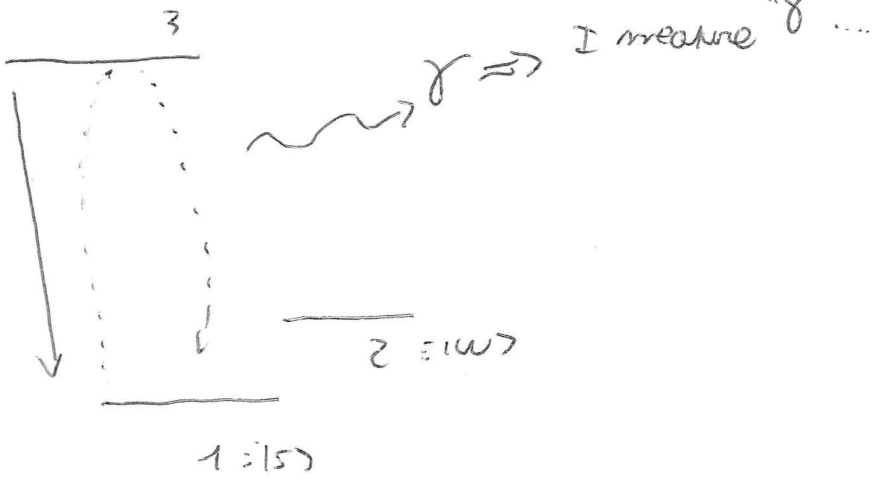
Then, we add the term H_{int}... this is achieved by a rf e.m. field with frequency $\omega = \Delta E = \frac{E_2 - E_1}{\hbar} = E_2 - E_1$.

We start with $|s\rangle$, then we can tune the field to have

$$P(t) = \cos^2(\Delta E\, t)$$

$$\bar{P}(t) = \sin^2(\Delta E\, t)$$

But how to do the measurement?



3 can only go to |S⟩ ... but not to |ω⟩

optical pulse with frequency $(E_3 - E_1)$. This brings $1 \equiv |S\rangle$ to "3"...

It then goes from "1 to 3", 3 goes back to 1 and a photon is emitted.

And so on and so forth...

So, if the state is in 1 → and we apply the optical pulse....

we measure some photons γ with frequency $(E_3 - E_1)$. → collapse to |S⟩

But what if the ion was in "2"? Then "nothing"! No transition.

"Null measurement."

The wf collapses

$\omega_{31} >> \omega_{21}$ ....

[subtlety: the presence of the rf field brings the two energies of |S⟩ and |ω⟩ together... but how do I see it in terms of Rabi?

Well, this is another mode... when the rf is "on", the real eigenvalues are E_1 and E_2, but as soon as we switch it off... it is gone, I get back |S⟩ and |ω⟩ again!

Measurement much shorter than the Rabi-pulse ...

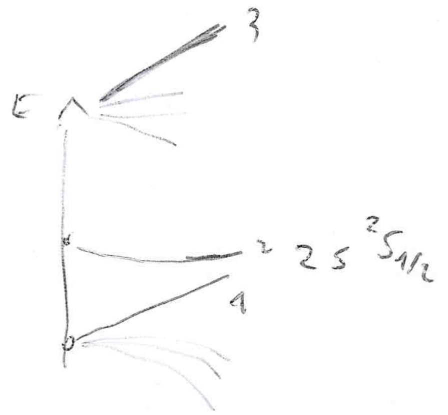
3 electrons

$^{9}Be^{+}$

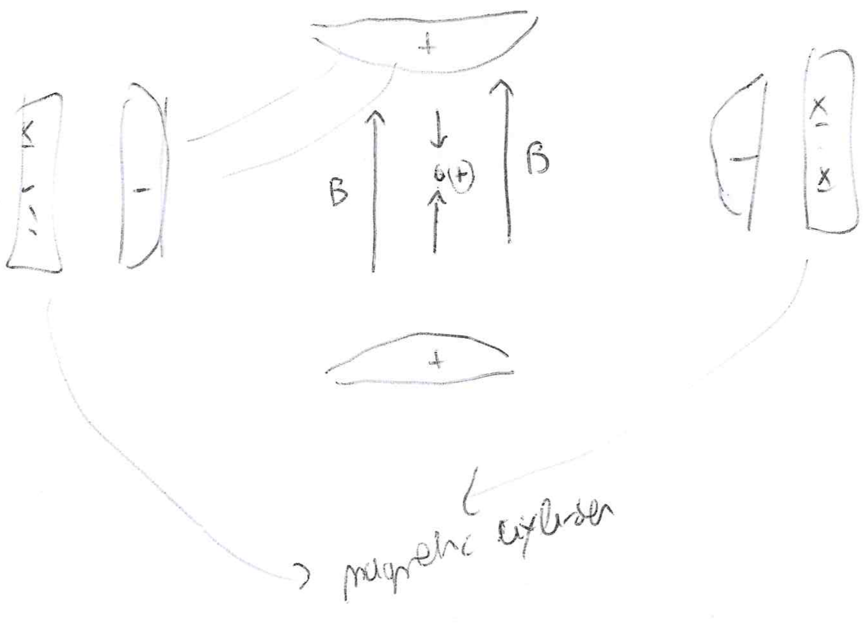
4 protons,
5 neutrons

Levels "1" and "2" $(m_I, m_J) = (\frac{3}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ in $2s\ ^{2}S_{1/2}$

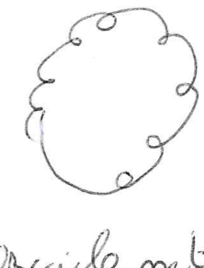
$B = 0.8194\ T$



Be → in a Penning trap. (Magnetic and electric fields are used to confine the ions).



Electric quadrupole
Magnetic hom. field



Epicycle motion of the particles!

Storage for some hours! 5000 independent ions!!!

Laser 1 -> 1 -> 3 transition! (313 nm)

~> 72 scattered photons! (Detected ≪ 1!)

But what does generate the collapse?

The 72 photons scattered away or the "ion" which has been measured?

$T = 256$ ms

~> Duration of the measurement: Few ms.