Let us consider two states: $|s\rangle$ and $|w\rangle$.

$$H_0 = M_0 |s\rangle\langle s| + \omega |w\rangle\langle w|$$

$|s\rangle$ and $|w\rangle$ are "obviously" eigenstates of H_0:

$$\begin{cases} H_0|s\rangle = M_0|s\rangle & M_0 \text{ is the "energy" (or the "mass") of } |s\rangle \\ H_0|w\rangle = \omega|w\rangle \end{cases}$$

If at $t=0$ the state is given by $|s\rangle$, which is the state of the system at the instant t?

The time-evolution operator is

$$U = e^{-iH_0 t} \qquad (\hbar = 1)$$

Ergo

$$|s(t)\rangle = e^{-iH_0 t}|s\rangle = e^{-iM_0 t}|s\rangle \propto |s\rangle$$

We see that if we start with the state $|s\rangle$ at $t=0$, we still have $|s\rangle$ at each time $t>0$. The only effect of the time-evolution is a phase e^{-iMt}.

The survival probability amplitude for the state $|S\rangle$ is

defined as

$$a(t) = \langle S| e^{-iHt} |S\rangle$$

where H is the full Hamiltonian of

of

The survival probability amplitude

$$p(t) = |a(t)|^2$$

is the probability that the system, prepared in the state $|S\rangle$

at $t=0$, is still in the state $|S\rangle$.

In the present example with $H = H_0$:

$$a(t) = \langle S| e^{-iH_0 t} |S\rangle = e^{-iM_0 t}$$

$$p(t) = |a(t)|^2 = 1 \quad \forall t$$

In this case the state $|S\rangle$ is found with probability "1" at each

$t > 0$ after the preparation at $t=0$. This is indeed expected:

$|S\rangle$ is indeed a 'stable state', which is an eigenstate of the Hamiltonian

Let us then introduce

$$H_1 = g\left(|s\rangle\langle w| + |w\rangle\langle s|\right)$$

↳ necessary in order to have an Hermitian H_1.

and study: $H = H_0 + H_1$.

$|s\rangle$ and $|w\rangle$ are not any longer eigenstates.

H_1 'mixes' $|s\rangle$ with $|w\rangle$:

$$\begin{cases} H_1|s\rangle = g|w\rangle \\ H_1|w\rangle = g|s\rangle \end{cases}$$

We need to find the eigenvalues $|E_1\rangle$ and $|E_2\rangle$.

Considering that $\{|w\rangle, |s\rangle\}$ is a orthonormal basis, one gets $|E_1\rangle$ and $|E_2\rangle$ via a $SO(2)$ rotation:

$$\begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} |w\rangle \\ |s\rangle \end{pmatrix}$$

$c = \cos\theta$

$s = \sin\theta$

$c^2 + s^2 = 1$

$$\begin{pmatrix} |w\rangle \\ |s\rangle \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix}$$

The full Hamiltonian is

$$H = H_0 + H_1 = \omega|\omega\rangle\langle\omega| + M_0|S\rangle\langle S| + g\left(|S\rangle\langle\omega| + |\omega\rangle\langle S|\right)$$

Let us plug in $|E_1\rangle$ and $|E_2\rangle$:

$$\begin{aligned} H = {} & M_0\left(s|E_1\rangle + c|E_2\rangle\right)\left(s\langle E_1| + c\langle E_2|\right) \\ & + \omega\left(c|E_1\rangle - s|E_2\rangle\right)\left(c\langle E_1| - s\langle E_2|\right) \\ & + g\left(s|E_1\rangle + c|E_2\rangle\right)\left(c\langle E_1| - s\langle E_2|\right) \\ & + g\left(c|E_1\rangle - s|E_2\rangle\right)\left(s\langle E_1| + c\langle E_2|\right); \end{aligned}$$

$$\begin{aligned} = {} & |E_1\rangle\langle E_1|\left(M_0 s^2 + \omega c^2 + g s c + g s c\right) \\ & + |E_2\rangle\langle E_2|\left(M_0 c^2 + \omega s^2 - g s c - g s c\right) \\ & + |E_1\rangle\langle E_2|\left(M_0 s c - \omega s c - g s^2 + g c^2\right) \\ & + |E_2\rangle\langle E_1|\left(M_0 c s - \omega s c + g c^2 - g s^2\right); \end{aligned}$$

Thus, $|E_1\rangle$ and $|E_2\rangle$ are eigenstates of H if the terms $|E_1\rangle\langle E_2|$ and $|E_2\rangle\langle E_1|$ disappear.

$$M_0 s c - \omega s c + q (c^2 - s^2) = 0 .$$

This means:

$$(M_0 - \omega) \sin\theta \cos\theta + q (\cos^2\theta - \sin^2\theta) = 0 .$$

$$\begin{cases} \sin(2\theta) = 2 \sin\theta \cos\theta \\ \cos(2\theta) = \cos^2\theta - \sin^2\theta \end{cases}$$

$$\frac{(M_0 - \omega)}{2} \sin(2\theta) = - q \cos(2\theta)$$

$$\boxed{\tan(2\theta) = \frac{-2q}{M_0 - \omega}}$$

$$\boxed{\theta = \frac{1}{2} \arctan\left(\frac{-2q}{M_0 - \omega}\right)}$$

For this value of θ →

$$H_0 = E_1 |E_1\rangle\langle E_1| + E_2 |E_2\rangle\langle E_2|$$

with

$$\begin{cases} E_1 = \omega c^2 + M_0 s^2 + 2 q s c \\ E_2 = M_0 c^2 + \omega s^2 - 2 q s c \end{cases}$$

Alternative way → Matrix form

$$H = H_0 + H_1 = (|w\rangle, |s\rangle) \underbrace{\begin{pmatrix} \omega & g \\ g & M_0 \end{pmatrix}}_{M} \begin{pmatrix} \langle w| \\ \langle s| \end{pmatrix}$$

We realise that

$$\begin{pmatrix} |w\rangle \\ |s\rangle \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix} = B \begin{pmatrix} |E_1\rangle \\ |E_2\rangle \end{pmatrix}$$

Implies that

$$H = (|E_1\rangle, |E_2\rangle)\, B^t M B \begin{pmatrix} \langle E_1| \\ \langle E_2| \end{pmatrix}$$

$$B^t M B = D = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}; \quad \rightarrow H = E_1 |E_1\rangle\langle E_1| + E_2 |E_2\rangle\langle E_2|$$

E_1, E_2 are the eigenvalues of M, i.e. they solve:

$$(\omega - \lambda)(M_0 - \lambda) - g^2 + \omega M_0$$

$$\lambda^2 - \lambda(\omega + M_0) - g^2 = 0$$

$$E_{1,2} = \frac{(\omega + M_0) \mp \sqrt{(\omega + M_0)^2 + 4(g^2 - \omega M_0)}}{2} \rightarrow \left\{ E_1 = \omega + M_0 - \sqrt{(\omega - M_0)^2 + 4g^2} \right.$$

Ergo:

$$E_{1,2} = \frac{\omega + M_0 \mp \sqrt{(\omega - M_0)^2 + 4g^2}}{2}$$

Note, if $g = 0$ and $M_0 > \omega$ we get:

$$E_{1,2} = \frac{\omega + M_0 \pm |M_0 - \omega|}{2}$$

$$\begin{cases} E_1 = \dfrac{\omega + M_0 - M_0 + \omega}{2} = \omega \\[2ex] E_2 = \dfrac{\omega + M_0 + M_0 - \omega}{2} = M_0 \end{cases}$$

✓

we have found two different expressions for $E_1(E_2)$:

$$E_1 = \frac{\omega + M_0 - \sqrt{(M_0-\omega)^2 + 4g^2}}{2}$$

and

$$E_1 = \omega c^2 + M_0 s^2 + 2gsc \qquad \text{with} \quad \vartheta = \frac{1}{2}\arctan\left(\frac{-2g}{M_0-\omega}\right)$$

show that they coincide!!!

Evolution of $a(t)$ and $p(t)$

survival amplitude:

$$a(t) = \langle S| e^{-iHt} |S\rangle$$

but

$$|S\rangle = \cos\theta |E_2\rangle + \sin\theta |E_1\rangle$$

$$\langle S| = \langle E_2| \cos\theta + \langle E_1| \sin\theta$$

then:

$$a(t) = c^2 \langle E_1| e^{-iHt} |E_1\rangle + s^2 \langle E_2| e^{-iHt} |E_2\rangle =$$

$$= c^2 e^{-iE_1 t} + s^2 e^{-iE_2 t}$$

Ergo, the survival probability reads

$$p(t) = |a(t)|^2 = c^4 + s^4 + c^2 s^2 e^{-i(E_1 - E_2)t} + c^2 s^2 e^{-i(E_2 - E_1)t}$$

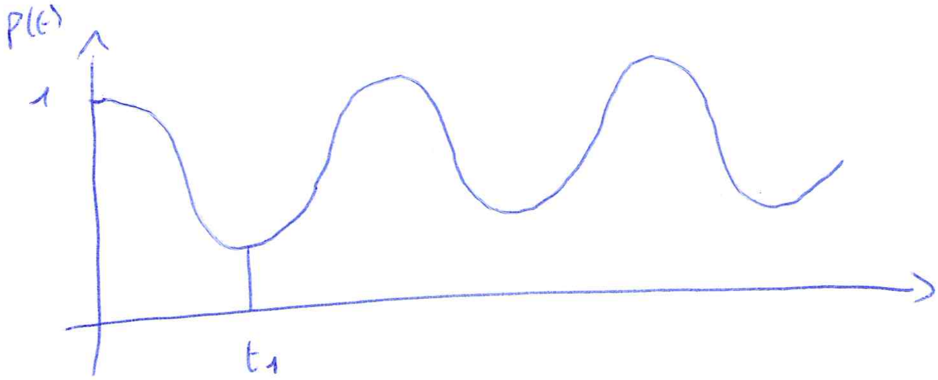
$$= c^4 + s^4 + 2c^2 s^2 \cos\left((E_2 - E_1)t\right)$$

$t = 0$

$$p(0) = c^4 + s^4 + 2c^2 s^2 = (c^2 + s^2)^2 = 1 \, !$$

$$t_1 = \frac{\pi}{\Delta E} = \frac{\pi}{E_2 - E_1}$$

$$p(t_1) = c^4 + s^4 - 2c^2 s^2 = (c^2 - s^2)^2 = P_{min}$$

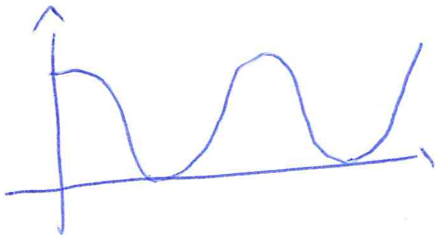


For

$$t_2 = \frac{2\pi}{E_2 - E_1} \longrightarrow |\psi(t_2)\rangle = e^{-iHt_2}|S\rangle = |S\rangle \;!!!$$

$$\underline{\underline{p(t_2) = 1}}.$$

Note, if $\varphi = \pi/4 \Rightarrow p(t_1) = 0 \rightarrow$ full cancellation.



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* note :

$$a(t) = \langle S | e^{-iHt} | S \rangle$$

$$1 = |E_1\rangle\langle E_1| + |E_2\rangle\langle E_2|$$

$$a(t) = \langle S | 1\, e^{-iHt}\, 1 | S \rangle =$$

$$= \langle S | \left(|E_1\rangle\langle E_1| + |E_2\rangle\langle E_2| \right) e^{-iHt} \left(|E_1\rangle\langle E_1| + |E_2\rangle\langle E_2| \right) | S \rangle =$$

$$= \sum_{i=1}^{2} |\langle E_i | S \rangle|^2 e^{-iE_i t} .$$

Short-time evolution

$$p(t) = \cos^4\Omega + \sin^4\Omega + 2\cos^2\Omega\,\sin^2\Omega\,\cos\big((E_2 - E_1)t\big)$$

For short time "t" we

$$\cos\big((E_2 - E_1)t\big) = 1 - \frac{1}{2}(E_2 - E_1)^2 t^2 + \dots$$

$$p(t) = \overbrace{\cos^4\Omega + \sin^4\Omega + 2\cos^2\Omega\,\sin^2\Omega}^{1}$$

$$- \cos^2\Omega\,\sin^2\Omega\,(E_2 - E_1)^2 t^2 + \dots =$$

$$= 1 - \cos^2\Omega\,\sin^2\Omega\,(E_2 - E_1)^2 t^2 + \dots$$

$$= 1 - \frac{t^2}{\Upsilon_z^2}$$

Υ_z is the "so-called" zero-time:

$$\Upsilon_z = \frac{1}{|\cos\Omega\,\sin\Omega\,(E_2 - E_1)|}.$$

'Quadratic coeffecient'

This is "important" for future evolutions.