QZE
induced by imperfect detectors

Bachelor thesis
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DEUTSCHSPRACHIGE ZUSAMMENFASSUNG:

Die vorliegende Bachelorarbeit trägt den Titel: "QZE - induced by imperfect detectors", und ist in englischer Sprache verfasst.

Sie untersucht die Realisierung des Quanten-Zenon-Effekts (QZE) durch sogenannte BANG-BANG Messungen eines unvollkommenden Detektors. Dabei bezieht sich die Unvollkommenheit insbesondere auf die mangelnde Fähigkeit Zustände beliebiger Energie zu detektieren. Die Messung erfasst folglich nur Zustände innerhalb eines endlichen Energie-Intervals. Als BANG-BANG Messung bezeichnen wir eine Folge von kurz nacheinander durchgeführten Messungen. Ist die Zeitspanne zwischen den einzelnen Messungen hinreichend gering, so stellt sich in Kombination mit einem unvollkommenen Detektor ein Quanten-Zenon-Effekt ein. Läuft der zeitliche Abstand zwischen den Messungen gegen 0, so kann der Zerfall eines Systems sogar verhindert werden.

Die erwähnten Zeitfenster werden in dieser Bachelorthesis einer statistischen Schwankung unterworfen, ringsum einen hinreichend kurzen Erwartungswert $t = \tau$, um einen realistischeren Detektor zu simulieren.

Die Grundmotivation zur Untersuchung des QZE auf diese Art und Weise ist, weitere Informationen über die Wechselwirkung zwischen quantenmechanischem System und der Messung zu sammeln. Der QZE basiert nämlich letzten Endes auf den Kollaps der Wellenfunktion.

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Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die im Literaturverzeichnis angegebenen Quellen benutzt habe. Ich habe die Regeln der guten wissenschaftlichen Praxis eingehalten.

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Frankfurt, den . . . .

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1 Introduction

The subject of this bachelor thesis is the realization of the Quantum Zeno Effect (QZE) induced by so-called BANG-BANG measurements of an 'imperfect' detector.

Usually, the QZE occurs due to the non-exponential nature of decays at short times. Namely, the survival probability has a initial flat behavior followed by oscillations, and only at later times it is very well approximated by the exponential decay law.

If the decaying quantum state is measured at very short time intervals (BANG-BANG), that is when the survival probability is still in its initial non-exponential part, the decaying process can be slowed down and, in the limit in which the interval between the measurements tends to zero, even stopped.

Hence, the QZE is a fascinating phenomenon which follows from the collapse of the wave-function. The mysteries and questions related to the collapse are one of the main motivations of this thesis. The phenomenon of the collapse follows certain well formalized rules. However, there are still some aspects that are unique and atypical for a physical process, such as the violation of time reversal. It is also possible that our understanding of the collapse is not yet complete, [11]. Studying the QZE by observing how and under which circumstances it appears can be helpful to gain more insights on the collapse process. For example, we will show that QZE can be realized in two different ways, one which follows from the non-exponential decay and one from the presence of an imperfect detector. We will mostly focus on the later case.

This thesis starts as a short summary of what the QZE is about and than concentrates on QZE induced by imperfect detectors. In particular, the BANG-BANG case as shown by F. Giacosa and G. Pagliara in Ref. [16], "Influence of the measurement on the decay law: the BANG-BANG case", will be the starting point for the study of this thesis. There, the measurements of the imperfect detector took place at equal time intervals ($\tau, 2\tau$, ...) We will go further by considering *randomized* BANG-BANG, which seems closer to what a real detector does.

WHO IS ZENO?:
The phenomenon of QZE is named after the famous greek philosopher Zeno of Elea, which constructed the paradox of Achilles and the tortoise. This and other paradoxes convinced Zeno that movement is a pure illusion. Only relatively recently, the paradox of Achilles and the tortoise was solved by using modern calculus, which explains that an infinite sum can be convergent. But Zeno's paradoxes contain some other physical premises that also deserve a careful look as discusses in "Zeno meets modern science" by Z.K. Silagadze [1]. According to Zeno, the footrace of Achilles and the tortoise assumes some observation

procedures:

- check the position of the contenders in the race;
- check again when Achilles reach the position the tortoise occupied at previous step;
- repeat the previous instructions until Achilles catch the tortoise (and this is an infinite loop because he never does).

As mentioned earlier, calculus teaches us that Achilles is behind the tortoise just for only a finite time interval despite of its infinitely many steps: even if infinitely many steps occurs, the total duration is finite. During this time interval the tortoise will be indeed always a step ahead of Achilles. All this is correct in the framework of classical mechanics, where external observer have no influence whatsoever on the system. Still, that is not the solution to the quantum version of the paradox (which indeed is at the basis of our real physical world, hence in ultimate analysis also responsible for macroscopic bodies such as Achilles and the tortoise). Now, things get directly linked to our QZE. Zeno implicitly assumes an ability to perform position measurements. Therefore two questions remain: is it even possible to perform infinitely frequent measurements, and how will the race be affected by back-reaction from these measurements? Indeed QZE shows us that with a frequently infinite number of measurements ($\tau \to 0$), the time development of a decaying system seems to be stopped. So, the QZE resembles the idea of the old Zeno paradox. In this case time is the flowing variable which seems to be stopped rather than physical movement. Therefore, in the tradition of a "Zeno-like" argumentation, time flow is just an illusion.

Even if QZE was first found and described as a symptom of a non-exponential decay, in Ref. [16] it was shown that a similar effect is realized also when the decay law is perfectly exponential (this is only an approximations since deviations are always present, however they can be very small and negligible for pour purposes). This different kind of QZE resulted from an imperfect detector, meaning that the measuring detector is not able to detect particles of every possible energy. Their results is the starting point of my bachelor thesis, which is a continuation of their work. I will investigate the case of randomized bang-bang measurements. This means that the measurements do not take place exactly at $\tau, 2\tau$, ..., but are smeared by randomized distributions (averagely, there is one measurement in the time interval 2τ, but the distance between two measurements is not necessarily τ, but can be smaller or larger depending on a random fluctuation). Then, I will compare the randomized BANG-BANG simulation to the results of Ref. [16].

COURSE OF ACTION:
To do so, we first introduce the physics of measurements in general. What is a measurement in a quantum-mechanical way, and how does it differ to the

classical view of a measurement? After a short recap of how those quantum-mechanical measurements are mathematically described, we will look at the main characteristics of a decaying process in both the classical and the quantum mechanical frameworks. With this theoretical foundation we will summarize the fundamental properties of the QZE as a consequence of a non-exponential decay. Then we move to the QZE generated by an imperfect detector and finally to its randomized version.
The results are derived from analytical calculations and from the application of statistical methods using the program 'Mathematica'. Below you can see an example of our main outcome. The randomized bang-bang imperfect detector generates a faster decay than the Bang-Bang detector acting at equal time intervals. The precise result depends on the statistical fluctuations and on the underlying probability distribution that a measurement takes place (see Sec. 5.2 for details), but the qualitative feature is general.

The results are derived from purley analytical calculations and the application of statistical mathematics using the program Mathematica. Below you can see an example of further introduced outcomes.

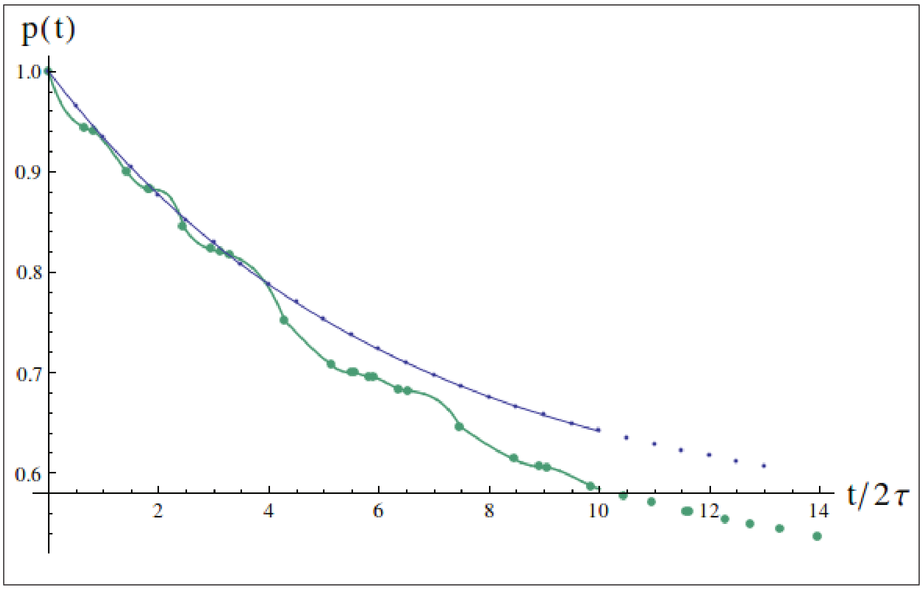

Figure 1: Sample of a later discussed randomized Bang-Bang measurements.

2 Theoretical foundation

2.1 About the physics of measurements

This chapter follows the book of D. Griffiths “Introduction to Quantum Mechanics” [2] which introduces the fascinating character of quantum mechanics in a simple way, so it is easy for students who encounter this subject for the first time to reflect the revolutionary quality of modern physics.
Other underlying references are [3, 4].

An essential concept of quantum mechanics is the description of a quantum through the wave function Ψ whose dynamics is subject to the famous Schrödinger equation (for simplicity one-dimensional):

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi \ , \tag{1}$$

where $\hbar$ stands for the well-known Planck’s constant, i for the imaginary unit, and m for the mass of the particle. $V(x)$ is the potential acting on it.
Once Ψ is known (given an appropriate initial condition $\Psi(t_0, x) = f(x)$), one is confronted with the task to interpret it. The answer to this question was given by Max Born, who established the *probabilistic interpretation*, in which $|\Psi|^2$ denotes the probability density to find a particle at a given time t at specific place x. Therefore it has to be normalized:

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, \mathrm{d}x = 1 \ . \tag{2}$$

Now let’s consider a given wave function with three disjunct segments A, B, C in which the probability to find the particle is nonzero. Let A be an interval with the highest probability, B is almost impossible to occur and C has a probability in between.
Suppose that the particle was measured at point C (this implies that measuring a particle’s location is possible). Where has the particle been right before our measurement? There are two main mindsets which I want to present.

First, there is the REALISTIC POSITION:
The most famous proponent of this assessment is Albert Einstein, who believed that the particle de facto was at point C. But if Einstein was correct, that would imply that the theory of quantum mechanics is incomplete. The indeterminacy would be a consequence of lack of information, and besides Ψ, other so called *hidden variables* would be needed. The french physicist Bernard d’Espagnat embodied this point of view in the following sentence: “*According to this argument the position of the particle was never undetermined, but was merely unknown to the experimenter*”.

Second, there is the ORTHODOX POSITION:
This position is a really fascinating one. The particle was nowhere! According to

Bohr: "*Physics concerns only what we can say about Nature*". The experiment is the cause that the particle appeared at a certain location. This way of thinking is known as the Copenhagen interpretation, and is particularly represented by Niels Bohr and his followers. This interpretation suggests that a particle just doesn't have a location before the measurement. The measurement itself creates the outcome, and that outcome is only at a probabilistic level specified by Ψ.

In 1964 John Bell published his inequalities, which should be fulfilled by any local deterministic theory. Quantum Mechanics violates them. Experimentally, violations of these inequalities were observed repeatedly in the last decades. That means that all theories which contain hidden variables and are locally deterministic can be ruled out. Yet, deterministic theories with nonlocal correlations are still compatible with the present knowledge: the most prominent example is the Bohm's mechanic, in which the hidden variables are the positions [5, 6, 7].

Let's go back to the mysteries that Ψ is giving us. If you measure the location of the particle a second time, right after the first measurement, it stays at point C. Obviously the wave function has drastically changed under the influence of the first measurement. This phenomenon is called the *collapse* of the wave function. Yet, the second measurement must take place shortly after the first one, because the wave function quickly changes form due to the time evolution. This *collapse* raises the question of why and especially how this breakdown of unitarity and time-reversal works.

The results of this bachelor thesis shall give some more hints about the process of a collapsing wave function. As we will observe how the collapse influences the lifetime of an unstable particle, we might get one step closer to understanding, or at least get a better feeling, of how a measurement interacts with a particle through the collapse.

2.1.1 Von Neumann measurements

In this subsection we use the work "Quantum Zeno effect by general measurements" by Kazuki Koshino and Akira Shimizu [8]. Especially, the fourth chapter of this article, in which they reviewed the quantum measurement theory and thus the *ideal* 'Von Neumann' measurements, is relevant for us. See also [9] for a deeper comprehension.

As we have seen in the previous subsection, we know that quantum systems exhibit a probabilistic nature. Therefore one has to perform many independent measurements: one resets the system before each individual experiment operation in order to have the same quantum state $|\psi\rangle$ for all runs. The alternative would be to prepare an ensemble of equivalent states and perform the same measurement independently for each system.

In this work, we are also interested on the *post-measurement state* $|\psi'\rangle$, because the further evolution is relevant to the study of decays. To calculate $|\psi'\rangle$, the so-called projection postulate must be used. Details are presented later on.

It should be stressed that many studies in the last several decades have further developed the quantum measurement theory. Although Landau and Lifshitz were pessimistic about the possibility, quantum measurement theory has turned out into a powerful tool. Instead of ideal measurements, the theory talks about *general measurements, imperfect measurements* or *non-ideal measurements*. The usefulness of these concepts has been confirmed by many experiments, the majority in quantum optics. For instance, studying the Zeno effect in general requires the quantum measurement theory. Yet, for the purposes of this thesis, we still use the projection postulate, since it works sufficiently well in dealing with a simple realization of the Zeno effect.

In particular, the subject of this bachelor-thesis is: "QZE induced by imperfect *detectors*". Namely, we consider detectors which can't measure the whole energy of emitted particles but only a limited energy interval (this is a step toward a more realistic treatment of decays). While in the previous work of [16] the measurements occurred after equal time interval τ (that is, first measurement at τ, second at 2τ, and so on: BANG-BANG), we randomized their occurrence: first measurement at $t_1 \subset (0, 2\tau)$, second at $t_2 \subset (t_1, t_1 + 2\tau)$, and so on: *randomized* BANG-BANG (details later on).

Despite our attempt to generate a more realistic simulation of detectors, we will assume instantaneous measurements in our model. Therefore we must introduce the very basic properties of ideal measurements.

The theoretical interpretation of a measurement process in line with quantum mechanics demands three steps:

- Producing main units of particles, each of one represented by a wave function Ψ (preparation at $t = 0$).
- A temporal development of each wave function Ψ between 0 and τ according to the Schrödinger equation (unitary evolution).
- Measurement and registration of the result at the time τ. Here the detector plays an important role and the aforementioned collapse takes place.

Note, not only the third step is difficult. As Willis Eugene Lamb pointed out, the first step is also important. Preparing a system means fixing the external conditions that define the initial situation of a system (for example that a particle is trapped within a certain area or has a certain magnetic momentum). Indeed, the measurement can be also regarded as a preparation of a different quantum state. This will be particularly important in this work, since a series of measurements will be considered.

In the end, the result of a single measurement is a real number or a characteristic property of the examined quantity. If the measurement is repeated, you don't necessary get the same result, but at best a statistical distribution of values that the relevant quantity can possibly obtain. With a given preparation, the probability of certain outcome measurement which is bound to the time dependent wave function can be predicted.

John Von Neumann was the one who first formalized this concept 1932 in his textbook about mathematical quantum mechanics. Von Neumann's aim was to describe both the measured system as well as the measuring apparatus and their interaction in the framework of QM. To this end, he has been the first who has taken into account the wave function of the detector into the mathematical formalism (see below).

In quantum mechanics, observable values like momentum, spin or energy of a physical system are described with eigenvalues of Hermitian operators. All of the actual measured quantities are within the real numbers due to the Hermitian nature of the operators. Possible eigenvalues can be discrete, what leads to a so called discrete spectrum. Though, if the eigenvalues are spread over an interval of real numbers, one has a continuos spectrum.

Preparation:
Preparing a quantum mechanical state means to specify it at a certain time ($t = 0$) before the measurement occurring at later time ($t = \tau > 0$). That specification is formally taken in the abstract Hilbert-Space which is attributed to the physical quantity. Let $\hat{Q}$ be the Hermitian operator of the physical quantity that we aim to measure and q_n and $|\phi_n\rangle$ the associated eigenvalues and the eigenvectors ($\hat{Q}\,|\phi_n\rangle = q_n\,|\phi_n\rangle$). One may use $\{|\phi_n\rangle\}$ as a basis of the Hilbert-Space $\mathcal{H}_{\mathcal{Q}}$. Every arbitrary state $|\psi\rangle$ prepared at $t = 0$ can be explicitly depicted within that basis

$$|\psi\rangle = \sum c_n\,|\phi_n\rangle \ , \tag{3}$$

where $c_n = \langle\phi_n|\psi\rangle$ stands for the components of the vector of state.

Measurement:
Upon measuring the observable $\hat{Q}$, the probability of getting the value q_n is given by the square of the absolute value of the corresponding component of the state vector: $|c_n|^2 = |\langle\phi_n|\psi\rangle|^2$ (Born's postulate). The approach for physical systems with a continuous spectrum of eigenvectors is similar, although it requires more advanced mathematics.
von Neumann went further and described also the macroscopic measurement apparatus as basis vectors $\{|M_n\rangle\}$ in a Hilbert-Space $\mathcal{H}_{\mathcal{M}}$. The apparatus (e.g. a detector) displays the state of the observed system after interacting with it at $t = \tau$. The "pointer position" n of the measuring apparatus is shown by the detector state $|M_n\rangle$, if the system's state was $|\phi_n\rangle$ before the interaction.

Before the measurement, the state of the apparatus is defined as $|M_0\rangle$, which shows that there was no measurement yet. The interaction of the system and the apparatus results according to the linear time-evolution of the Schrödinger equation. Of before measurement $|\psi\rangle = |\phi_n\rangle$ (eigenstate), then:

$$\text{at } t = \tau^-\text{:} \;\; |\phi_n\rangle\,|M_0\rangle \xrightarrow{t} |\phi_n\rangle\,|M_n\rangle \;\; \text{at } t = \tau^+ \; , \tag{4}$$

where the product is the so called tensor product of the two states of the entire system. It can be simply viewed as a logical conjunction. According to this scheme we get an explicit correlation of the possible states of the system and the possible pointer positions of the apparatus. Given that this process happens instantaneously, it is called an *ideal measurement.* Then, for a general initial state $|\psi\rangle = \sum c_n |\phi_n\rangle$, one has:

$$\text{at } t = \tau^-\text{:} \;\; \left(\sum c_n\,|\phi_n\rangle\right)|M_0\rangle \xrightarrow{t} \sum c_n\,|\phi_n\rangle\,|M_n\rangle \;\; \text{at } t = \tau^+ \; , \tag{5}$$

hence one arrives at the conclusion that a superposition of macroscopic states exists (Schroedinger's cat). This property is matter of ongoing discussions. Models which employ the collapse of the wave function as a physical process have been developed in order to answer this question [10]. Having established the general theoretical basis of QM, we need to focus on decays.

2.2 About the physics of decay

For the following part, we refer to the review article "Decay theory of unstable quantum systems" by L. Fonda, G. Ghirardi and A. Rimini [11].
A quantum decay describes a physical process in which and unstable particle transforms in something else (new particles can be emitted in particle decays, daughter nuclei in nuclear decays, etc.). The description of this physical procedure is of great relevance because almost all known elementary particles have an unstable nature. Besides the elementary particles, even a considerable number of unstable and therefore radioactive nuclei, both naturally and artificially produced, are known in nuclear physics. The aim to account for natural radioactivity actually first led physicists to the elaboration of a quantitative theory for the decay. Although the classical approach of describing a decay can provide us with good approximate results for the decay law, quantum effects need to be considered in order to understand why particles start to decay at all.

The proper characterization of an unstable system requires a critical analysis of the theoretical description of the naive approach of earlier times in quantum mechanics. This analysis can be seen in the given review [11], in which advanced quantum mechanical effects are described. For this bachelor thesis we need only general and quite understandable formulas.

First, we will shortly cover THE CLASSICAL THEORY:
The discovery of natural radioactivity in 1896 by H. Bequerel, marks the beginning of the studying of the decaying processes. He observed that the potassium

uranite sulfate emitted penetrating radiation which could be revealed by a photographic plate. Rutherford could identify the emitted radiation as a composition of α-particles and free flying electrons which are referred to as β-radiation. The third well known radiation was the so called γ-radiation which was shown later by Villard. The famous Mme Sklodowska-Curie was one of the first of many scientists searching for other radioactive elements, and in 1898 she introduced polonium and radium as two other radioactive elements.

The classical theory of decay is simple. It is based on the assumption that there is a certain probability of radioactive nuclei to undergo a decaying process and that this probability doesn't depend on the past history of the individual decaying nuclei. One can say, that decaying processes were thought to be independent and that they wouldn't keep no memory of the past. It follows immediately that the variation of $N(t)$ of radioactive nuclei, which gives us the number of decaying nuclei at a certain time t during an infinitesimal interval of time dt must be proportional to said $N(t)$. That leads us to

$$dN(t) = -\frac{1}{\tau} N(t)\ dt. \tag{6}$$

Radioactivity is a frequently given example for exponential decay, since the decay process for $N(t)$ is based on following formula,

$$N(t) = N_0 e^{-\lambda t}. \tag{7}$$

In this case τ is the *lifetime* of a radioactive nucleus, whereat λ is the reciprocal of it. It is called the *characteristic constant*, sometimes referred to as *decay rate*. N_0 is the number of particles at the time the observation of the radioactive decay started.

As mentioned earlier, the classical approach above is just a phenomenological description of the process, since no effort is made to understand the mechanism that trigger the radioactive decay. Anyway, it has to be remarked that the classical equation accounts very well for the experimentally observed facts.
At this point we should also mention the so-called tunnel effect, which is at the basis of many quantum decays. An α-particle, which is a preformed state inside the nucleus, lies in a potential well and has a non-zero probability of crossing the barrier and be therefore emitted. The first theoretical description of this mechanism was published in 1928, by R. W. Gurney, E. U. Condon, and G. Gamow as an application of quantum mechanism to the referred problem of nuclear stability which can be overcome by the tunnel-effect, [12];[13]. However, the first formal approach to the problem of decay was given by Weisskopf and Wigner in 1930, who tried to formalize the problem independently from the shape of the interaction, [14];[15].

We now focus on the QUANTUM THEORY of decay:
For the quantum description of the decay process, we start with the key question of determining the probability of finding, by doing a measurement at time

t, the quantum system in the same initial state. In particular:

- $|S\rangle$ always stands for an unstable state prepared at $t = 0$.

- The survival probability amplitude at $t > 0$ is given by ($\hbar = 1$):

$$a(t) = \langle S| e^{-iHt} |S\rangle \ , \tag{8}$$

where H is the Hamiltonian that lays underneath the dynamics of the observed quantum system.

- The survival probability is then given by

$$p(t) = |a(t)|^2 \, . \tag{9}$$

The number of particles at a given time t is still $N(t) = N_0 p(t)$.

The survival probability $a(t)$ is calculated as

$$a(t) = \langle S| e^{-iHt} |S\rangle = \int_{-\infty}^{\infty} d_S(E) e^{-iEt} \mathrm{d}E, \tag{10}$$

where $d_S(E)$ is the probability distribution that the unstable state $|S\rangle$ has an energy between E and $E + \mathrm{d}E$.
In the limit in which $|S\rangle$ is stable, the distribution $d_S(E)$ is:

$$d_S(E) = \delta(E - M_0), \tag{11}$$

which results to

$$a(t) = e^{-iM_0 t} \to p(t) = 1. \tag{12}$$

In general, however, one may have decays: using the Schwarz inequality one has

$$|a(t)| = \left| \langle S| e^{-iHt} |S\rangle \right| \leq |\langle S|S\rangle| = 1 \tag{13}$$

Usually, the Breit-Wigner distribution is a very good approximation:

$$d_{BW}(E) = \frac{\Gamma}{2\pi} \frac{1}{(E - M_0)^2 + \Gamma^2/4}. \tag{14}$$

Using the technique of complex residual integration we get following life expectancy,

$$a(t) = e^{-iM_0 t - \Gamma t/2} \to p(t) = e^{-\Gamma t} = e^{-\frac{t}{\tau}}, \tag{15}$$

where Γ is the width of the distribution and $\tau = \frac{1}{\Gamma}$ is the lifetime of the particle.

Notice that for $\Gamma \to 0$ one correctly recovers $d_{BW}(E) = \delta(E - M_0)$. In general, however, the BW distribution cannot be correct because it extends to infinite negative energies and does not contain some kind of form factor at high energies. These modifications lead to deviations from the exponential law at large and small times, respectively.

3 Common QZE

This chapter will be referring to the work of F. Giacosa and G. Pagliara [16], [17], as well as the paper of B. Misra and E. Sudarshan [18]. Furthermore the references [19, 20, 21, 22] are of importance.

3.1 Basic definitions

In later cases $|S\rangle$ will always stand for an unstable state that is prepared at $t = 0$.

The previously mentioned survival probability amplitude at $t > 0$ looks in this formalism as followed, $a(t) = \langle S| e^{-iHt} |S\rangle$ with $\hbar = 1$.

The survival probability is still given by $p(t) = |a(t)|^2$.

In agreement with chapter 2.2 of this thesis, the Number of particles at a given time t is still $N(t) = N_0 p(t)$.

3.2 Non-exponential decay

As mentioned before, the exponential solution is the most used mathematical description for radioactive decay. However, in quantum mechanics that doesn't seem to be fully consistent. Deviations from exponential decay have been predicted for short as well as long times. And such deviations have already been confirmed by the Team of the Department of Physics, at Austin, Texas [20].

The Team around Steven R. Wilkinson presented their experimental evidence in a quantum tunnelling experiment. The simplicit of their concept enables detailed comparison with existing theoretical predictions. Ultra-cold sodium atoms were trapped in an accelerating periodic optical potential created by a standing wave of light. This trapped state can be considered as an unstable quantum system that decays into a reservoir through tunneling. Atoms could escape the wells and the remaining number of particles could be measured as a function of interaction time for a fixed value of the well depth and acceleration. They observed that in a short time scale the survival probability is initially constant before developing the characteristics of an exponential decay. The predicted deviation from exponential decay in this short time scales arises from the circumstance of reversibility in this time frame of the coupling between the system and the reservoir.

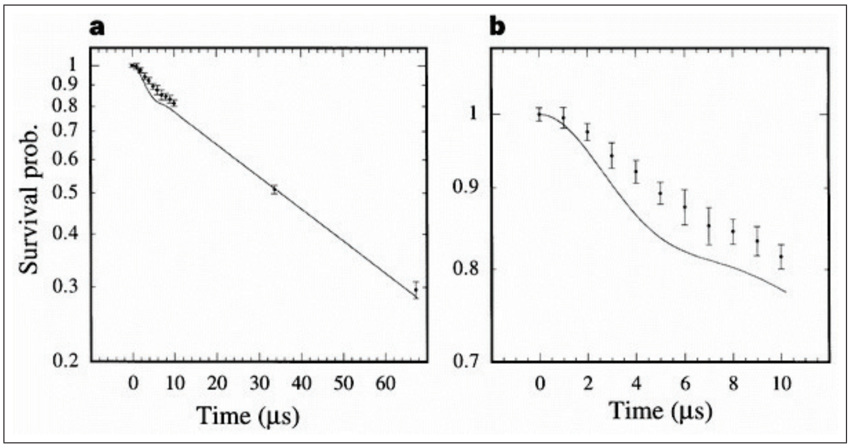

Figure 2: The experimental data are shown with error bars. The solid line signifies the theoretical prediction. Potential $V_0/h = 74kHz$ [20].

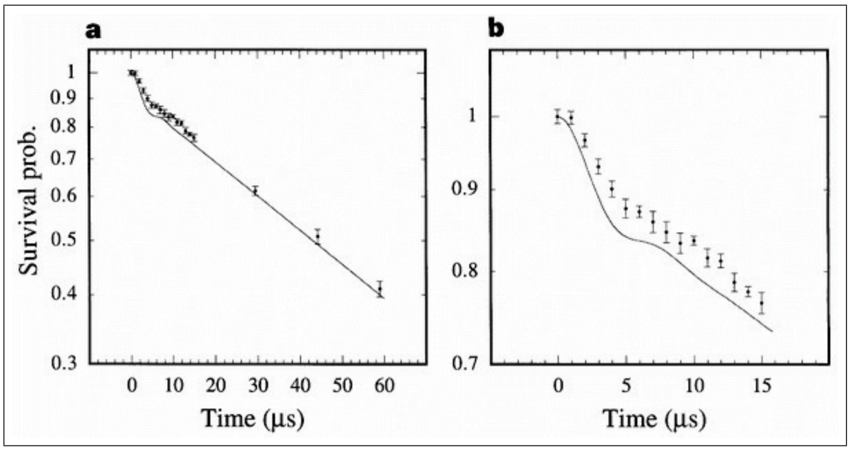

Figure 3: The experimental data are shown with error bars. The solid line signifies the theoretical prediction. Potential $V_0/h = 92kHz$ [20].

As you can see, the distribution of experimental data show a slightly swinging schema before they run over to an exponential one. At the right time, this cirumstance even can produce an anti-zeno-effect, where particles decay faster than expected after the exponential concept.

3.2.1 Deviatons from exponential law at short times

Besides the experimental fact, that radioactive decay has an initially non-exponential character, we now take a look at the theoretical background of the phenomenon. Let us calculate the survival probability for short times by using the taylor expansion for the amplitude:

$a(t) = \langle S| e^{-iHt} |S\rangle = 1 - it \langle S| H |S\rangle - \frac{t^2}{2} \langle S| H^2 |S\rangle + ...$

$a^*(t) = \langle S| e^{+iHt} |S\rangle = 1 + it \langle S| H |S\rangle - \frac{t^2}{2} \langle S| H^2 |S\rangle + ...$
The survival probability we are looking for then looks like this,

$$p(t) = |a(t)|^2 = a^*(t)a(t) = 1 - t^2 \left(\langle S| H^2 |S\rangle - \langle S| H |S\rangle^2 \right) + ... = 1 - \frac{t^2}{\tau_Z^2} + ... \tag{16}$$

where τ_Z is the 'Zeno-time' and given by

$$\tau_Z = \frac{1}{\sqrt{\langle S| H^2 |S\rangle - \langle S| H |S\rangle^2}}. \tag{17}$$

$p(t)$ decreases according to this formula quadratically. Keep in mind that this calculation only works for short times.

3.3 Resulting QZE

Now let's apply Von Neumann measurements to our non-exponential decay. We perform N measurements at time intervals $\tau = \frac{t}{N}$, to check if the system is still in its initial unstable state $|S\rangle$. Each measurement will 'project' back the system's state onto its initial state $|S\rangle$. Then the evolution starts anew according to Schrödinger's equation with the condition $|S\rangle$.

The survival probability $p^{(N)}(t)$ at the final time $t = N\tau$ is given by

$$p^{(N)}(t) = p(\tau)^N = p\left(\frac{t}{N}\right)^N \simeq \left[1 - \left(\frac{t}{N\tau_Z}\right)^2\right]^N \xrightarrow{N\ large} exp\left(\frac{-t^2}{N\tau_Z^2}\right) \xrightarrow{N\to\infty} 1. \tag{18}$$

If we send N against infinity $N \to \infty$ we talk about infinitely frequent measurements, BANG-BANG measurements. As we can see, the quantum mechanical evolution is then 'frozen' in its initial state. This is what we call the Quantum-Zeno-Effect (QZE). It's a consequence of the short-time behaviour we have previously shown.

4 Influence of an imperfect detector

Ref. [16], entitled “Influence of the measurements on the decay law: the bang-bang case” and in Ref. [23], entitled “Pulsed and continuous measurements of exponentially decaying system” , are the basis of this chapter.

4.1 Pulsed measurements

Now that we have examined the Quantum Zeno Effect (QZE) due to the non-exponential decay, we turn our attention to the purely exponential case. As shown before, in that circumstance no QZE appears (if the detector works perfectly). The full time-evolution of the state $|S\rangle$ reads

$$e^{-iHt}\,|S\rangle = a(t)\,|S\rangle + \int_{-\infty}^{+\infty} \mathrm{d}k b(k,t)\,|k\rangle\,, \tag{19}$$

with

$$a(t) = e^{-\Gamma t/2}, \Gamma = g^2 \text{ and } b(k,t) = \sqrt{\frac{\Gamma}{2\pi}}\frac{e^{-ikt} - e^{-\Gamma t/2}}{k + i\Gamma/2}. \tag{20}$$

The ket $|k\rangle$ represents the decay product of $|S\rangle$. This can be, for instance, a two-photon state (emitted back-to-back) if $|S\rangle$ is identified with a neutral pion. The quantity k denotes the the momentum (and hence the energy) of one of the outgoing photons. In the exponential limit, we obtain the previously mentioned survival probability of a single particle

$$p(t) = |a(t)|^2 = e^{-\Gamma t}. \tag{21}$$

But to get this result, we tacitly assumed that our detector is perfect. In reality, one typically measures the decay product $|k\rangle$. The probability that our detector ‘sees’ the state $|k\rangle$ at the instant t (the probability to hear the measuring apparatus clicking) is given by

$$w(t) = \int_{-\infty}^{\infty} \mathrm{d}k\,|b(k,t)|^2 = 1 - e^{-\Gamma t}\ , \tag{22}$$

hence the non-decay probability is $p(t) = 1 - w(t) = e^{-\Gamma t}$.

However, in reality the boundaries of the previous integral over $\mathrm{d}k$ are not extending from minus infinity to infinity. Real, and therefore imperfect detectors, can only detect final states in a certain energy range $(-\lambda, \lambda)$. If we bring in this new consideration, we need to modify the function $w(t)$. if we assume to perform a measurement with an imperfect detector at the time t, the updated probability to hear the measuring apparatus clicking is given by

$$w_\lambda(t) = \int_{-\lambda}^{\lambda} \mathrm{d}k\,|b(k,t)|^2\,. \tag{23}$$

Now let us talk about performing multiple (N) measurements at the times τ, 2τ, ..., $t = N\tau$. The following conclusion is not easy to derive, so for a technical derivation, see Refs. [16, 17]:

$$p^{BB}_{no-click}(t = N\tau) = 1 - w_\lambda(\tau)\frac{1 - e^{-\Gamma t}}{1 - e^{-\Gamma\tau}} \ . \tag{24}$$

The previous equation is fundamental for the foundation of the QZE induced by imperfect detectors and hence for the present thesis.

Intuitively, the reasoning goes as follows. The probability to hear click at the N-th measurement is equal to

$$p^{BB}_{click}(t = N\tau) = p((N-1)\,\tau)w_\lambda(\tau) \ . \tag{25}$$

This is so because to hear the click at the N-th step, one needs to have the state $|S\rangle$ after $N-1$ measurements. Namely, only the state $|S\rangle$ can generate the click exactly at the N-th step. Then:

$$p^{BB}_{no-click}(t = N\tau) = 1 - \sum_{k=1}^{N} p^{BB}_{click}(t = k\tau) \ . \tag{26}$$

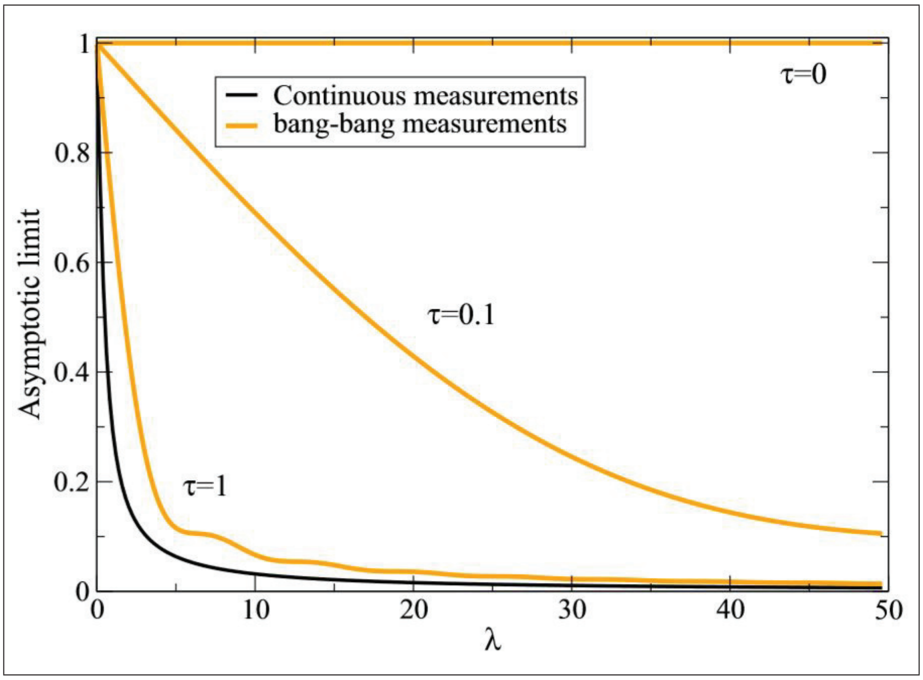

Figure 4: The impact of bang-bang-measurements as shown in the paper 'Pulsed and continous measurements of exponentially decaying systems' by Francesco Giacosa and Giuseppe Pagliara [23]. The continuous measurement refers to a process non described in this work, for details see Ref. [16].

As the plot shows, for finite λ (that is, imperfect detector) one can increase the decay time: the state does not decay if τ is short enough. Hence, we obtain a QZE which is driven by the imperfect detector (even if the intrinsic decay law was considered to be exactly exponential). Indeed, in the limit $\tau \to 0$ one obtains $p^{BB}_{no-click}(t = N\tau) = 1$: no decay at all.

5 Randomized Bang-Bang

5.1 Course of action

In Ref. [16] the first measurement occurs at τ, the second at 2τ, and so on. Our approach is to randomize the time which passes between two measurement. Naturally, the mean time should still be τ. The intention is that a randomized BANG-BANG measurement seems to be closer to reality, since the exact way how a measuring apparatus works is not fully understood. We use the computer program Mathematica to simulate our BANG-BANG measurements by implementing the formula that we introduced in chapter 3.1. We simulate an imperfect detector in three different ways:

- Homogenous distribution
- Parabolic distribution
- Segmented homogenous distribution

HOMOGENOUS DISTRIBUTION:
First, we are going to use a generated table of random numbers which are distributed homogeneously around our mean time τ. The bottom line of the interval is set at 0 and the upper one at 2τ. (That is, first measurement at $t_1 \subset (0, 2\tau)$, the second one at $t_2 \subset (t_1, t_1 + 2\tau)$, etc. We also our mean time to $\tau = 0.5$ [arbitrary time units], so our interval extends from 0 to 1. This setting and time frame will be used for the other distributions as well. We show a diagram with the points of the randomly picked time frames given from the generated random table as well as the corresponding plot of the quantity $p_{no-click}^{BB\text{-randomized}}(t = N\tau)$.This function is then compared to the result of Eq. (24) introduced in the previous chapter.

PARABOLIC DISTRIBUTION:
Here we use a different distribution for our generated table of random numbers. We choose one that has a higher probability around τ and flattens out to the edges of the allowed time frame. Naturally one may use take a Gaussian curve as a distribution for the table, but we use an easier parabola instead (because we want the probability for the number to be exactly zero at the edges of our interval). We will use following formula to ensure the null points and the peak at our mean time τ,

$$b_P(y) = y + c \cdot y(y - \frac{1}{2})(y - 1) \ , \tag{27}$$

where c is the constant we use to determine how much we want to increase the probability of measurements near our mean time τ. For the graphs below, we chose $c = 3.75$. Just like in the homogenous case, we show the point diagram and the corresponding plot.

SEGMENTED HOMOGENOUS DISTRIBUTION:
Our third approach can be seen as a tool to interpolate both the homogenous and the parabolic cases. We again use a homogenous table with our familiar mean time τ, but we also segment our interval of possible time frames in which our detector can measure. In this way we ensure that there is a certain minimum time interval before our detector can be ready to measure again. This is done by using the following formula as the distribution for the generated table of randomized numbers,

$$b_S(y) = c \cdot y + \frac{(1-c)}{2} \ , \tag{28}$$

where the parameter c defines the intrinsic boarders of our time frame (in relation to the previously (first case) entirely "populated" interval $[0,1]$). For our purpose, c of course ranges from 0 to 1. If we chose c to be 0, we force every "randomly" picked number to be at 0.5. This will naturally gives us the same results as the non-randomized case [23]. (This is the case of Eq. (24) that we used for comparison in all our examples). On the other hand, if we chose c to be 1, we get exactly the same results of case 1 (first case). If we pick a number between 0 and 1, the results can be seen as an interpolation of our homogenous and parabolic distributed cases. It generate more time-dots near to our mean time τ and less dots to the boarders. In this formula, we use the intermediate value $c = 0.7$, which simulates a QZE-influence right in between our non-segmented homogenous and our parabolic cases.

5.2 Results

5.2.1 Homogenous distribution

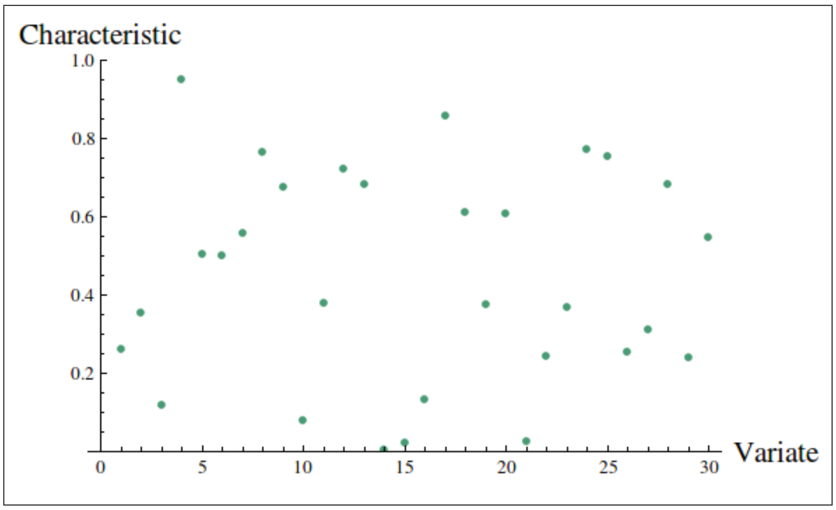

Figure 5: Random table of an homogenous distribution.

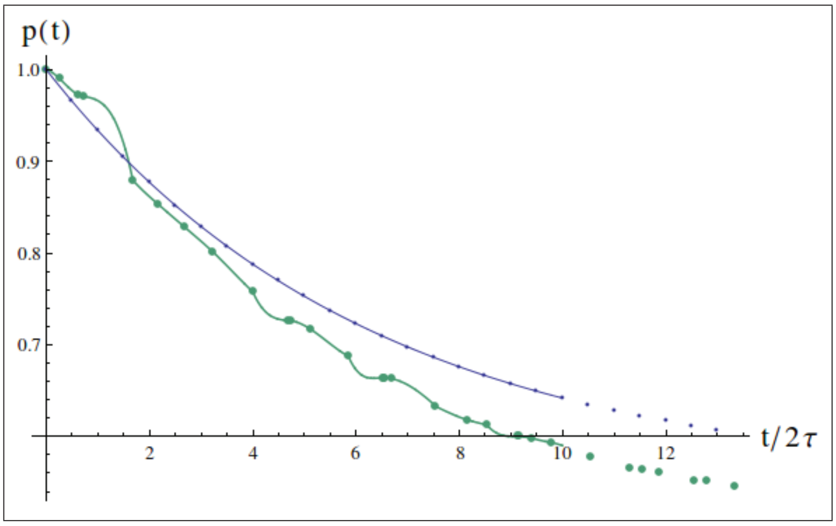

Figure 6: The survival probability belonging to figure 4.

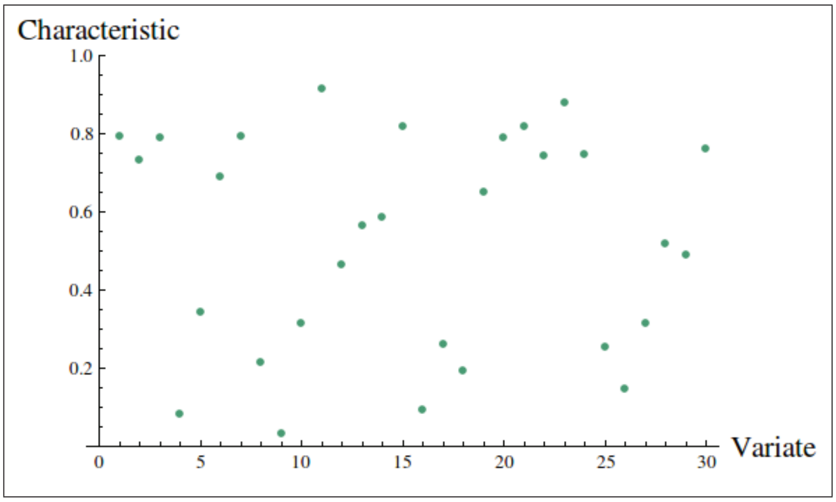

Figure 7: Random table of an homogenous distribution.

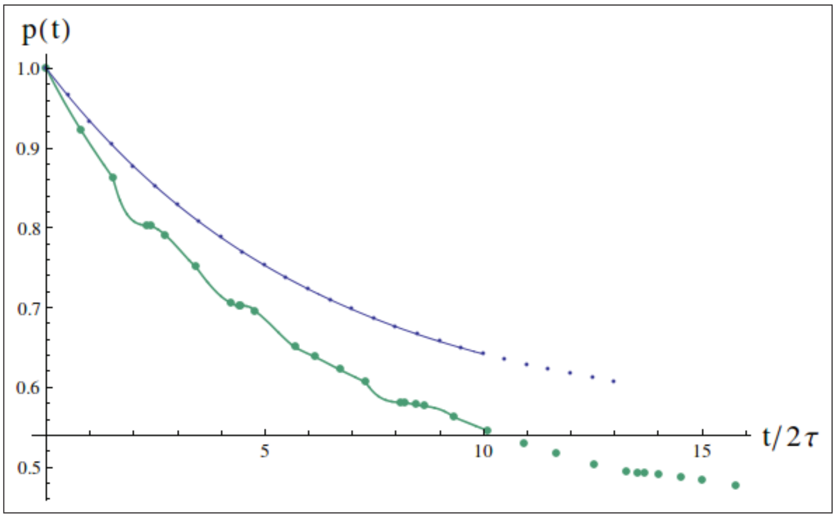

Figure 8: The survival probability belonging to figure 6.

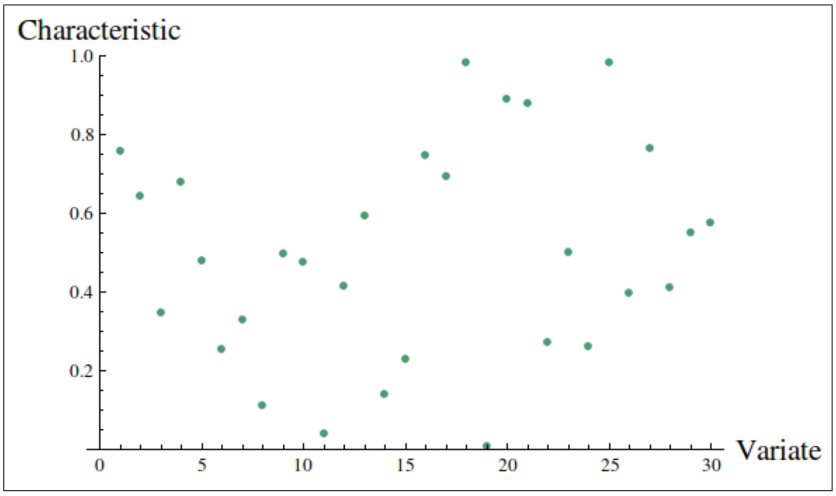

Figure 9: Random table of an homogenous distribution.

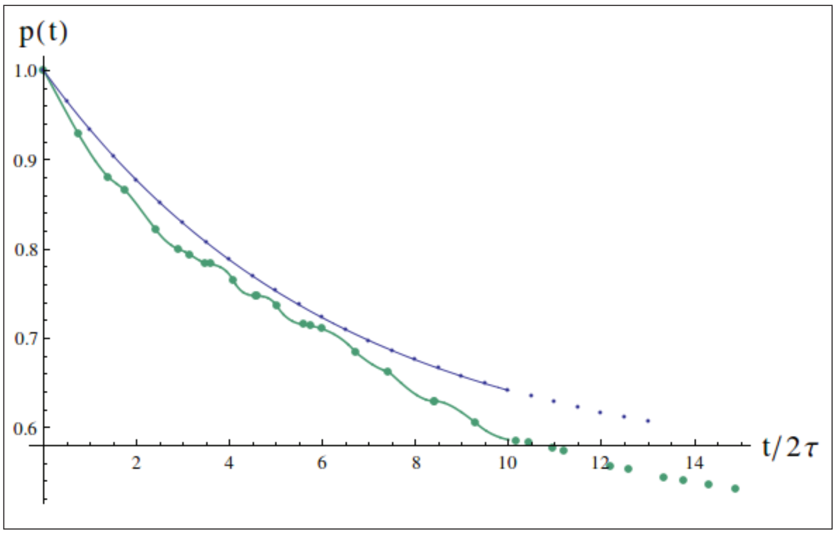

Figure 10: The survival probability belonging to figure 8.

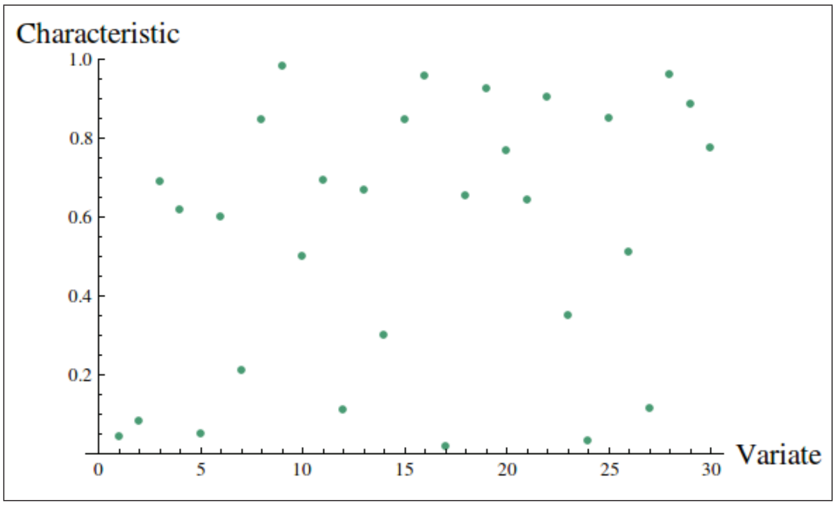

Figure 11: Random table of an homogenous distribution.

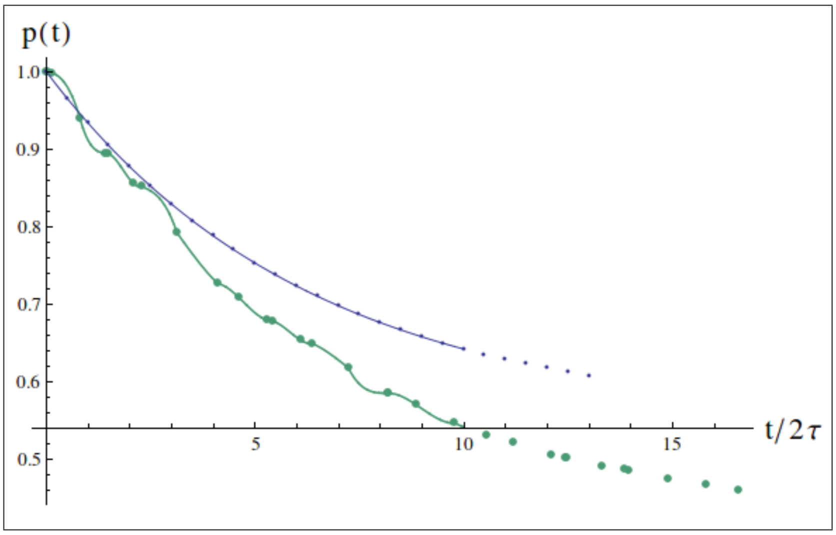

Figure 12: The survival probability belonging to figure 10.

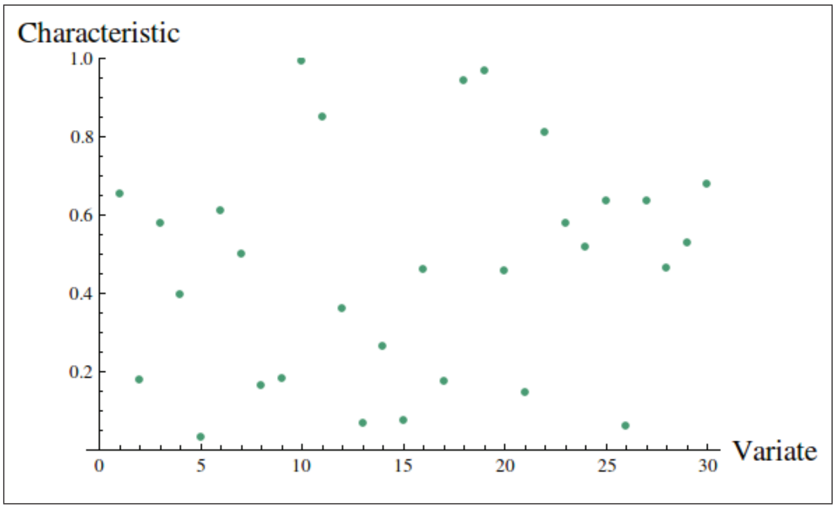

Figure 13: Random table of an homogenous distribution.

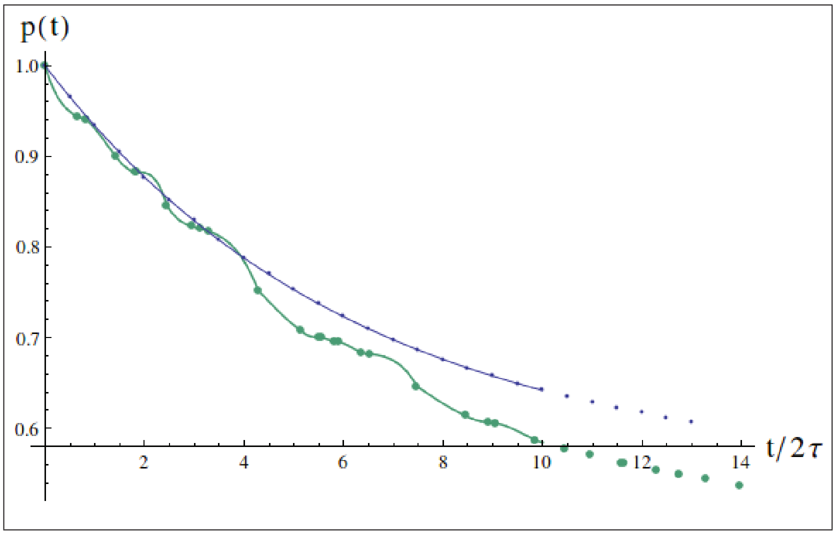

Figure 14: The survival probability belonging to figure 12.

5.2.2 Parabolic distribution

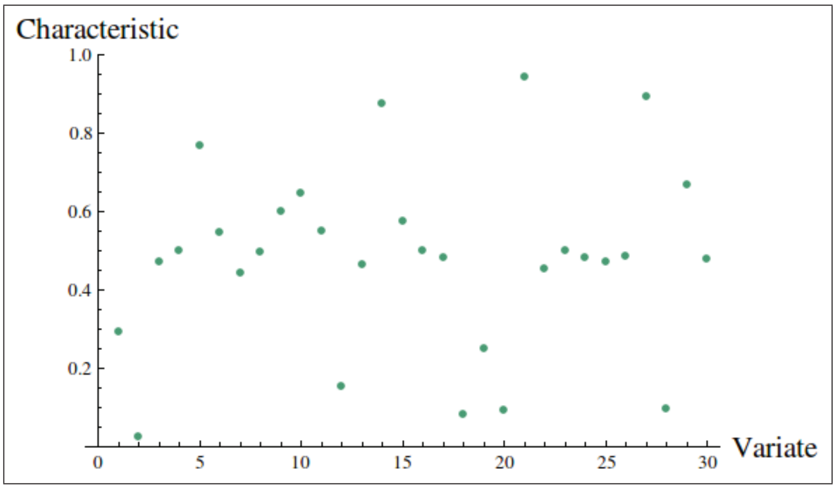

Figure 15: The distribution of the first parabolic plot.

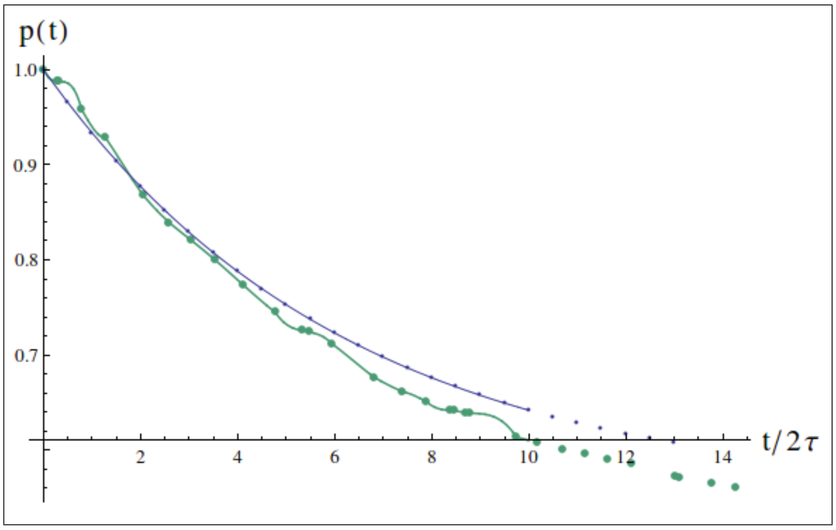

Figure 16: The survival probability of the first parabolic distribution.

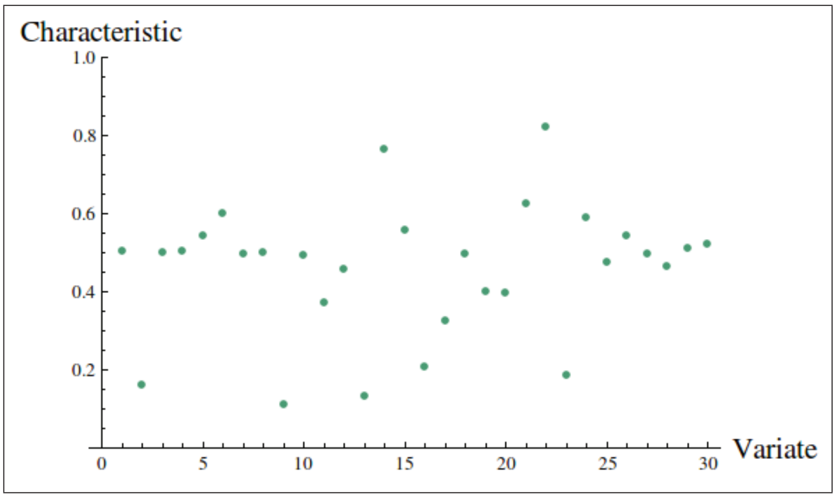

Figure 17: The distribution of the second parabolic plot.

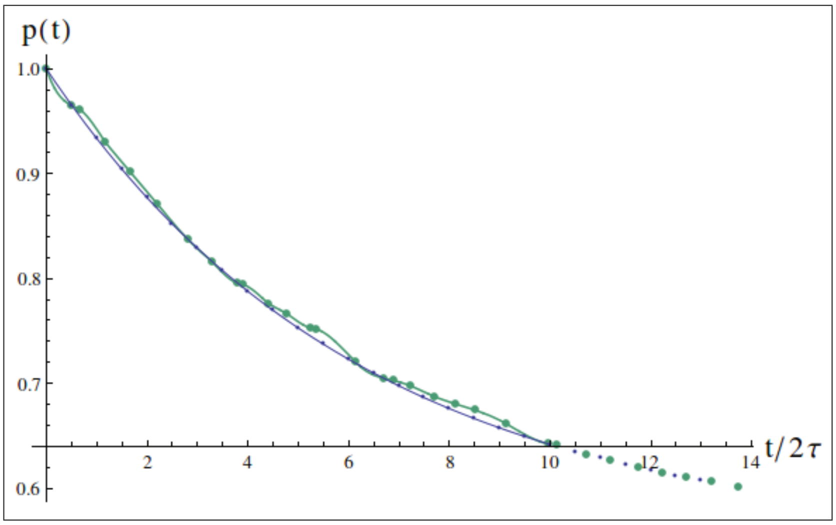

Figure 18: The survival probability of the second parabolic distribution.

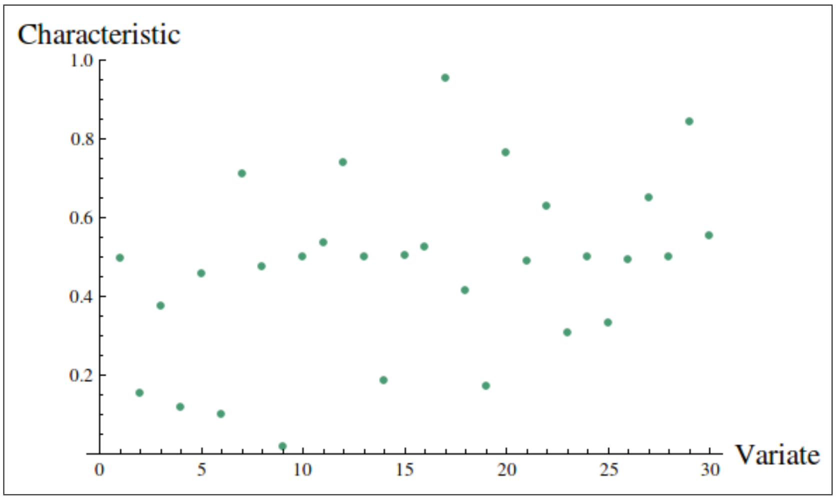

Figure 19: The distribution of the third parabolic plot.

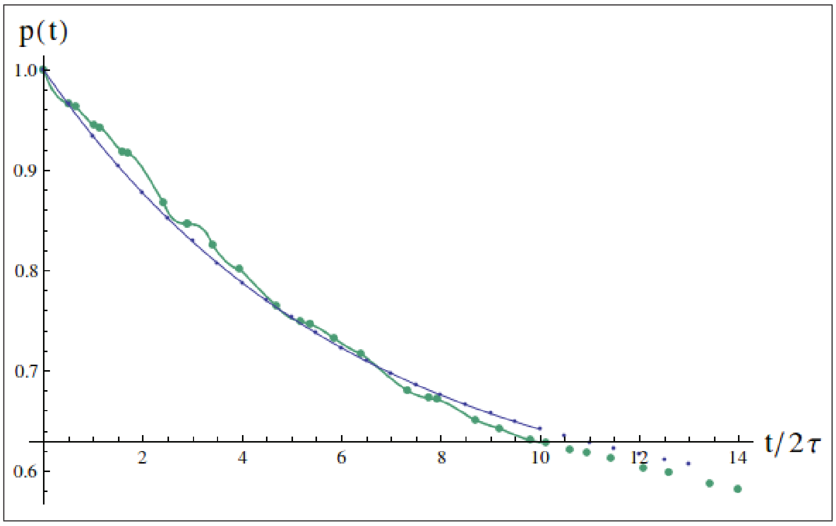

Figure 20: The survival probability of the third parabolic distribution.

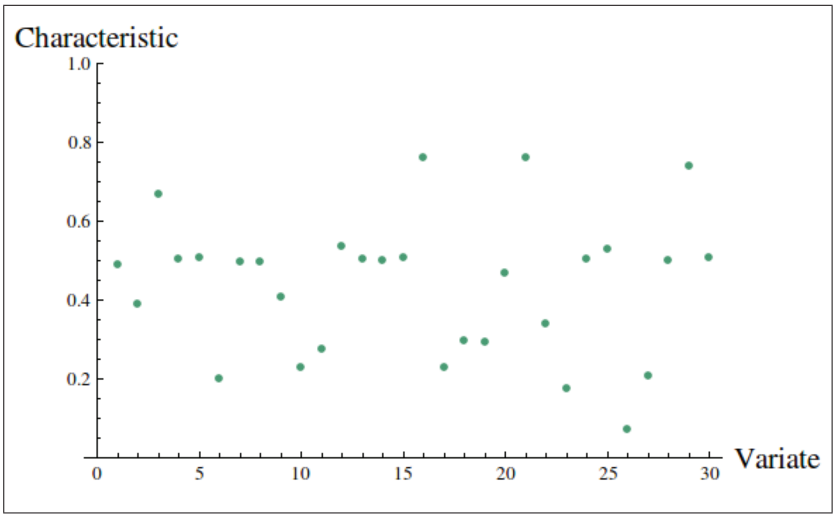

Figure 21: The distribution of the fourth parabolic plot.

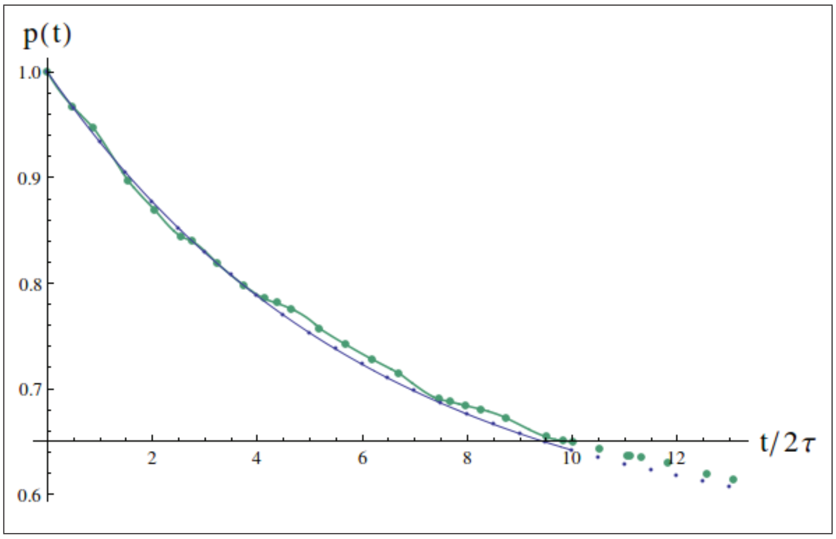

Figure 22: The survival probability of the fourth parabolic distribution.

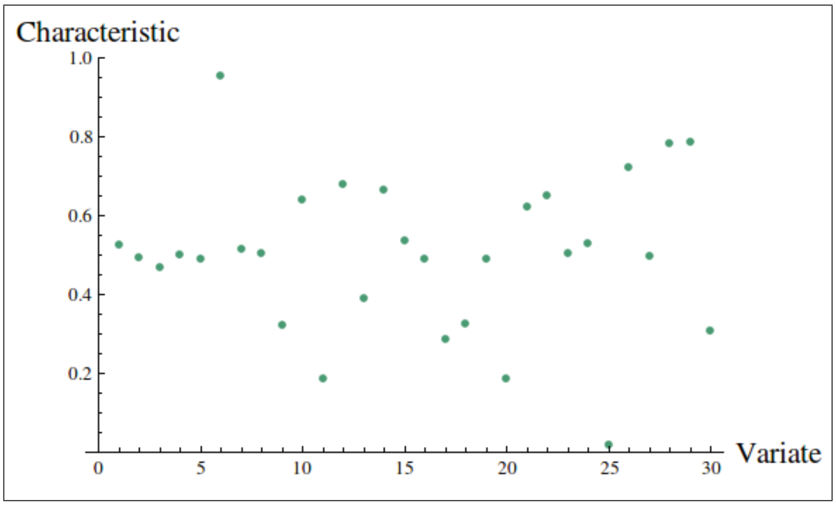

Figure 23: The distribution of the fifth parabolic plot.

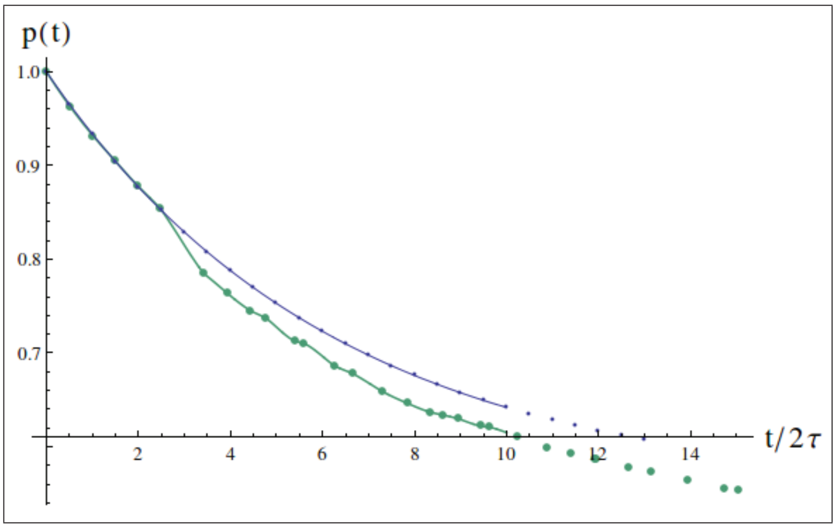

Figure 24: The survival probability of the fifth parabolic distribution.

5.2.3 Segmented distribution

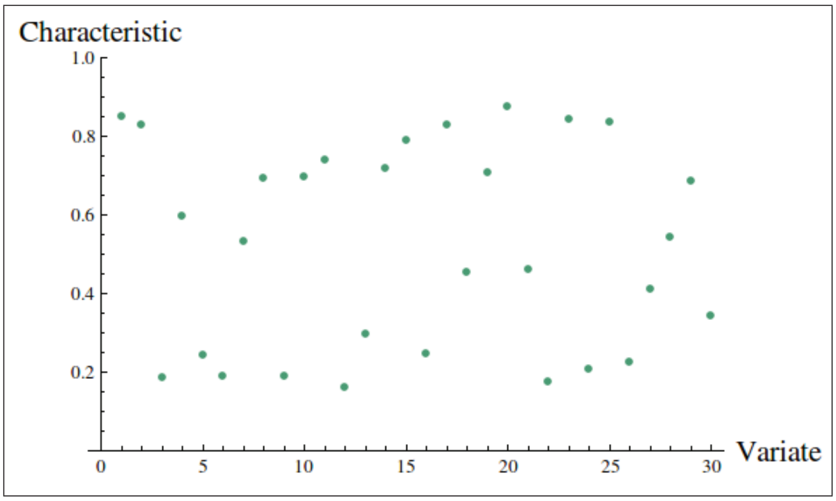

Figure 25: The distribution of the first segmented plot.

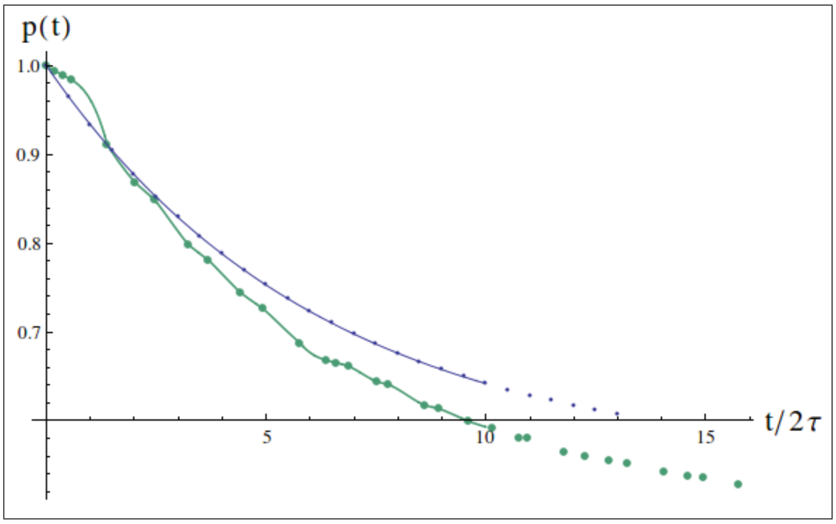

Figure 26: The survival probability of the first segmented distribution.

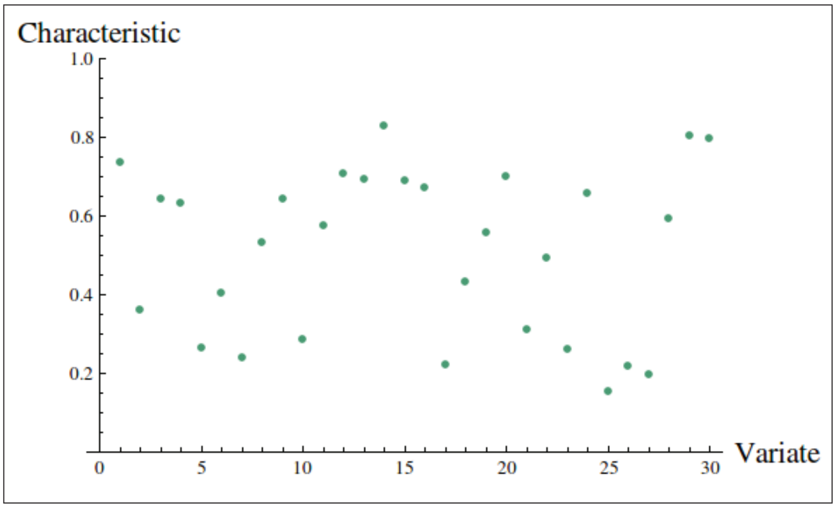

Figure 27: The distribution of the second segmented plot.

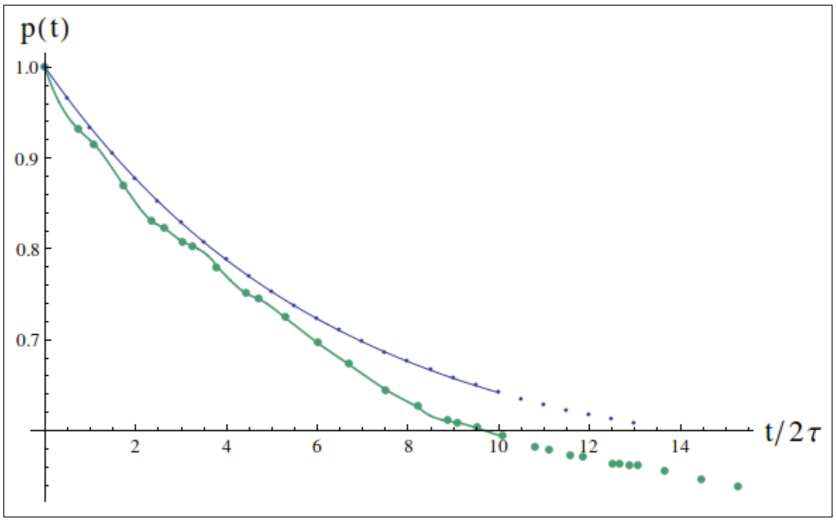

Figure 28: The survival probability of the second segmented distribution.

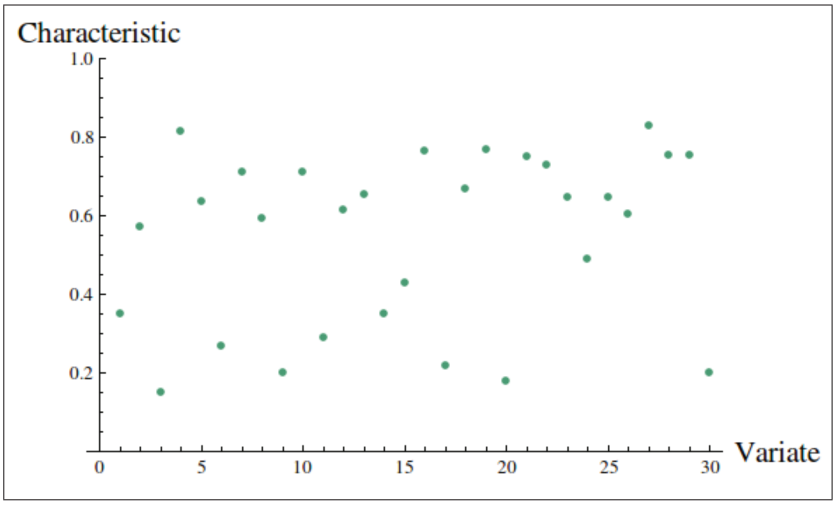

Figure 29: The distribution of the third segmented plot.

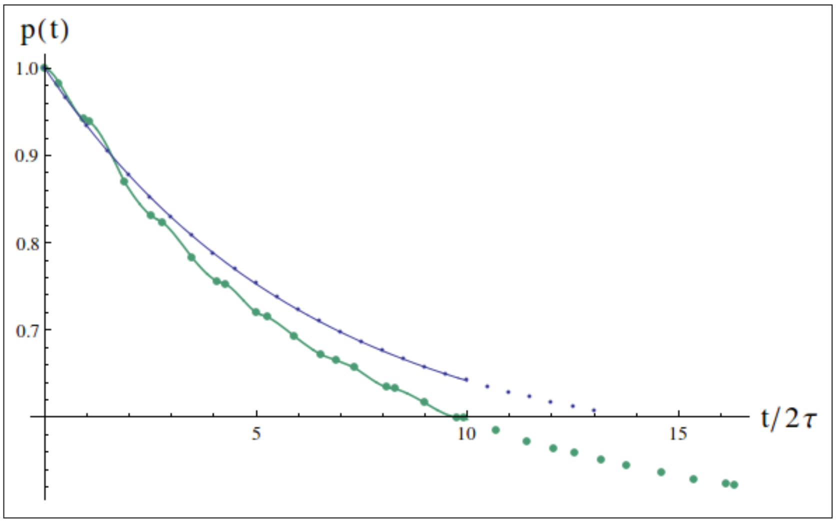

Figure 30: The survival probability of the third segmented distribution.

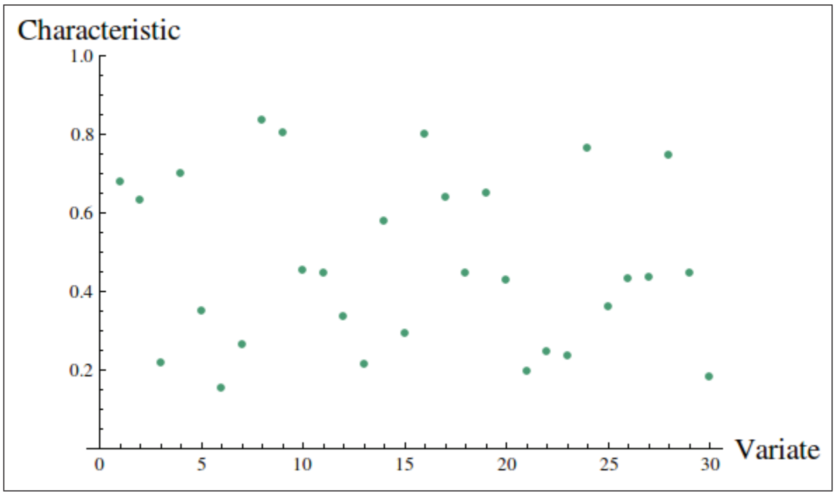

Figure 31: The distribution of the fourth segmented plot.

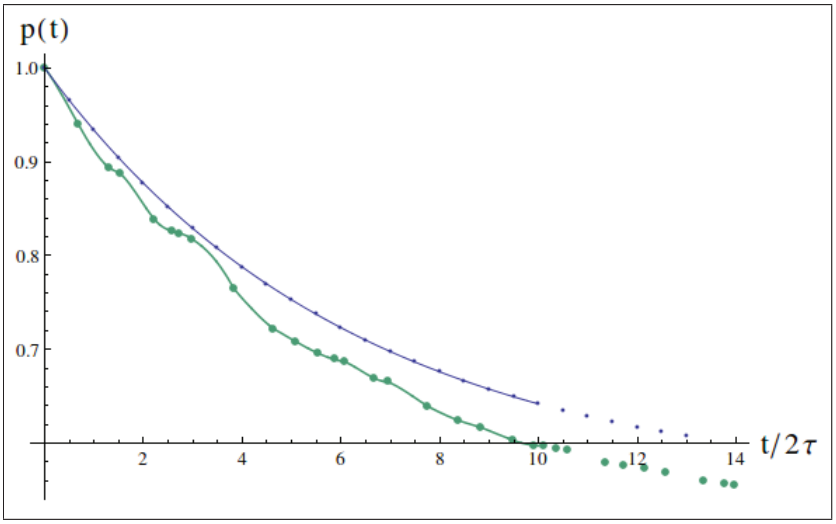

Figure 32: The survival probability of the fourth segmented distribution.

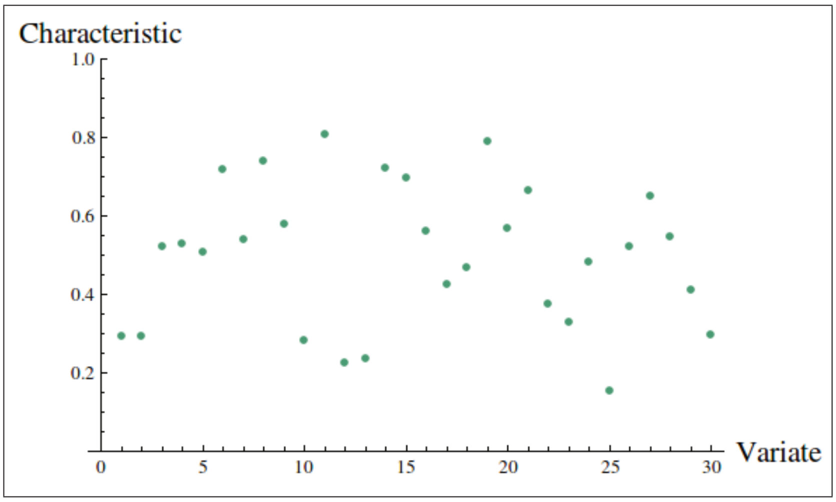

Figure 33: The distribution of the fifth segmented plot.

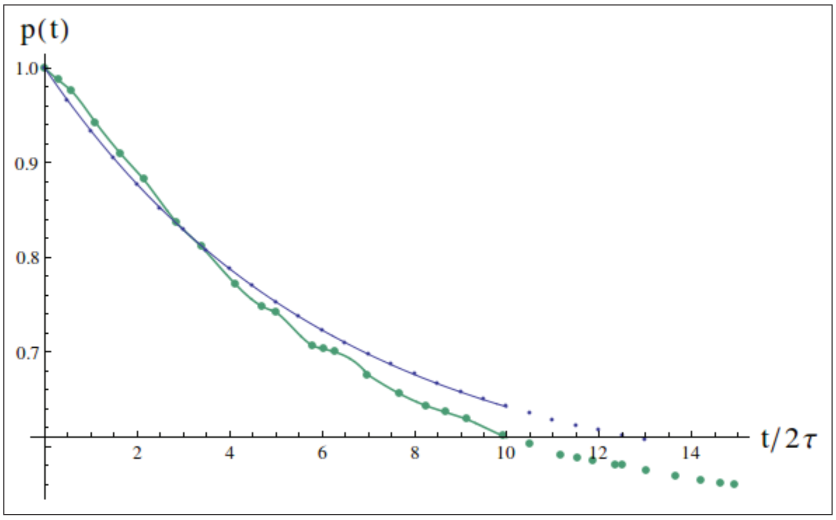

Figure 34: The survival probability of the fifth segmented distribution.

6 Conclusion

6.1 Main outcomes of the thesis

The motivation to deal with measurements was to gather information about the interaction between particles and detectors for a better understanding of the phenomenon of the wave function collapse. Restricting the energy range of the detectors cause another type of Zeno effect. We knew this result from non-randomized BANG-BANG measurements with imperfect detectors. In this thesis we have enlarged this study to randomized BANG-BANG measurements: the same type of quantum Zeno effect is realized.

However, the non-decay probability curve has lost its smoothness and shows a zigzag behavior. If the time interval between two distinct measurements is smaller than our mean time (because of statistical fluctuations), there is an even stronger contribution to the Zeno effect than in the non-randomized case. For time intervals longer than our mean time, the opposite is true. In most cases (but it is not always so), we find that – if we wait long enough – the decay of the particle is faster than in the non-randomized cases.
Our statistical approach creates time intervals that are smaller and bigger than our mean time τ, but averagely the time interval is still τ (it is symmetric). So naively one would expect that the zigzag result would oscillate around our non-randomized smooth curve, which we show for comparison in every graph. This is clearly not the case.

The most extreme cases that can occur are when a time interval goes to zero, and therefore stops the decaying process completely. The other extreme takes place when the time interval is twice as long as our mean time. Therefore, the movement of our zig zag motion is bound to a certain angle. However, our non-randomized curve doesn't necessary cuts this angle in half, so the oscillating probability curve of our randomized distribution does not stick to it.

As mentioned above, in most of our cases we could observe that on the long run the survival probability is smaller than our non-randomized curve. We will now comment our results for each of our three different distribution models in a more detailed way because we notice that each of them shows some peculiar distinctive characteristics.

HOMOGENOUS DISTRIBUTION:
The homogenous distribution is the model which shows us the 'zigzag' nature most evidently. The variance compared to the other distribution models is larger. By observing the size of the of oscillation, we get a pretty good feeling on how strong the QZE actually influences the survival probability. If we effectively get an oscillating dispersion around our non-randomized guideline it seems to only occur at short times. The longer our BANG-BANG measurement is running, the clearer our new survival probability lies below the guideline.

Parabolic distribution:
In this case we can still observe the ‘zigzag’ nature, but less strongly. The offsets to the guideline are much smaller than in the homogenous distribution. A part from that, it seems that the character of the randomized Bang-Bang case has not changed much (for the parameters that we used). Such a distribution confirms in average the results of the non-randomized case.

Segmented distribution:
Averagely we observe that the plots (in comparison to our homogenous distributions) differ the most from the non-randomized version. With our segmented distribution we could interpolate between different limits. Opening up our segment from 0 to 1 gives us exactly the same distribution as in the homogenous case. Closing the segment to just one possible outcome, which than is $\tau = 0.5$ gives us a completely non-randomized distribution. Everything in between can be also considered as an approximation of our parabolic distribution, although the nature is not exactly the same. But still, with our segmented distribution tool we can easily control how much ‘randomness’ is allowed into in the model and which is his quantitative effect.

In the end, our statement is that randomized Bang-Bang measurements, which are thought to mimic detectors in a more realistic way, induce a survival probability which is qualitatively comparable to the case of non-randomized measurements. Still, we also observe that the randomized QZE is typically weaker.

6.2 Outlook

Certainly, we already knew that the collapse of the wave-function follows certain principles. The idea and motivation for this bachelor-thesis was to describe some different types of collapse in connection to imperfect detectors. Unique information about how the particles interact with the detector could not yet be produced or rather extracted by us. The ultimate goal would be to achieve a higher understanding on how the detector exactly triggers the collapse. The collapse of the wave function, being the only non-reversible physical process, obviously still remains a big mystery.

We modelled the effect of an imperfect detector: in future we could combine the energy-range restricted detector with real measurements (which take into account some amount of time for the collapse) to better investigate the phenomenon. Additionally, our plots can be improved if one would perform significantly more runs for each distribution model and make an average over many curves with specification of the error. Also, one could change the parameters. Since each individual plot has an underlying statistical distribution, the coding in mathematica would not be as simple as in our case. More time and investigation would be needed, but that goes beyond the scope of a bachelor thesis.

In conclusions, a natural continuation of the present work is to apply more complex quantum measurement theory to the randomized or even non-randomized Bang-Bang measurements and expand the statistical approach. The hope is that it would help us in understanding the big picture of the interaction between a quantum system and its detector.

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