Hyperons in a Chiral Effective Approach

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Lisa Olbrich
aus Aschaffenburg

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Dekan:

Prof. Dr. Owe Philipsen

Gutachter:

PD Dr. Francesco Giacosa

Prof. Dr. Dirk. H. Rischke

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Für uns.

Zusammenfassung

Alle gewöhnliche Materie ist aus Atomen aufgebaut. Diese wiederum bestehen aus wechselwirkenden subatomaren Teilchen: einem Kern aus Protonen und Neutronen als auch einer Wolke aus Elektronen. Die elektrische Wechselwirkung bindet die negativ geladenen Elektronen an den positiv geladenen Atomkern. Doch gleichzeitig ruft sie auch eine Abstoßung der im Kern eng zusammengedrängten ($\leq$ 10 fm) Protonen hervor. Dies würde den Atomkern instabil machen. Unumstritten existiert jedoch stabile Materie im Universum. Aufschluss gibt die Berücksichtigung einer weiteren, uns weniger bekannten Wechselwirkung, die sog. starke Wechselwirkung. Da sie sehr kurzreichweitig ist und erst ab mikroskopischen Skalen (wie dem Atomkerndurchmesser oder kleiner) relevant wird, entzieht sie sich den Erfahrungen unseres alltäglichen Lebens. Jedoch innerhalb des Atomkerns ist sie die dominierende Wechselwirkung. Sie ruft eine attraktive Kraft zwischen Nukleonen bzw. zwischen deren Bestandteilen, den sogenannten Quarks, hervor.

Die mathematische Theorie der starken Wechselwirkung heißt Quantenchromodynamik (QCD). Sie beschreibt die Kraft zwischen Quarks als einen Austausch von Gluonen und gewährleistet, dass diese in gebundener Form, in sogenannten Hadronen, zu beobachten sind. Diese gebundenen Zustände als analytische Lösung der QCD zu berechnen, ist allerdings sehr schwer (bis heute sogar unmöglich). Zudem erlaubt die Theorie keine perturbativen Rechnungen, da die Kopplung im niederenergetischen Bereich zu stark ist.
Die bekanntesten nicht-perturbativen Herangehensweisen sind zum einen numerische Lösungen auf einem Gitter, welche die Theorie auf einem vierdimensionalen Hyperwürfel diskretisiert, zum andern effektive Theorien bzw. Modelle. Letztere weisen dieselben Symmetrien der zugrunde liegenden Quantenchromodynamik auf, ihre Freiheitsgrade jedoch sind nicht länger Quarks und Gluonen, sondern Hadronen. Im niederenergetischen Bereich sind Hadronen tatsächlich die einzigen relevanten Freiheitsgrade, da Quarks und Gluonen hier nicht einzeln zu beobachten sind, sondern nur noch in gebundener Form als farbneutrale hadronische Zustände ("color confinement").

Diese Arbeit untersucht das sogenannte erweiterte Lineare Sigma Modell (eLSM), welches eine chirale effektive Beschreibung hadronischer Massen und Wechselwirkungen im niederenergetischen Bereich darstellt. Es basiert auf einer linearen Realisierung der chiralen $U(N_f)_L \times U(N_f)_R$ Symmetrie (sowie Skaleninvarianz), beinhaltet aber auch anomale, explizite sowie spontane Brechungen der chiralen Symmetrie.

Das Modell wurde erstmals in Ref. [1] konstruiert und beschreibt (pseudo)skalare und (Axial-)Vektor–Mesonen, sowie Glueblle [2] für den Fall $N_f = 2$. Später wurde dieser mesonische Sektor erweitert auf den Fall $N_f = 3$ [3, 4, 5, 6, 7, 8] und $N_f = 4$ [9]. Massen und Zerfälle von Mesonen bis zu 1.7 GeV werden mit ausreichender Genauigkeit (5%) reproduziert. Des Weiteren erlaubt es, interessante Schlussfolgerungen zu ziehen, wie z.B. dass skalare Quark-Antiquark-Zustände schwerer als 1 GeV sind und $f_0(1710)$ überwiegend gluonisch ist [4]. Die Resonanz $f_0(1370)$ wird als Folge daraus als chiraler Partner des Pions identifiziert [und nicht das leichtere skalare $f_0(500)$], d.h. es besitzt bis auf Parität und G-Parität dieselben Quantenzahlen wie das Pion.

Das ursprüngliche sogenannte Lineare Sigma Modell (LSM) für $N_f = 2$ enthält zudem auch Baryonen: das Nukleon, welches das Proton und Neutron repräsentiert. Um chirale Symmetrie zu gewährleisten, konnte die Nukleonmasse ausschließlich (abgesehen von kleinen Korrekturen von nicht-verschwindenden Quarkmassen) durch den Effekt der spontanen chiralen Symmetriebrechung erzeugt werden, d.h. $m_N \propto \langle \bar{q}q \rangle$.
Die Situation ändert sich, wenn wir auch den chiralen Partner des Nukleons einbeziehen. Nehmen wir an, dass sich der chirale Partner unter chiralen Transformationen, im Vergleich zum Nukleon, genau "gespiegelt" verhält ["mirror assignment" [10, 11]], so ist es möglich, chiral invariante Massenterme zu konstruieren. Physikalisch parametrisieren diese Terme Beiträge zur Nukleonmasse, welche nicht aus dem chiralen Kondensat $\langle \bar{q}q \rangle$ resultieren. Dies wurde im Rahmen des eLSM in Refs. [12, 13, 14, 15] untersucht. Es resultierte ein nicht zu verachtender Massenbeitrag aus anderen Quellen.

Wie im mesonischen Sektor bereits geschehen, verallgemeinert die vorliegende Arbeit [16, 17] nun auch den baryonischen Sektor des eLSM [12] von $N_f = 2$ auf $N_f = 3$. Diese Erweiterung geht im baryonischen Sektor nicht ganz so einfach vonstatten wie im mesonischen Sektor. Mesonen werden im Zwei-Flavor-Fall durch 2×2 Matrizen beschrieben, welche für den Drei-Flavor-Fall auf 3×3 Matrizen erweitert werden. Dagegen werden im baryonischen Sektor im Zwei-Flavor-Fall die Nukleonen durch ein Spinor Isodublett beschrieben $\Psi_N = (p, n)^T$, wobei p und n das Proton bzw. das Neutron beschreiben. Für den Drei-Flavor-Fall muss die Repräsentation auf eine 3×3 Matrix erweitert werden, welche das $J^P = \frac{1}{2}^+$ Baryon-Oktett beschreibt:

$$\begin{pmatrix} \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} . \tag{0.1}$$

Auch die Einbindung des chiralen Partner-Multipletts $J^P = \frac{1}{2}^-$ verallgemeinert sich nicht ganz einfach von der Zwei- auf die Drei-Flavor-Diskussion.

Um diese Herausforderung anzugehen, konstruieren wir zunächst Baryon-Felder unter Ausnutzung der Quark-Diquark–Darstellung [16]. Wir kombinieren ein Quark mit einem Diquark, welches ein (pseudo)skalarer Zustand mit antisymmetrischer Flavor- und Farb-Struktur ist. Im Drei-Flavor-Fall transformiert letzteres wie ein Antiquark. Daher ähnelt

die Konstruktion der Baryonen der bereits ausgearbeiteten Konstruktion der Mesonen als Quark-Antiquark-Zustände. Die 3×3 Matrix-Struktur ergibt sich folglich ganz natürlich. In Kombination mit der Forderung nach chiral-invarianten Massentermen im Lagrangian ("mirror assignment") werden wir schließlich notwendigerweise zur Einbeziehung von vier Baryon-Multipletts geleitet.
Zwei dieser Repräsentationen bezeichnen wir mit N_1 und N_2. Sie verhalten sich unter chiralen Transformationen wie folgt:

$$N_{1R} \to U_R N_{1R} U_R^\dagger \ , \quad N_{1L} \to U_L N_{1L} U_R^\dagger \ , \tag{0.2}$$

$$N_{2R} \to U_R N_{2R} U_L^\dagger \ , \quad N_{2L} \to U_L N_{2L} U_L^\dagger \ . \tag{0.3}$$

Die anderen beiden Felder M_1 und M_2 weisen eine chirale Transformation auf, welche von links "gespiegelt" gegenüber den vorgenannten Feldern erscheint:

$$M_{1R} \to U_L M_{1R} U_R^\dagger \ , \quad M_{1L} \to U_R M_{1L} U_R^\dagger \ , \tag{0.4}$$

$$M_{2R} \to U_L M_{2R} U_L^\dagger \ , \quad M_{2L} \to U_R M_{2L} U_L^\dagger \ . \tag{0.5}$$

Es ist zu beachten, dass sich diese Felder aufgrund ihres definiten Verhaltens unter chiralen Transformationen sehr gut eignen, um eine chiral invariante Lagrangedichte zu konstruieren. Allerdings stellen sie keine Eigenzustände der Parität dar und können daher nicht physikalischen Zuständen zugeordnet werden. Solche Paritätseigenzustände sind dagegen durch folgende Linearkombinationen gegeben:

$$B_N = \frac{1}{\sqrt{2}}(N_1 - N_2) \ , \quad B_{N_*} = \frac{1}{\sqrt{2}}(N_1 + N_2) \ , \tag{0.6}$$

$$B_M = \frac{1}{\sqrt{2}}(M_1 - M_2) \ , \quad B_{M_*} = \frac{1}{\sqrt{2}}(M_1 + M_2) \ . \tag{0.7}$$

Die Einführung der baryonsischen Felder und ihrer Transformationseigenschaften erlaubt eine baryonische eLSM-Lagrangedichte zu konstruieren [16], welche die Drei-Flavor-Oktett-Baryonen und ihre Wechselwirkung mit Mesonen beschreibt. Neben der Invarianz unter Paritäts- und Ladungskonjugationstransformationen weist der baryonische Teil (zunächst) auch eine chirale $U(3)_L \times U(3)_R$ Symmetrie auf. (Später werden wir einen weiteren Term einführen, der die $U(1)_A$ Symmetrie bricht.) Die baryonische eLSM-Lagrangedichte enthält zwölf Parameter, von denen zehn die Wechselwirkung mit (pseudo)skalaren und (Axial-)Vektor-Mesonen beschreiben. Die zwei verbleibenden Parameter $m_{0,1}$ und $m_{0,2}$ ähneln Massenparametern und brechen formell die Dilatationsinvarianz. Wir nehmen aber an, dass diese aus (dilatationsinvarianten) Wechselwirkungen eines Glueballs und/oder eines Vier-Quark-Zustands mit einem baryonischen Feld entstehen, wobei der Mechanismus der spontanen Symmetriebrechung einen nicht-verschwindenden Vakuumerwartungswert des Glueballs und/oder des Vier-Quark-Zustands hervorruft. Die resultierende Größe der beiden Massenparameter gibt einen Hinweis darüber, zu welchem Teil die Baryon-Massen aus anderen Quellen als nur der chiralen Symmetriebrechung (z.B. einem Glueball-Kondensat) erzeugt werden.

Da eine Studie des Modells für $N_f = 3$ sehr aufwändig ist, untersuchen wir den einfacheren Zwei-Flavor-Fall. Durch die Reduzierung des Models auf $N_f = 2$ bleiben nur noch vier Dubletts nukleonischer Zustände. Mischungen dieser Repräsentationen beschreiben das experimentell beobachtete Nukleon $N(939)$, und die Resonanzen $N(1440)$, $N(1535)$ und $N(1650)$. Mittels einer gewöhnlichen χ^2-Prozedur passen wir die elf Parameter des Models an zehn experimentelle Größen und drei Größen aus Gitter-QCD–Rechnungen an: die vier Massen der nukleonischen Zustände, fünf Zerfallsbreiten der nukleonischen Resonanzen in ein Nukleon und ein pseudoskalares Meson und vier axiale Kopplungskonstanten der nukleonischen Zustände.

Die numerische Minimierung des χ^2-Funktionals liefert drei akzeptable und ähnlich tiefe Minima. Interessanterweise ergeben sich in zwei dieser Minima kleine Werte für $m_{0,1}$ und $m_{0,2}$, was signalisiert, dass alle Massen hauptsächlich aus der chiralen Symmetriebrechung stammen, während das dritte Minimum Werte aufweist, die fast genauso groß sind wie die Vakuummasse des Nukleons, was darauf hindeutet, dass die Massen wesentliche Beiträge aus anderen Quellen enthalten.

Mit allen drei Parameterwertmengen lassen sich fast alle experimentell beobachteten Größen in guter Übereinstimmung mit den Experimenten beschreiben. Die Verfolgung der resultierenden Massen zu dem Limes, an dem die chirale Symmetrie restauriert ist, erlaubt die Identifizierung der Paare

$$N(939) \text{ und } N(1535) \quad \text{bzw.} \quad N(1440) \text{ und } N(1635) \tag{0.8}$$

als chirale Partner. Im Fall der ersten beiden Minima nehmen die Massen von $N(939)$ und $N(1535)$ einen Wert von circa 140 MeV an, während sie im dritten Minimum eine gemeinsame Masse von circa 900 MeV aufweisen, wenn die chirale Symmetrie wieder hergestellt ist. Für die Resonanzen $N(1440)$ und $N(1650)$ ergibt sich in diesem Limes eine gemeinsame Masse von circa 200 MeV (Minimum 1 und 2) bzw. circa 1100 MeV (Minimum 3).

Als wichtige Unstimmigkeit muss die $N(1535) \to N\eta$ Zerfallsbreite besprochen werden. In allen drei Fällen ergibt sich ein Wert, der um circa eine Größenordnung zu klein ist, wenn wir ihn mit experimentellen Beobachtungen vergleichen. Diese Größe konnte bereits in den früheren Studien aus Ref. [12] nicht reproduziert werden. Die dort vorgeschlagene Einbindung von vier Multipletts konnte das Problem nicht lösen.

Interessanterweise ist die experimentelle $N(1535) \to N\eta$ Zerfallsbreite generell größer als intuitiv zu erwarten wäre. Sie ist nämlich fast genauso groß wie die Zerfallsbreite von $N(1535) \to N\pi$, was zunächst sehr überraschend ist, da wir von Flavor-Symmetrie ausgehend eigentlich folgendes Verhältnis erwarten würden:

$$\frac{\Gamma_{N(1535)\to N\eta}}{\Gamma_{N(1535)\to N\pi}} \simeq \frac{1}{3}\cos^2\theta_P \simeq 0,17\ , \tag{0.9}$$

wobei der Faktor 3 die drei Isospin-Zustände des Pions berücksichtigt und $\cos^2\theta_P$ mit $\theta_P \simeq -44,6°$ [3] die Mischung der pseudoskalaren Isosinglets η_N und η_S beachtet. Offensichtlich widerspricht die Vorhersage der Flavor-Symmetrie den experimentellen Befunden. (Die Berücksichtigung des Phasenraums würde das Verhältnis sogar noch weiter verkleinern.)
Diese Überlegung kann leicht auf eine Flavor-Symmetriestudie ausgedehnt werden, in der die Zerfälle des gesamten Baryon-Oktetts $\{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$ in die Baryon-Grundzustände $\{N(939),\ \Lambda(1116),\ \Sigma(1193),\ \Xi(1338)\}$ und ein pseudoskalares Meson untersucht werden [17]. Dazu konstruieren wir ein Modell, welches ausschließlich auf Flavor-Symmetrie beruht [d.h. ohne vollständige chirale Symmetrie und ohne Terme, die die axiale $U(1)_A$ Anomalie parametrisieren]. Das Modell beschreibt die meisten Zerfallsbreiten in guter Übereinstimmung mit experimentellen Befunden, allerdings ergeben sich zwei wichtige Unstimmigkeiten im Zusammenhang mit dem η Meson. Die Zerfälle $N(1535) \to N\eta$ und $\Lambda(1670) \to \Lambda(1116)\eta$ werden zu niedrig eingeschätzt (circa eine Größenordnung zu klein). Wir folgern daraus, dass die Flavor-Symmetrie nicht ausreicht, um die Zerfälle des Oktetts $\{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$ in ein Grundzustandsbaryon und ein η-Meson zu beschreiben.
Wenn wir auf diese Weise die Zerfälle des Oktetts $\{N(1650),\ \Lambda(1800),\ \Sigma(1750),\ \Xi(?)\}$ untersuchen, ergibt sich im Gegensatz dazu keine Unterschätzung des Zerfalls $\Lambda(1800) \to \Lambda\eta$. Der Zerfall $N(1650) \to N\eta$ wird nur um einen Faktor von 1.4 zu klein beschrieben (wenn wir den maximalen theoretischen Wert und den minimalen experimentellen Wert vergleichen). Allerdings ist zu bemerken, dass in Ref. [18] ein kleinerer Wert zwar aufgeführt, aber bei der Mittelwertsbildung des $N\eta$-Verzweigungsverhältnisses nicht berücksichtigt wurde.

Diese Studie veranschaulicht, warum das eLSM in obiger Form die $N(1535) \to N\eta$ Zerfallsbreite nicht richtig reproduzieren konnte. Da es auf der chiralen Symmetrie basiert, ist es auch flavor-symmetrisch und unterschätzt deshalb gemäß Gl. (0.9) diese Zerfallsbreite. Die Resonanz $N(1535)$ [bzw. das komplette Oktett um $N(1535)$] muss stärker an das η-Meson koppeln als es aus reinen Flavor-Symmetriebetrachtungen folgt.

Eine Möglichkeit, diese verstärkte Kopplung einiger angeregter Baryonen an das η-Meson zu erreichen, bietet die Einbindung des QCD-Phänomens der $U(1)_A$ Anomalie im baryonischen Sektor des eLSM [17]. Es erlaubt eine Kopplung von $N(1535)$ an $N(939)$ durch die Emission zweier Gluonen im isoskalaren-pseudoskalaren Kanal $I(J^{PC}) = 0(0^{-+})$. Diese Gluonen wiederum koppeln mit der gleichen Intensität an alle Quark-Antiquark-Paare $\bar{u}u$, $\bar{d}d$ und $\bar{s}s$ und damit fast ausschließlich an η und η'. Der neue Wechselwirkungsmechanismus vergrößert also (unter anderem) die $N(1535) \to N\eta$ Zerfallsbreite.

Die Realisierung eines "mirror assignments" erlaubt die Konstruktion eines solchen anomalen Terms im eLSM, ohne die chirale $SU(N_f)_L \times SU(N_f)_R$ Symmetrie zu zerstören. Wie erwähnt, ermöglicht das "mirror assignments" die Konstruktion eines chiral-invarianten Massenterms. Ganz ähnlich kann ein pseudoskalarer Term konstruiert werden, welcher das Nukleon und seinen chiralen Partner koppelt. In Kombination mit folgendem Term negativer Parität,

$$\det\Phi - \det\Phi^\dagger \ , \tag{0.10}$$

der (pseudo)skalare Mesonen via Φ einbindet, kann ein $U(1)_A$ anomaler Term konstruiert werden. Die chirale $SU(N_f)_L \times SU(N_f)_R$ Symmetry bleibt erhalten, er bricht aber die Symmetrie unter $U(1)_A$ Transformationen, da die Determinante nicht invariant unter letzteren Transformationen ist. Nach spontaner Symmetriebrechung ruft dieser Term eine weitere Kopplung der Baryonen an η und η' hervor.

Im Zwei-Flavor Fall wird der Anomalie-Term durch eine Kopplungskonstante der Dimension [Energie^{-1}] parametrisiert. Wir konnten zeigen, dass unter Annahme von $N(1535)$ als chiralem Partner des Nukleons der anomale Term die Wahrscheinlichkeit des Zerfalls $N(1535) \to N\eta$ in Überstimmung mit experimentellen Daten erhöht, wenn die neue Kopplungskonstante geeignet gewählt wird. Diese $N_f = 2$ Studie ist interessant, da sie die Möglichkeit eines neuen Zerfallsmechanismus zeigt, allerdings erlaubt sie keine weiteren Vorhersagen. Aus diesem Grund verallgemeinern wir die Untersuchung auf den Fall $N_f = 3$.

Wenn wir das oben eingeführte Drei-Flavor-eLSM mit vier Baryon-Oktetts im "mirror assignment" betrachten, können wir analoge Terme konstruieren, welche die $U(1)_A$ Anomalie parametrisieren. Die explizite Brechung der $U(1)_A$ Symmetrie wird abermals durch Determinanten wie in Gl. (0.10) erzeugt. Im Drei-Flavor-Fall tauchen zwei Kopplungskonstanten mit den Dimensionen [Energie^{-2}] auf. Wir betrachten jedoch nur einen bestimmten Term, der nichtverschwindende Beiträge zu den untersuchten Zerfallsbreiten liefert (welche zuvor im Modell basierend auf Flavor-Symmetrie unterschätzt wurden). Es bleibt nur ein einziger Parameter, $\lambda_A^{N_f=3} \equiv (\lambda_{A1} + \lambda_{A2})/2$, welchen wir aus einem Fit an die $N(1535) \to N\eta$ Zerfallsbreite bestimmen. Der Zerfall $\Lambda(1670) \to \Lambda\eta$ wird dann in Übereinstimmung mit experimentellen Daten beschrieben.
Wir folgern, dass die durch den Anomalie-Term verursachte Kopplungsverstärkung also in Übereinstimmung mit den experimentellen Befunden ist.

Als weiteres Resultat diesen Ansatzes folgt eine starke $N(1535)N\eta'$-Kopplung:

$$g_{\eta NN_*} \simeq 1.9 \ , \qquad g_{\eta' NN_*} \simeq 7.2 \ , \qquad g_{\pi NN_*} \simeq -0.7 \ . \tag{0.11}$$

Ohne Anomalie ergäbe sich $g_{\eta NN_*} \simeq |g_{\eta' NN_*}| = 0.5$. Diese Kopplungsverstärkung von $N(1535)$ an N und η ist in qualitativer Übereinstimmung mit Ref. [19], deren Studie von Streuprozessen $g_{\eta' NN_*} \simeq 3.7$ ergab. Die Untersuchungen von Streuprozessen im Rahmen des eLSM bietet sich als eine mögliche künftige Erweiterung vorliegender Arbeit an.

Des Weiteren liefert die Anomalie einige Wechselwirkungen zwischen den Baryonen, die uns erlauben, das Oktett $\{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$ als chirale Partner der

Grundzustandsbaryonen zu identifizieren (in Übereinstimmung mit dem Resultat aus obiger Zwei-Flavor-Studie).

Die Suche nach Glueböllen ist ein interessantes Thema der Hadronen-Physik, insbesondere da diese Zustände im aktuellen BESIII [20] und zukünftigen PANDA [21] Experiment beobachtet werden können. In Rahmen des eLSM wurden Glueballe bereits in den Refs. [2, 4, 22, 23] untersucht. (Der pseudoskalare Glueball ist direkt verknüpft mit der chiralen Anomalie.)

Die mathematische Struktur, die die Kopplung des vermeintlichen pseudoskalaren Glueballs an Baryonen beschreibt, ist dem Anomalie-Terms im baryonischen Sektor sehr ähnlich. Daher bietet sich eine Studie dieser Glueball-Wechselwirkung im bereits diskutierten Model an.
Das pseudoskalare Glueball-Feld $\tilde{G}$ ist invariant unter chiralen $U(3)_L \times U(3)_R$ Transformationen und besitzt negative Parität (Ladungskonjugation ist positiv). Daher ergeben sich die Kopplungen der Baryonen mit dem pseudoskalaren Glueball, indem wir im Anomalieterm den Determinantenfaktor (0.10) durch das pseudoskalare Glueballfeld $\tilde{G}$ ersetzen. Die Untersuchung der resultierenden Lagrangedichte ergibt, dass der pseudoskalare Glueball stark an $N(1535)N$ und vermutlich auch an $N(1440)N$ koppelt. Wir erwarten daher, dass der pseudoskalare Glueball $\tilde{G}$ im zukünftigen PANDA Experiment in der Streuung $p + \bar{p} \to \tilde{G} \to p + \bar{p}(1535)$ beobachtet werden kann.

Kommen wir zurück auf die oben entwickelte eLSM-Lagrangedichte für Baryonen. Eine volle Drei-Flavor-Analyse wäre hier der nächste Schritt, da für diesen Fall deutlich mehr experimentelle Daten in Form von Hyperonmassen und -zerfallsbreiten zum Vergleich bereitstehen. Allerdings, wie oben bereits diskutiert, sollten diese Studien auch anomale Terme im baryonischen Sektor einschließen. Deshalb empfehlen wir als wichtigen Ausblick dieser Arbeit einen neuen unabhängigen Fit der Parameter für den Fall $N_f = 3$ durchzuführen. In diesem Zusammenhang wird es auch interessant sein, die Notwendigkeit der “large-N_c” unterdrückten effektiven Vierpunktwechselwirkungen zu untersuchen, welche in dieser Arbeit zwangsläufig eingeführt werden mussten, um eine unphysikalische paarweise Entartung der Baryonmassen zu verhindern. Des Weiteren sollte auch die Auswirkung des Massenunterschieds zwischen skalarem und pseudoskalarem Diquark untersucht werden (welcher experimentell klar ist, aber durch die effektive Herangehensweise nicht berücksichtigt wurde).

Abschließend erkennen wir, dass die Beschreibung der Baryonen innerhalb eines chiralen effektiven Models ein interessantes aber auch herausforderndes Thema darstellt. In dieser Arbeit haben wir eine Erweiterung auf den Drei-Flavor-Fall entwickelt, indem wir vier $J = 1/2$ Baryon-Multipletts eingebunden haben und anomale Terme im baryonischen Sektor konstruiert haben. Ungeachtet dessen, bietet die Arbeit viele Punkte, an die in Zukunft noch angeknüpft werden kann: Neben der oben erwähnten vollen Drei-Flavor-Studie sollten auch noch weitere Resonanzen, wie zum Beispiel $J = 3/2$ Baryonen, eingebunden werden und verschiedene Meson-Nukleon- und Nukleon-Nukleon-Streuungen mit Strangeness untersucht werden. Letztendlich ist das entwickelte Modell auf Fälle mit nichtverschwindender Dichte anzuwenden, um nukleare Materie und Neutronensterne zu beschreiben. Insbesondere werden wir durch die erst kürzlich erfolgte Beobachtung von Gravitationswellen, die von kollidierenden Neutronensternen emittiert wurden, detaillierte Informationen über solche Objekte erlangen. Dies wird neue Forderungen an die Zustandsgleichung der Kernmaterie stellen und daher auch neue Einschränkungen an Modelle zur Beschreibung von Baryonen liefern, wie zum Beispiel das in dieser Arbeit vorgestellte extended Linear Sigma Model.

Der Dissertation liegen folgende Veröffentlichungen zugrunde:

L. Olbrich, M. Zétényi, F. Giacosa and D. H. Rischke, *"Three-flavor chiral effective model with four baryonic multiplets within the mirror assignment"*,
Phys. Rev. D **93**, 034021 (2016) [arXiv:1511.05035 [hep-ph]];

L. Olbrich, M. Zétényi, F. Giacosa and D. H. Rischke, *"Influence of the axial anomaly on the decay $N(1535) \to N\eta$"*,
Phys. Rev. D **97**, 014007 (2018) [arXiv:1708.01061 [hep-ph]].

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Abbreviations

QFT	quantum field theory
QCD	quantum chromodynamics
LSM	linear σ model
eLSM	extended linear sigma model
SSB	spontaneous symmetry breaking
VEV(s)	vacuum expectation value(s)

West-coast metric We use the west-coast sign convention, i.e., the metric in flat space-time is given by

$$g_{\mu\nu} = g^{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Natural units For the same reason as we no longer use a certain royal cubit to measure distances, it is convenient to use natural units in high-energy physics , i.e., we set

$$\hbar = c = \epsilon_0 = 1 .$$

Then, for instance, mass has dimension [energy].

Einstein sum rule Repeated (not only co- and contravariant) indices are understood to be summed over (unless indicated otherwise).

Vectors Space-time coordinates are usually labelled by Greek indices ($\mu = 0, 1, 2, 3$). The shorthand notation $\partial_\mu = \partial/\partial x^\mu$ of the 4-gradient is used frequently. Three-vectors (and other vectors), such as the position vector $\boldsymbol{r} = (x^1, x^2, x^3)^T$, are written in bold letters.

Language of quantum field theory As it is convenient in (quantum) field theory, we work predominantly with the Lagrangian $\mathcal{L}$ instead of the Lagrange function $L = \int \mathrm{d}^3x\, \mathcal{L}$.

Abstract

We investigate the so-called extended linear sigma model (eLSM), which is an low-energy effective approach to quantum chromodynamics. It is based on a linear realization of chiral symmetry (as well as dilatation invariance), but features also the anomalous, explicit, and spontaneous breaking of chiral symmetry. Originally constructed for the case $N_f = 2$, in the mesonic sector the extension from $N_f = 2$ to $N_f = 3$ has already been performed. In this work we generalize the baryonic sector to $N_f = 3$.

We construct four $J = \frac{1}{2}$ baryon fields utilizing the quark-diquark picture in the chiral representation as well as a "mirror assignment". We implement these field into the eLSM via a baryonic Lagrangian which exhibits (initially) a global chiral $U(3)_L \times U(3)_R$ symmetry. (Later on, we include a further term incorporating the $U(1)_A$ anomaly.) We investigate the two-flavor reduction of the Lagrangian, where only four nucleonic doublets are present, which mix to describe positive-parity N and $N(1440)$, as well as the negative-parity $N(1535)$ and $N(1650)$. Using a standard χ^2 procedure, the parameters are fitted ten experimental and lattice data. Tracing the masses to where chiral symmetry is restored allows to identify N and $N(1535)$ as chiral partners, as well as the pair of $N(1440)$ and $N(1650)$. However, in this form the model is not able to describe the decay width of $N(1535) \to N\eta$. Indeed, the experimental $\Gamma_{N(1535)\to N\eta}$ [as well as $\Gamma_{\Lambda(1670)\to\Lambda(1116)\eta}$] are larger than predicted by flavor symmetry as we show by investigating model based on $U(3)_V$ symmetry only. We propose that the enhanced coupling of some baryons to the η meson is induced by the axial $U(1)_A$ anomaly. Thus, we include a new chirally-invariant but $U(1)_A$ anomalous term in the baryonic sector of eLSM. Indeed, such a term enhances the decay of the chiral partners into the ground-state baryons, such as $N(1535) \to N\eta$. Furthermore, we obtain a strong $N(1535)N\eta'$ coupling through the inclusion of the anomaly, which is interesting for η' production studies. The approach predominately assigns $N(1535)$ as chiral partner of the nucleon and $\Lambda(1670)$ as chiral partner of $\Lambda(1116)$. Finally, we use the introduced mathematical structure of the anomaly term to couple the pseudoscalar glueball to baryons. Thus, in search of the pseudoscalar glueball $\tilde{G}$, we propose the investigation of the process $p + \bar{p} \to \tilde{G} \to p + \bar{p}(1535)$, which can be studied at the future PANDA experiment [21].

The present thesis is based on the following publications:

L. Olbrich, M. Zétényi, F. Giacosa, and D. H. Rischke, Phys. Rev. D **93**, 034021 (2016) [arXiv:1511.05035 [hep-ph]];

L. Olbrich, M. Zétényi, F. Giacosa, and D. H. Rischke, Phys. Rev. D **97**, 014007 (2018) [arXiv:1708.01061 [hep-ph]].

Chapter 1.

Introduction

The present work aims to exploit a phenomenological approach to describe baryons and their properties which are determined by the strong interaction. The investigated chiral effective model is the so-called extended linear sigma model. In this chapter, we first give an overview of the considered particles and resonances in Sec. 1.1. Then, Sec. 1.2 provides an introduction to quantum field theory as well as symmetries in physical laws. Subsequently, Sec. 1.3 is specifically devoted to quantum chromodynamics and its symmetries. Finally, Sec. 1.4 leads over to effective approaches describing low-energy hadronic bound states.

1.1. Mesons and baryons

1.1.1. The concept of quarks

One might say that the story of particle physics dates back to the early 1960s. Protons and neutrons were believed to be elementary. However, physicists began to dispute it, since doubts arose when they started to examine the force that binds the atomic nuclei. They shot protons with more and more energy onto nuclei but when they reached a certain energy, something unexpected happened: new elementary particles appeared. First they found pions, then lambdas, sigmas, and rhos, etc.:

$$\pi \;,\; \Lambda \;,\; \Sigma \;,\; \rho \;,\; \ldots \tag{1.1}$$

This continued until they ran out of letters, but as they found more and more particles, they began to use numbers which refer to the masses of the particles, such as e.g.

$$\Sigma(1190) \;,\; \Sigma(1385) \;,\; \ldots \;, \tag{1.2}$$

where the mass of the first Sigma is around 1190 MeV and of the second one around 1385 MeV [18].

Finally, in the middle of the sixties about a hundred strong interacting particles (so-called hadrons) were found in various experiments.

This "zoo of particles" has been structured in 1964, when Gell-Mann, Neeman, and Zweig independently introduced a theoretical classification scheme, which explains the proton and the neutron as well as other "baryons" (hadrons with half-integer spin) as a combination of just three fundamental particles, called quarks [24]. (Hadrons with integer spin, so-called mesons are described as quark-antiquark states.) The name of this fundamental particle was chosen by Gell-Mann and is a nonsensical word used by James Joyce in the novel Finnegan's Wake "Three quarks for Muster Mark!":

Three quarks for Muster Mark!
Sure he hasn't got much of a bark
And sure any he has it's all beside the mark.

Nowadays, we expect that six different quarks exist, which are distinguished by their "flavor" (up, down, strange, charm, bottom, and top):

$$u \ , \ d \ , \ s \ , \ c \ , \ b \ , \ t \ . \tag{1.3}$$

Note, each quark flavor can come in three different "colors" (red, green, and blue):

$$r \ , \ g \ , \ b \ . \tag{1.4}$$

This additional quantum number will be justified and discussed at the end of this subsection.

The quark flavors are conserved in processes of the strong interaction and the related quantum numbers are called upness U, downness D, strangeness S, charmness C, bottomness $\mathcal{B}$, and topness $\mathcal{T}$, see Tab. 1.1. (The strangeness of s and the bottomness of b are negative because of historical reasons only.) However, instead of U and D it is convenient to introduce the following quantum numbers:

$$T_3 = \frac{U - D}{2} \qquad \text{and} \qquad \mathcal{Y} = B + S + C + \mathcal{B} + \mathcal{T} \ . \tag{1.5}$$

The first one, T_3 is the z component of isospin I, which is defined such that the u and d quarks form the isodoublet[1] ($I = \frac{1}{2}$). The second quantum number $\mathcal{Y}$ is the so-called hypercharge, which is defined such that the Gell-Mann–Nishijima formula holds:

$$Q = T_3 + \frac{\mathcal{Y}}{2} \ , \tag{1.6}$$

where Q is the electrical charge in units of the elementary charge e. The quantity B in Eq. (1.5) is the baryon number which is defined to be $B = \frac{1}{3}$ for every quark, such that any baryon consisting of three quarks carries the baryon number $B = 1$ (and for each "meson" being a quark-antiquark state is $B = 0$).
Table 1.1 summarizes the quantum numbers and other properties (including the mass and spin) of the different quarks.

[1]Compared to the typical hadronic mass scale the u and the d quark can be assumed to have equal masses.

	flavor$^{\text{color}}$	spin	mass [MeV]	$Q[e]$	$\mathcal{Y}$	T_3	S	C	$\mathcal{B}$	$\mathcal{T}$	B
I	u^r, u^g, u^b	1/2	$2.3\ ^{+0.1}_{-0.5}$	2/3	1/3	1/2	0	0	0	0	1/3
	d^r, d^g, d^b	1/2	$4.8\ ^{+0.5}_{-0.3}$	-1/3	1/3	-1/2	0	0	0	0	1/3
II	s^r, s^g, s^b	1/2	95 ± 5	-1/3	-2/3	0	-1	0	0	0	1/3
	c^r, c^g, c^b	1/2	$(1.275 \pm 0.025) \cdot 10^3$	2/3	4/3	0	0	1	0	0	1/3
III	b^r, b^g, b^b	1/2	$(4.18 \pm 0.03) \cdot 10^3$	-1/3	-2/3	0	0	0	-1	0	1/3
	t^r, t^g, t^b	1/2	$(173.07 \pm 1.14) \cdot 10^3$	2/3	4/3	0	0	0	0	1	1/3

Table 1.1. Properties of quarks [18].

All hadrons are constructed as combinations of quarks with appropriate flavors, such that the resulting hadron carries the desired quantum numbers. For instance, the proton with half-integer spin and charge $+e$ is interpreted as a uud state, while the neutron with charge 0 is explained as a udd state (prioritizing the minimum-quark-content part of the wave function).

As already mentioned, depending on their spin hadrons can be further classified into baryons with half-integer spin (fermions) or as mesons with integer spin (bosons). The most basic representatives of these two types are:

1. The conventional qqq baryons, which are combinations of three (constituent) quarks.
2. The conventional $\bar{q}q$ mesons, which are made of a (constituent) antiquark and a (constituent) quark.

Note, the concept of "constituent quarks" pays attention to the cloud of gluons and virtual quark-antiquark pairs generated and annihilated in the field of the strong interaction, which were surrounding the almost massless bare quarks. In this way, the "dressed" constituent u and d quarks become quasiparticles with an effective mass of about 300 MeV, while the effective mass of the heavier constituent s quark is about 450 MeV . (For the heavier $c, b,$ and t quarks the difference between the bare- and the constituent-quark mass is small.)

Besides these qqq nor $\bar{q}q$ states, also "exotic" hadrons are possible. For instance, charmed "pentaquark" states (baryons) with minimal quark content $c\bar{c}uud$ has been observed by the LHCb collaboration, see Ref. [25]. Mesonic states that do not contain any (anti)quark, but consist only of gluons (the transmitter particles of the strong interaction, see later) do also exist: so-called glueballs. Furthermore, one can also think of mesonic states that contain two quarks and two antiquarks, $(\bar{q}q)(\bar{q}q)$ "tetraquarks", as well as baryonic states which are built from two qqq combinations, $(qqq)(qqq)$ states. (The mandatory criterion following from "color confinement" is only that the hadronic states white objects in color space, i.e., color singlets, see below.)
An overview of all known hadrons can be found in Ref. [18], the so-called "particle data"

booklet of the "particle data group".

Note, mesons and baryons can be further distinguished by their orbital angular momentum L and their spin S, which couple to a total angular momentum J with the usual relation

$$|L - S| \leq J \leq |L + S| \ . \tag{1.7}$$

However, instead of giving L and S, it is more convenient to introduce the quantum numbers parity P and charge conjugation C. For mesons the following relations hold:

$$P = (-1)^{L+1} \quad \text{and} \quad C = (-1)^{L+S} \ , \tag{1.8}$$

where we have assumed that the intrinsic parity of a quark is $+1$ and of an antiquark is -1. For baryons the parity quantum number is related to L via

$$P = (-1)^{L} \ . \tag{1.9}$$

Parity reflects the behavior under spatial reflections and charge conjugation shows the effect of a particle-antiparticle exchange.
In the case of mesons being $\bar{q}q \equiv q\bar{q}$ states, it is just a matter of convention if they are designated as particles or antiparticles (they are eigenstates of the charge-conjugation operator), while a qqq baryon under charge conjugation transforms into a $\bar{q}\bar{q}\bar{q}$ antibaryon which is a genuine new particle.

By the late 1960s, the quark-model concept was already widely accepted. However, when discovering the Δ^{++} baryon, a uuu resonance with quantum numbers $I(J^P) = \frac{3}{2}(\frac{3}{2}^+)$, $\mathcal{Y} = 1$, see Ref. [26], physicists encountered a problem.
Since Δ^{++} is the lightest baryon with $J^P = 3/2^+$, it is expected that its angular momentum is zero, $L = 0$. That is, the spatial wave function (s wave) of Δ^{++} is symmetric. Since, $L = 0$ the spins of all three quarks have to parallel in order to obtain the total angular momentum of $J = 3/2$, i.e.

$$\left|\Delta^{++}\right\rangle_{\text{flavor-spin}} = \left|u^{\uparrow}u^{\uparrow}u^{\uparrow}\right\rangle \ , \tag{1.10}$$

which is symmetric in flavor and in spinor space. Consequently, the total wave function $\Psi_{\Delta^{++}} = \psi_{\text{space}}\ \psi_{\text{spin}}\ \psi_{\text{flavor}}$ is symmetric, which contradicts the spin-statistics theorem [Pauli exclusion principle [27]].

This inconsistency can be resolved by introducing a further quark quantum number. Namely, quarks have to carry a further hitherto unknown internal degree of freedom, called color. As it is already proven experimentally by the $\pi^0 \to \gamma\gamma$ decay [28, 29] or measuring the ratio of the cross section of e^+e^- annihilation into hadrons in comparison to the one into muons [30], quarks can come in three fundamental colors, called red, green, and blue. (In Tab. 1.1, we have already included this quantum number by the superscripts r, g, b.)

The total wave function of Δ^{++} has to be extended by an additional color part:

$$\Psi_{\Delta^{++}} = \psi_{\text{space}}\ \psi_{\text{spin}}\ \psi_{\text{flavor}}\ \psi_{\text{color}}. \tag{1.11}$$

Now, we can render the total wave function antisymmetric if we require ψ_{color} to be antisymmetric.

This is related to the concept of color confinement [31], which states that hadrons have to transform as singlets under the $SU(3)_c$ color symmetry of the strong interaction, see Sec. 1.3.1. That is, hadrons are invariant under $SU(3)_c$ transformations, which is why they are often said to be white objects. In the case of $\bar{q}q$ mesons color and anticolor combine to a white state, while in the case of qqq baryons, we have to work out the coupling rules of $SU(3)$ to obtain the color singlet. However, as an mnemonic aid it is as in chromatics: the combination of red, green, and blue yields a white state.
After this discussion it is clear that neither qq nor $qq\bar{q}$ (or similar) states exist. No matter how we choose the colors, such quark combinations will never form white objects in color space.

Note, in turn color confinement also entails that colored states cannot be isolated. This is why quarks cannot be experimentally studied in any more direct way than on a hadron level.

1.1.2. Baryons in the quark model

Baryons are hadrons with half-integer spin (fermions) and baryon number $B = 1$. As far as it is known [apart from some exotic states, e.g. pentaquarks [25]] all established baryons can be understood as bound states of three quarks surrounded by an arbitrary number of gluons and quark-antiquark pairs, or equivalently as a three-quark object in the concept of constituent quarks. In this work, we describe baryons as qqq states.

The baryonic wave function has to be antisymmetric under the exchange of any two quarks (anticommuting Grassmann variables) and decomposes as follows:

$$|qqq\rangle_A = |\text{color}\rangle_A \times |\text{space, spin, flavor}\rangle_S \ , \tag{1.12}$$

where the subscripts S and A indicate symmetric or antisymmetric wave functions. The antisymmetry of the color part is mandatory due to color confinement. In the case of baryons, where three quarks clump together, the color wave function has to combine all three colors as follows[2]

$$|\text{color}\rangle_A = \frac{1}{\sqrt{6}}(rgb + brg + gbr - rbg - bgr - grb) \ , \tag{1.13}$$

[2]It is the $SU(3)_c$ singlet state emerging from the coupling of three color triplets being the fundamental representations of $SU(3)_c$.

$$\frac{1}{\sqrt{6}}(uds + sud + dsu - usd - sdu - dus)$$

Figure 1.1. The antisymmetric baryon singlet $[1]_A$ in the $(I,\mathcal{Y})$ plane.

where the factor $6^{-1/2}$ accounts for normalization.

The combined space-, spin-, and flavor-wave function $|\text{space, spin, flavor}\rangle_S$ has to be symmetric under the exchange of two quarks. In the following, we separately consider these three parts step by step for the case $N_f = 3$.

Flavor part

We assume that to a good approximation the masses of the u, d, and s quarks can be considered to be equal when compared to the typical hadronic mass scale of ~ 1 GeV, see Tab. 1.1. This implies an approximate $SU(3)_V \times U(1)_V$ symmetry (more details are presented in Sec. 1.3.4), where the exact symmetry under $U(1)_V$ corresponds to the baryon-number conservation.
The remaining (approximate) $SU(3)_V$ flavor symmetry allows us to classify hadrons by the irreducible representations of the residual symmetry group.

Each state of an $SU(3)$ multiplet is uniquely determined by the eigenvalues of the Cartan generators $\hat{T}_3$ and $\hat{T}_8$. The latter, however, is usually replaced by the hypercharge operator $\hat{\mathcal{Y}} = 2\hat{T}_8/\sqrt{3}$. The eigenvalue of $\hat{T}_3$ is the z-component of the isospin $T_3 = I_3$ and the eigenvalue of the $\hat{\mathcal{Y}}$ is the hypercharge $\mathcal{Y}$. Since each state of the multiplets is characterized by these eigenvalues, they can be uniquely plotted in the $(I, \mathcal{Y})$ plane.

The fundamental triplet $[3]$ of $SU(3)_V$ is formed by the three quarks $\{u, d, s\}$, while the three antiquarks $\{\bar{u}, \bar{d}, \bar{s}\}$ form an anti-triplet $[\bar{3}]$. Using the basic group rules for $SU(3)$ multiplets, we construct all higher-dimensional flavor multiplets from the fundamental quark triplet $[3]$ or antiquark anti-triplet $[\bar{3}]$, where the number of used triplets and anti-triplets is determined by the number of constituent quarks (or antiquarks) of the considered hadron.

Baryons being three-quark states belong to the multiplets on the right-hand side of

$$[3] \otimes [3] \otimes [3] = [10]_S \oplus [8]_{M_S} \oplus [8]_{M_A} \oplus [1]_A \ . \tag{1.14}$$

Thus, baryons can be sorted either into a symmetric flavor decuplet $[10]_S$, into an antisymmetric flavor singlet $[1]_A$, or into flavor octets $[8]_{M_{S/A}}$ with "mixed symmetry", which means the symmetry (M_S) or antisymmetry (M_A) applies only to the first two quarks. The multiplets in the decomposition (1.14) are depicted in Figs. 1.1 to 1.4, where we omit the T_3 and $\mathcal{Y}$ axes, but give the flavor content of the different states.

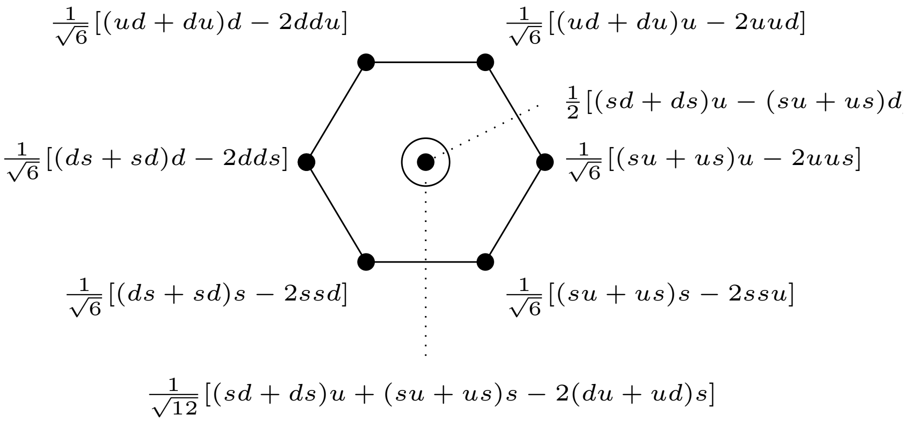


Figure 1.2. The mixed-symmetric flavor octet $[8]_{M_S}$ in the $(I,\mathcal{Y})$ plane.

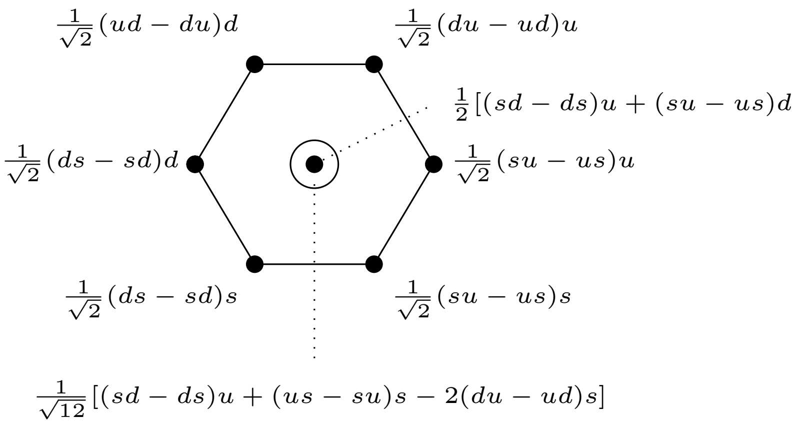


Figure 1.3. The mixed-symmetric flavor octet $[8]_{M_A}$ in the $(I,\mathcal{Y})$ plane.

Within these multiplets two similar *uds* states appear, one is the singlet $[1]_A$ state, and the other is contained in the $[8]_{M_A}$ octet. If these two states have equal spin and parity quantum numbers, they can mix.

Spin part

A quark carrying spin $\frac{1}{2}$ forms a spin doublet: $\{+\frac{1}{2},-\frac{1}{2}\}\equiv\{\uparrow,\downarrow\}$. Since baryons are three-quark states, we couple three spin doublets and obtain that baryons are arranged into a symmetric spin-quartet and two spin-doublets with mixed symmetry:

$$[2]\otimes[2]\otimes[2]=[4]_S\oplus[2]_{M_S}\oplus[2]_{M_A}\ . \tag{1.15}$$

Baryons with spin $\frac{3}{2}$ belong to the quartet, while those with spin $\frac{1}{2}$ belong to the doublets.

Spin-flavor part

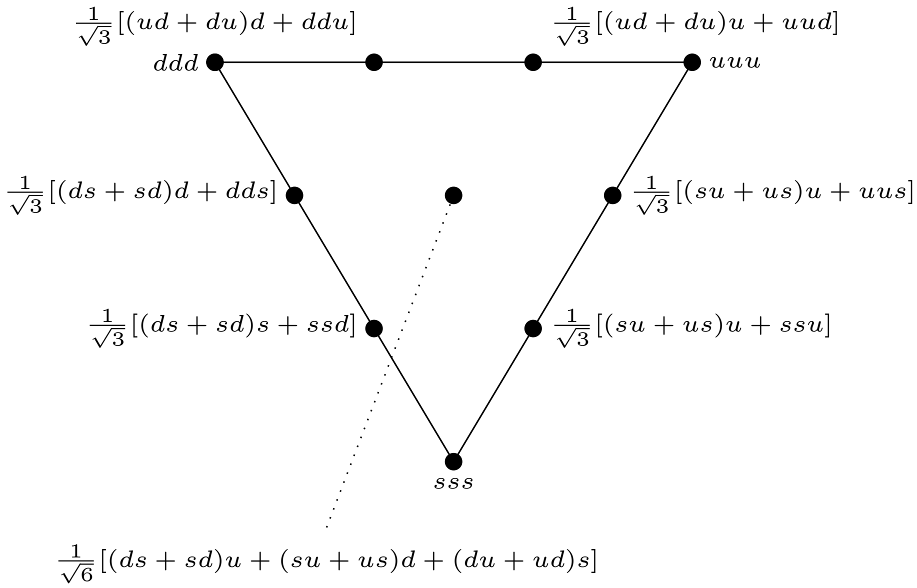


Figure 1.4. The symmetric baryon decuplet $[10]_S$ in the $(I,\mathcal{Y})$ plane.

It is possible to combine spin and flavor to an approximate $SU(6)$ symmetry. We denote the six basic states as

$$u^{\uparrow} \ , \ u_{\downarrow} \ , \ d^{\uparrow} \ , \ d_{\downarrow} \ , \ s^{\uparrow} \ , \ s_{\downarrow} \ . \tag{1.16}$$

Then, "ordinary" baryons can be sorted into the multiplets on the right-hand side of

$$[6] \otimes [6] \otimes [6] = [56]_S \oplus [70]_{M_S} \oplus [70]_{M_A} \oplus [20]_A \ , \tag{1.17}$$

which decompose into spin and flavor multiplets as follows,

$$[56]_S = ([4]^s_S, [10]^f_S) \oplus ([2]^s, [8]^f)_S \ , \tag{1.18}$$

$$[70]_{M_S} = ([2]^s_{M_S}, [10]^f_S) \oplus ([4]^s_S, [8]^f_{M_S}) \oplus ([2]^s_{M_S}, [8]^f_{M_S}) \oplus ([2]^s_{M_A}, [1]^f_A) \ , \tag{1.19}$$

$$[70]_{M_A} = ([2]^s_{M_A}, [10]^f_S) \oplus ([4]^s_S, [8]^f_{M_A}) \oplus ([2]^s_{M_A}, [8]^f_{M_A}) \oplus ([2]^s_{M_S}, [1]^f_A) \ , \tag{1.20}$$

$$[20]_A = ([2]^s, [8]^f)_A \oplus ([4]^s_S, [1]^f_A) \ , \tag{1.21}$$

where we introduced a "(spin, flavor)"-multiplet notation $(SU(2)_S, SU(3)_V)$ and the totally symmetric and antisymmetric flavor octets arise from the combinations

$$([2]^s, [8]^f)_S = \frac{([2]^s_{M_S}, [8]^f_{M_S}) \oplus ([2]^s_{M_A}, [8]^f_{M_A})}{\sqrt{2}} \ , \tag{1.22}$$

$$([2]^s, [8]^f)_A = \frac{([2]^s_{M_A}, [8]^f_{M_S}) \oplus ([2]^s_{M_S}, [8]^f_{M_A})}{\sqrt{2}} \ . \tag{1.23}$$

Spatial part

We distinguish between ground-state baryons with $L = 0$, and excited states with $L > 0$. While the spatial wave function of ground-state baryons is symmetric, the wave function of excited states remains to be further investigated. Usually these baryons are classified into multiplets of the combined spin-flavor-space group $SU(6) \times O(3)$.

As in Ref. [18], we introduce the notation

$$[D]_N^{LP} \ , \tag{1.24}$$

which allows us to further distinguish baryons into bands of equal quanta of excitations N, where D is the dimension of the $SU(6)$ spin-flavor representation.

$N = 0$: For ground-state baryons ($L = 0$, $P = +$), the spatial part of the wave function is symmetric. Thus, in order to render the combination of the spin-flavor and space wave function symmetric (and thus the full baryon wave function (1.12) antisymmetric), the ground-state baryons are collected in the symmetric

$$[56]_0^{0+}\text{-plet} \ .$$

That is, they are arrange into either a flavor octet $([2]^s, [8]^f)$ [Eq. (1.22)] with $J^P = \frac{1}{2}^+$ ($L = 0, S = \frac{1}{2}$), or a flavor decuplet $([4]^s, [10]^f)$, [Eq. (1.18)] with $J^P = \frac{3}{2}^+$ ($L = 0, S = \frac{3}{2}$). A ground-state singlet, however, is forbidden due to the Pauli principle.
The ground-state octet contains, for instance, the isospin-doublet of nucleons (proton p and neutron n), while the decuplet contains for example the isospin-quartet $\Delta^{[0,\pm,++]}$ of the $\Delta(1232)$ baryon. The arrangement of all ground-state baryons into the octet and decuplet is depicted in Fig. 1.5, where the representation is given in the strong isospin-hypercharge plane.

$N > 0$: The mixed-symmetric [70]plets and the antisymmetric [20]plet [Eqs. (1.19) to (1.21)] require excitations of the spatial part due to Pauli's principle. The excited states in the $N = 1$ band are arranged into the

$$[70]_1^{1-} - \text{plet} \ .$$

The $N = 2$ band contains

$$[56]_2^{0+} \ , \ [70]_2^{0+} \ , \ [56]_2^{2+} \ , \ [70]_2^{2+} \ , \ \text{and} \ [20]_2^{1+} \ ,$$

see Ref [18].

In this work, we focus on octet baryons. All of the ground-state baryons of the octet are believed to be experimentally known [18]. These are the nucleon $N(393)$, describing the proton and neutron, $\Lambda(1116)$, $\Sigma(1193)$, and $\Xi(1318)$, as already indicated in the left

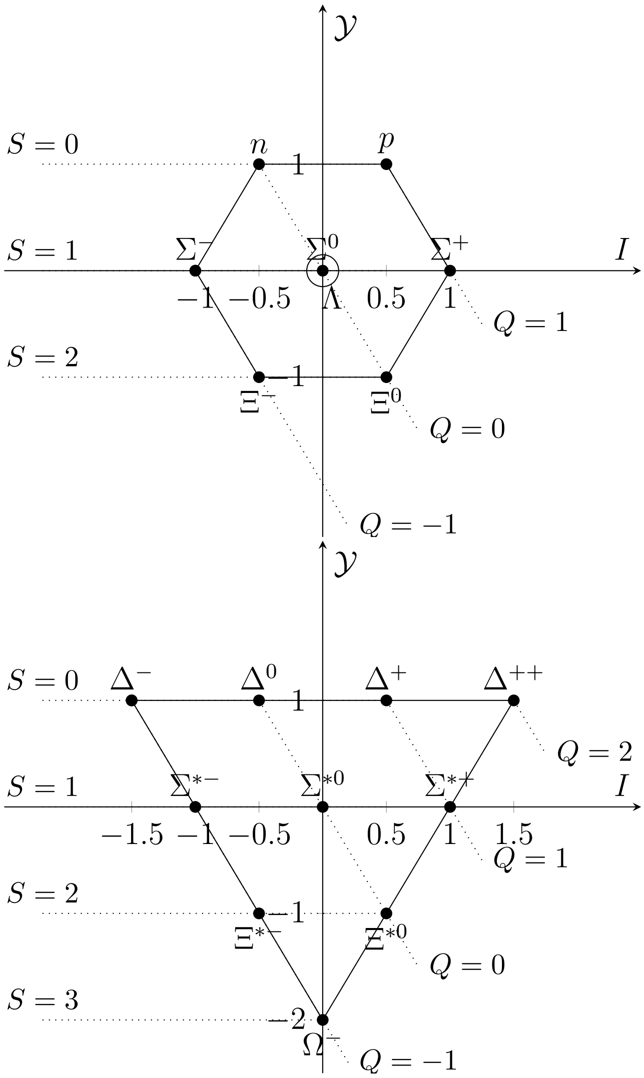


Figure 1.5. Ground-state baryons assigned to octet and decuplet.

diagram of Fig. 1.5. The assignment of excited octet members is less clear. In some cases candidates are missing or some states are only educated guesses. Nevertheless, Tab. 1.2 summarizes possible candidates that can be assigned to the four lowest $J^P = 1/2^{\pm}$ octet-baryon states, see Ref. [18].

1.1.3. Classification of mesons

Conventional mesons are hadrons with integer spin and baryon number $B = 0$. They are classified in multiplets of J^{PC}:

- scalars with $J^{PC} = 0^{++}$,
- pseudoscalars with $J^{PC} = 0^{-+}$,

	J^P	states	status	octet members of $[D]_N^{LP}$ $[56]_0^{0+}$	$[56]_2^{0+}$	$[70]_1^{1-}$	$[70]_1^{1-}$	mass [Mev]	quark content
$\mathcal{S}=0$, $I=\frac{1}{2}$	$\frac{1}{2}^+$	$N(939)$	****	×				$938.9187473 \pm 0.0000041$	
		$N(1440)$	****		×			1410 to 1450 (≈ 1430)	$p, N^+ = uud$
	$\frac{1}{2}^-$	$N(1535)$	****			×		1525 to 1545 (≈ 1535)	$n, N^0 = udd$
		$N(1650)$	****				×	1645 to 1670 (≈ 1655)	
$\mathcal{S}=-1$, $I=0$	$\frac{1}{2}^+$	$\Lambda(1116)$	****	×				1115.683 ± 0.006	
		$\Lambda(1600)$	***		×			1560 to 1700 (≈ 1600)	
	$\frac{1}{2}^-$	$\Lambda(1670)$	****			×		1660 to 1680 (≈ 1670)	$\Lambda^0 = uds$
		$\Lambda(1800)$	***				×	1720 to 1850 (≈ 1800)	
$\mathcal{S}=-1$, $I=1$	$\frac{1}{2}^+$	$\Sigma(1193)$	****	×				1193.15 ± 0.03	
		$\Sigma(1660)$	***		×			1630 to 1690 (≈ 1660)	$\Sigma^+ = uus$
	$\frac{1}{2}^-$	$\Sigma(1560)$	**			?		≈ 1560	$\Sigma^- = dds$
		$\Sigma(1620)$	*			×	?	≈ 1620	$\Sigma^0 = uds$
		$\Sigma(1750)$	***				×	1730 to 1800 (≈ 1750)	
$\mathcal{S}=-2$, $I=\frac{1}{2}$	$\frac{1}{2}^+$	$\Xi(1318)$	****	×				1318.28 ± 0.11	
		$\Xi(1690)$	***		?			1690 ± 10	$\Xi^0 = uss$
	$\frac{1}{2}^-$	$\Xi(?)$				×		?	$\Xi^- = dss$
		$\Xi(?)$					×	?	

Table 1.2. Octet-baryon assignment, where "?" entries are recent suggestions for (re-)assignments [18].

	L	S	J	P	C	J^{PC}
scalar	1	1	0	$+$	$+$	0^{++}
pseudoscalar	0	0	0	$-$	$+$	0^{-+}
vector	0	1	1	$-$	$-$	1^{--}
axial-vector	1	1	1	$+$	$+$	1^{++}

Table 1.3. Quantum numbers of (pseudo)scalar and (axial-)vector mesons.

- vector mesons with $J^{PC} = 1^{--}$,
- axial-vector mesons with $J^{PC} = 1^{++}$,
- or tensor states for $J \geq 2$,

see Tab. 1.3.

1.2. Quantum field theory and symmetries

In order to describe the behavior of the small(est) constituents of nature (may this be quarks or hadrons) we have to consider the mathematical approach of quantum field theory (QFT), because it merges special relativity and quantum mechanics. Only in this framework it is possible to describe the matter that particles can be created in the quantum vacuum and, in reverse, particles can also disappear when mass is transformed into energy. This can be understood as a combination of Heisenberg's uncertainty principle, which causes a turbulent and erratical appearance of the microscopic world allowing the energy to fluctuate over a small interval of time, and on the other hand, special relativity explaining that energy can be converted into mass and vice versa. Thus, it is a characteristic feature of QFT that energy fluctuations can be transformed into mass, and vice versa.

1.2.1. Lagrangian formalism and path integrals

The leading role in field theories is played by fields. In the framework of QFT, particles are no longer point-like elements described by generalized coordinates and velocities, but they are understood as excitations of a quantum field that is defined over the whole space and all time.[3]

[3]In Ref. [32] a simple motivation of this idea is explained: we can think of a mattress described by a two dimensional lattice of point masses connected to each other by springs. For simplicity we neglect horizontal deflections of the masses and consider only the vertical deflections. From the equations of motion one finds that the point masses can oscillate with certain eigenfrequencies and eigenmodes

To study the dynamics of fields, we use the Lagrangian formalism. Considering a theory that contains N fields, the Lagrange function is a functional of the field variables ϕ_a and their temporal partial derivatives $\dot{\phi}_a$ for $a = 1, 2, \ldots, N$. This functional can be written as a volume integral over the Lagrangian $\mathcal{L}$:

$$L[\phi_a, \dot{\phi}_a] = \int_V \mathrm{d}^3\boldsymbol{r}\ \mathcal{L}(\phi_a, \partial_\mu \phi_a) \ , \tag{1.25}$$

where $V \subset \mathbb{R}^3$ is the volume of the system.

The action is defined as the temporal integral over the Lagrange function or equivalently as the spacetime integral over the Lagrangian for a certain time interval $[t_i, t_f]$:

$$S[\phi_a] = \int_{t_i}^{t_f} \mathrm{d}t\ L[\phi_a, \dot{\phi}_a] = \int_\Omega \mathrm{d}^4x\ \mathcal{L}(\phi_a, \partial_\mu \phi_a) \ , \tag{1.26}$$

where we have introduced the spacetime volume $\Omega = [t_f, t_i] \times V \subset \mathbb{R} \times \mathbb{R}^3$.

Applying Hamilton's principle, we can extract all equations of motion (related to the considered system), if we determine which field configurations minimize the action.
The functional (1.26) is stationary, if its variation upon varying the fields inside the spacetime volume Ω vanishes (while the fields as well as the spacetime coordinates on the spacetime surface $\partial\Omega$ are kept fixed at a constant value):

$$\begin{aligned} 0 \stackrel{!}{=} \delta S &= \delta \int_\Omega \mathrm{d}^4x\ \mathcal{L}(\phi_a, \partial_\mu \phi_a) \\ &= \int_\Omega \mathrm{d}^4x\ \Big[\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a)\Big] \\ &= \int_\Omega \mathrm{d}^4x\ \Big\{\Big[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\Big]\delta\phi_a + \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\Big)\Big\} \\ &= \int_\Omega \mathrm{d}^4x\ \Big[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\Big]\delta\phi_a \ , \end{aligned} \tag{1.27}$$

where in the last line, we have used Gauss's theorem [divergence theorem, see e.g. Ref. [33]] to convert the 4-divergence into a surface integral, which vanishes, because the variation of the field is fixed on the surface. Since the fields are varied independently, each term of the integrand in Eq. (1.27) has to vanish itself:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \qquad \forall\ a \in \{1, 2, \ldots, N\} \ . \tag{1.28}$$

The resulting equations (1.28) are called Euler-Lagrange equations. They are the classical equations of motion of the fields ϕ_a. They determine the classical field configurations,

which by superposition form wave packets. After quantisation, we realize that these wave packets, wandering around on the mattress, behave like particles.

which under Hamilton's principle minimize the action in accordance with the given boundary conditions. [In QFT this is somewhat different: not only the solution that minimizes the action, but all possible field configurations contribute to the physical observables (although with different weighting factors proportional to the action).]

As mentioned above, particle-antiparticle pairs can be constantly generated out of the vacuum. As a consequence, it is necessary to use a many-body approach to QFT.[4]

In quantum mechanics, it is known that many-particle systems are described by the introduction of creation and annihilation operators. This is called "second quantization" (in contrast to the "first quantization", which gives the quantum-mechanical description of a single-particle system by a wave equation, such as the Schrödinger equation).
In the case of field theories a similar procedure exists, see e.g. Ref. [34], where fields of the theory are promoted to operators[5] and suitable canonical commutation relations as imposed. This implicates the occurrence of creation and annihilation operators, which create or annihilate particles when acting on a known quantum state. However, in the following we use another approach to QFT where one utilizes the path-integral formalism of quantum mechanics, see Ref. [35].

Due to Heisenberg's uncertainty principle, it is not possible to know the position and momentum of a particle at the same time. The central quantum-mechanical quantity is the overlap $\langle \boldsymbol{q}_f, t_f | \boldsymbol{q}_i, t_i \rangle$, which corresponds to the amplitude of a transition from the state $|\boldsymbol{q}_i, t_i\rangle$ at time t_i to the state $\left|\boldsymbol{q}_f, t_f\right\rangle$ at time t_f. To build the bridge to QFT, we consider the transition amplitude of an initial field configuration at t_i to a final field configuration at t_f,

$$\langle \phi_{1,f}, \ldots, \phi_{N,f}, t_f | \phi_{1,i}, \ldots, \phi_{N,i}, t_i \rangle \; . \tag{1.29}$$

For almost all physical theories [when a Hamiltonian contains terms which are at maximum of quadratic power in the conjugate fields $\pi_a(x)$], this transition can be expressed as follows utilizing the path-integral formalism, see e.g. Refs. [34, 36]:

$$\langle \phi_{1,f}, \ldots, \phi_{N,f}, t_f | \phi_{1,i}, \ldots, \phi_{N,i}, t_i \rangle = \mathcal{N} \int \mathcal{D}\vec{\phi}(x) \exp\Big\{ i \int \mathrm{d}^4x \; \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)) \Big\} \; , \tag{1.30}$$

[4] We recall the mechanism of the Dirac sea, i.e., the theoretical interpretation of the solutions of the free Dirac equation. One assumes that in the vacuum all negative energy states are occupied with fermions, which build the Dirac sea. Due to the Pauli principle, this prevents a fermion on a positive energy level from continuously emitting energy and occupying deeper and deeper negative energy levels. The generation of particle-antiparticle pairs corresponds to the lifting of a particle in the Dirac sea into a state with positive energy. The missing particle in the Dirac sea can be seen as a hole, which in turn is interpreted as the antiparticle. Due to this constant presence of an infinite number of fermions in the Dirac sea, it is clear that a relativistic description of fermions cannot be a single-particle theory.

[5] The Fourier components are promoted to creation and annihilation operators.

where $\mathcal{N}$ is a normalization constant and the path-integral measure is defined as

$$\mathcal{D}\vec{\phi}(x) \equiv \lim_{\substack{n\to\infty \\ \tau\to 0}} \prod_{k=1}^{n} \prod_{\boldsymbol{r}} \mathrm{d}\vec{\phi}_k(\boldsymbol{r}) \ , \tag{1.31}$$

with $\mathrm{d}\vec{\phi}_k(\boldsymbol{r}) = \mathrm{d}\phi_{1,k}(\boldsymbol{r}) \ldots \mathrm{d}\phi_{N,k}(\boldsymbol{r})$. The integral measure

$$\prod_{k} \prod_{\boldsymbol{r}} \mathrm{d}\vec{\phi}_k(\boldsymbol{r}) \tag{1.32}$$

represents the integration over all $\phi_{a,k}(\boldsymbol{r})$ at each point in space $\boldsymbol{r}$ for all (discrete) points in time

$$t_k = k\tau + t_i \ , \text{where} \ \ n\tau = t_f - t_i \ . \tag{1.33}$$

The notation in Eq. (1.31) has to be understood symbolically. If we discretize the (infinite) space, the set of all space points is countably infinite. In this case, the integration measure is well defined. Next, we perform the limit to the continuum, that is, we shrink the gaps between points in space to zero (and also $\tau \to 0, n \to \infty$, with $n\tau = t_f - t_i$ being fixed).

The expression (1.30) can be understood as a generalization of Hamilton's principle of stationary action.
In classical field theory a system evolves from an initial to a final configuration on a "path" that minimizes the action.
In the quantum field theoretical approach the transition amplitude (1.30) appears to be a sum over an infinity of possible configuration paths weighted by a phase factor $\sim \exp(iS)$ with the action (1.26) in the exponent. The system takes all possible field configurations leading from an initial to a final configuration. However, for the non-classical paths the different phase factors are such that their contributions are suppressed. The classical configuration is important because many paths close to it interfere constructively. In turn, that is why Hamilton's principle of the stationary action works.

1.2.2. Decay-width calculation

Further exploiting Eq. (1.30), one can compute physical quantities of interest, such as decay widths, which are related to the lifetime τ of a particle by

$$\Gamma = 1/\tau \ . \tag{1.34}$$

This quantity that can be compared to experimentally known data and will be considered frequently in this work.

For the decay of an initial particle (with momentum k) into n final ones (with momenta $p_f\ \forall\ f \in \{1, 2, \dots n\}$) the following formula for the differential decay width holds, see Ref. [34]:

$$\mathrm{d}\Gamma = \frac{1}{2m_i}\left(\prod_f \frac{\mathrm{d}\boldsymbol{p_f}}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}_{m_i\to p_f}\right|^2 (2\pi)^4\delta^{(4)}\left(k - \sum_f p_f\right) , \tag{1.35}$$

where the reference frame is the one in which the initial particle is at rest, i.e.,

$$k = (m_i, 0, 0, 0)^T . \tag{1.36}$$

The out-going particles carry the four-momentum

$$p_f = (E_f, \boldsymbol{k}_f)^T . \tag{1.37}$$

The invariant matrix element $\mathcal{M}_{m_i\to p_f}$ can be understood as analogon of the scattering amplitude of one-particle quantum mechanics. It contains all the physics depending on the details of the specific Lagrangian (so to say the dynamics), while everything else in the expression (1.35) contains only the general kinematic information.

As an example we consider a theory that includes the Dirac fields $B'(x)$ and $B(x)$, as well as a scalar Klein-Gordon field $P(x)$. [Later on, we will exploit the result of this study by assigning baryons and a (pseudo)scalar meson to these fields.] The Lagrangian contains the following interaction terms which include also derivative couplings :

$$\mathcal{L} = -ig_{B'PB}\ \bar{B}\Upsilon P B' \pm g_{B'\partial PB}\ \bar{B}\Upsilon\gamma_\mu(\partial^\mu P)B' , \tag{1.38}$$

where γ^μ are the usual Dirac matrices in spinor space, $g_{B'PB}$ and $g_{B'\partial PB}$ are dimensionless coupling constants. The matrix Υ is defined as

$$\Upsilon = \begin{cases} \gamma_5 & \text{, if } B' \text{ and } B \text{ have the same parity,} \\ 1 & \text{, if } B' \text{ and } B \text{ have opposite parity,} \end{cases} \tag{1.39}$$

and the upper and lower sign in Eq. (1.38) is valid for $\Upsilon = \gamma_5$ and 1, respectively, such that the Lagrangian is invariant under parity transformations. We further rewrite Eq. (1.38) as

$$\begin{aligned}\mathcal{L} &= \bar{B}\Upsilon\left[-ig_{B'PB}\ P \pm g_{B'\partial PB}\ \gamma_\mu\left(\partial^\mu P\right)\right]B' \\ &= -i\bar{B}\Upsilon\left(g_{B'PB} \pm g_{B'\partial PB}\ \not{p}_1\right)PB' ,\end{aligned} \tag{1.40}$$

where we have replaced

$$\partial_\mu P \to -ip_{1,\mu}P , \tag{1.41}$$

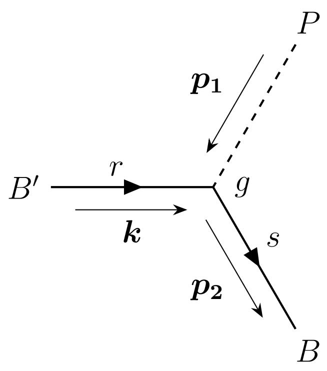


Figure 1.6. Tree-level Feynman diagram corresponding to the decay $B' \to BP$.

with momentum p_1^μ of the (pseudo)scalar particle and the Feynman-slash notation $\gamma_\mu p_1^\mu = \not{p}_1$.

The tree-level Feynman diagram of the corresponding decay

$$B' \to BP \tag{1.42}$$

is depicted in Fig. 1.6. Note, we assigned momenta (k, p_1, p_2) and spins (r, s) as follows:

initial Dirac field B' :	k , r
final Klein-Gordon field P:	p_1
final Dirac field B:	p_2 , s

We assume that the decay $B' \to BP$ is kinematically allowed, i.e., $m_{B'} > m_B + m_P$ holds, and compute the decay width according to Eq. (1.35) as

$$\Gamma_{B'\to BP} = \int \frac{\mathrm{d}\boldsymbol{p_1}\mathrm{d}\boldsymbol{p_2}}{(2\pi)^2} \frac{1}{8m_{B'}E_B E_P} \left|\mathcal{M}_{B'\to BP}\right|^2 \delta^{(4)}(k - p_1 - p_2) \ . \tag{1.43}$$

In the rest frame of the initial particle, i.e.,

$$k = (m_{B'}, 0, 0, 0)^T \ , \ p_1 = (E_P, \boldsymbol{p_1})^T \ , \ p_2 = (E_B, \boldsymbol{p_2})^T$$

the four-dimensional delta distribution can be rewritten as

$$\delta^{(4)}(k - p_1 - p_2) = \delta(m_{B'} - E_B - E_P)\delta^{(3)}(\boldsymbol{p_1} + \boldsymbol{p_2}) \ . \tag{1.44}$$

The three-dimensional delta distributions cause a collapse of the $\boldsymbol{p_2}$ integral, such that Eq. (1.43) becomes

$$\Gamma_{B'\to BP} = \int \frac{\mathrm{d}\boldsymbol{p_1}}{32\pi^2 m_{B'}} \frac{\left|\mathcal{M}_{B'\to BP}\right|^2}{\sqrt{m_B^2 + \boldsymbol{p_1}^2}\sqrt{m_P^2 + \boldsymbol{p_1}^2}} \ \delta\left(m_{B'} - \sqrt{m_B^2 + \boldsymbol{p_1}^2} - \sqrt{m_P^2 + \boldsymbol{p_1}^2}\right) \ , \tag{1.45}$$

where we have used the relativistic energy-momentum relation, $E^2 = m^2 + \boldsymbol{p}^2$, to replace E_B and E_P. We evaluate the remaining delta distribution utilizing the following general property of delta distributions, see Ref [33]:

$$\delta[f(x)] = \sum_j \frac{\delta(x - x_j)}{|f'(x_j)|} \ , \tag{1.46}$$

where $f(x_j) = 0$ and $f'(x_j) \neq 0$. For the delta distribution in Eq. (1.45) we find

$$\delta\left(m_{B'} - \sqrt{m_B^2 + \boldsymbol{p_1}^2} - \sqrt{m_P^2 + \boldsymbol{p_1}^2}\right) = \frac{\sqrt{m_B^2 + p_f^2}\sqrt{m_P^2 + p_f^2}}{p_f m_{B'}} \delta(|\boldsymbol{p_1}| - p_f) \ , \tag{1.47}$$

where

$$p_f = \frac{1}{2m_{B'}} \sqrt{(m_{B'}^2 - m_B^2 - m_P^2)^2 - 4m_B^2 m_P^2} \tag{1.48}$$

is the magnitude of the three-momentum of the particles in the final state[6].
With Eq. (1.47), the decay width in Eq. (1.45) becomes

$$\begin{aligned} \Gamma_{B' \to BP} &= \int \frac{\mathrm{d}\boldsymbol{p_1}}{32\pi^2 m_{B'}} \frac{|\mathcal{M}_{B' \to BP}|^2}{\sqrt{m_B^2 + \boldsymbol{p_1}^2}\sqrt{m_P^2 + \boldsymbol{p_1}^2}} \frac{\sqrt{m_B^2 + p_f^2}\sqrt{m_P^2 + p_f^2}}{p_f m_{B'}} \delta(|\boldsymbol{p_1}| - p_f) \\ &= \int \frac{\mathrm{d}|\boldsymbol{p_1}|\mathrm{d}\Omega \ |\boldsymbol{p_1}|^2}{32\pi^2 m_{B'}} \frac{|\mathcal{M}_{B' \to BP}|^2}{p_f m_{B'}} \delta(|\boldsymbol{p_1}| - p_f) \ , \end{aligned} \tag{1.50}$$

where we have introduced spherical coordinates in the second line and we have assume that there is no angular dependence in $|\mathcal{M}_{B' \to BP}|$. Then, the integrals of the polar and azimuth angles separate and give $\int \mathrm{d}\Omega = 4\pi$. Finally, the delta distribution causes the remaining integral in Eq. (1.50) to collapse and we obtain

$$\Gamma_{B' \to BP} = \frac{p_f}{8\pi m_{B'}^2} |\mathcal{M}_{B' \to BP}|^2 \ . \tag{1.51}$$

[6]The expression (1.48) follows from the conservation of energy and momentum. Since $\boldsymbol{k} = 0$, momentum conservation implies $\boldsymbol{p_1} = -\boldsymbol{p_2} \equiv \boldsymbol{p}$. In detail:

$$\begin{aligned} 0 &= m_{B'} - E_B - E_P \\ 0 &= m_{B'} - \sqrt{\boldsymbol{p}^2 + m_B^2} - \sqrt{\boldsymbol{p}^2 + m_P^2} \\ m_{B'}^2 &= \boldsymbol{p}^2 + m_B^2 + 2\sqrt{\boldsymbol{p}^2 + m_B^2}\sqrt{\boldsymbol{p}^2 + m_P^2} + \boldsymbol{p}^2 + m_P^2 \\ (m_{B'}^2 - m_B^2 - m_P^2 - 2\boldsymbol{p}^2)^2 &= 4(\boldsymbol{p}^2 + m_B^2)(\boldsymbol{p}^2 + m_P^2) \\ (m_{B'}^2 - m_B^2 - m_P^2)^2 + 4\boldsymbol{p}^4 - 4\boldsymbol{p}^2(m_{B'}^2 - m_B^2 - m_P^2) &= 4\boldsymbol{p}^4 + 4\boldsymbol{p}^2(m_B^2 + m_P^2) + 4m_B^2 m_P^2 \\ (m_{B'}^2 - m_B^2 - m_P^2)^2 - 4\boldsymbol{p}^2 m_{B'}^2 &= 4m_B^2 m_P^2 \ , \end{aligned} \tag{1.49}$$

The solutions of this equations are $\pm p_f$, see Eq. (1.48).

The invariant matrix element $\mathcal{M}_{B'\to BP}$ can be obtained from the Feynman diagram in Fig. 1.6 by using the respective Feynman rules corresponding to the theory in Eq. (1.40). In momentum space they are given by

- external scalar and pseudoscalar particles with arbitrary momentum contribute with a factor 1,
- external fermion lines pointing into the vertex with three-momentum $\boldsymbol{k}$, spin r, and isospin j contribute with the Dirac spinor $u^j(\boldsymbol{k}, r)$,
- external fermion lines pointing away from the vertex with three-momentum $\boldsymbol{p_2}$, spin s, and isospin i are represented by the Dirac spinor $\bar{u}^i(\boldsymbol{p_2}, s)$.

Without closer defining the isospin of the involved particles, we can write down the following general expression[7] for the invariant matrix element:

$$
\begin{aligned}
i\mathcal{M}^{\pm}_{B'\to BP} &= -i\bar{u}(\boldsymbol{p_2}, s)\Upsilon g u(\boldsymbol{k}, r) \\
&= -i\bar{u}(\boldsymbol{p_2}, s)\Upsilon(g_{B'PB} \pm g_{B'\partial PB}\, \not{p}_0)u(\boldsymbol{k}, r) \\
&= -i\bar{u}(\boldsymbol{p_2}, s)\Upsilon\left[g_{B'PB} \pm g_{B'\partial PB}(\not{p}_1 - \not{k})\right] + u(\boldsymbol{k}, r) \\
&= -i\bar{u}(\boldsymbol{p_2}, s)\left[\Upsilon g_{B'PB} \pm g_{B'\partial PB}(\mp\not{p}_1\Upsilon - \Upsilon\not{k})\right]u(\boldsymbol{k}, r) \\
&= -i\bar{u}(\boldsymbol{p}, s)\left[g_{B'PB} \pm g_{B'\partial PB}(\mp m_B - m_{B'})\right]\Upsilon u(\boldsymbol{k}, r) \\
&= -i\left[g_{B'PB} - g_{B'\partial PB}(m_B \pm m_{B'})\right]\bar{u}(\boldsymbol{p}, s)\Upsilon u(\boldsymbol{k}, r) \ ,
\end{aligned}
\tag{1.52}
$$

where in the second last line we have used the Dirac equation for the spinors and in the rest frame $\boldsymbol{k} = 0$ momentum conservation implies

$$
\boldsymbol{p_1} = -\boldsymbol{p_2} \equiv \boldsymbol{p} \ . \tag{1.53}
$$

We compute the squared amplitude and average over spin, i.e., we sum over all possible final spin states and divide it by the number of possible initial spin states:

$$
\overline{\left|\mathcal{M}^{\pm}_{B'\to BP}\right|^2} = \frac{1}{2}\sum_{r,s}\left[g_{B'PB} - g_{B'\partial PB}(m_B \pm m_{B'})\right]^2 \bar{u}^{\alpha}(\boldsymbol{p}, s)\Upsilon_{\alpha\beta}u^{\beta}(\boldsymbol{k}, r)\bar{u}^{\delta}(\boldsymbol{k}, r)\bar{\Upsilon}_{\delta\kappa}u^{\kappa}(\boldsymbol{p}, s) \ . \tag{1.54}
$$

[7] Note, momenta are assigned as depicted in Fig. 1.6.

This expression can be simplified by further evaluating the spin sums over spinor products:

$$
\begin{aligned}
&\frac{1}{2}\sum_s \bar{u}^\alpha(\boldsymbol{p},s)\Upsilon_{\alpha\beta}u^\beta(\boldsymbol{k},r)\bar{u}^\delta(\boldsymbol{k},r)\bar{\Upsilon}_{\delta\kappa}u^\kappa(\boldsymbol{p},s)\\
&\quad= \frac{1}{2}\sum_s \bar{u}^\alpha(\boldsymbol{p},s)\Upsilon_{\alpha\beta}(\not{k}+m_{B'})_{\beta\delta}\bar{\Upsilon}_{\delta\kappa}u^\kappa(\boldsymbol{p},s)\\
&\quad= \frac{1}{2}\Upsilon_{\alpha\beta}(\not{k}+m_{B'})_{\beta\delta}\bar{\Upsilon}_{\delta\kappa}(\not{p}+m_B)_{\kappa\alpha}\\
&\quad= \frac{1}{2}\,\mathrm{Tr}[\Upsilon(\not{k}+m_{B'})\bar{\Upsilon}(\not{p}+m_B)]\\
&\quad= \frac{1}{2}[\mathrm{Tr}(\Upsilon\not{k}\bar{\Upsilon}\not{p}) + m_{B'}m_B\,\mathrm{Tr}(\Upsilon\bar{\Upsilon}))]\\
&\quad= \frac{1}{2}[\mp k^\mu p^\nu\,\mathrm{Tr}(\gamma_\mu\Upsilon\bar{\Upsilon}\gamma_\nu) + m_{B'}m_B\,\mathrm{Tr}(\mp\mathbb{1}_4))]\\
&\quad= \frac{1}{2}[k^\mu p^\nu\,\mathrm{Tr}(\gamma_\mu\gamma_\nu) \mp 4m_{B'}m_B)]\\
&\quad= \frac{1}{2}[k^\mu p^\nu g_{\mu\rho}g_{\nu\sigma}\,\mathrm{Tr}(\gamma^\rho\gamma^\sigma) \mp 4m_{B'}m_B)]\\
&\quad= \frac{1}{2}[4k^\mu p^\nu g_{\mu\rho}g_{\nu\sigma}g^{\rho\sigma} \mp 4m_{B'}m_B)]\\
&\quad= \frac{1}{2}[k_\mu p^\mu \mp m_{B'}m_B)]\\
&\quad= 2[m_{B'}E_B \mp m_{B'}m_B)]\\
&\quad= 2(E_B \mp m_B)m_{B'}\ ,
\end{aligned}
\tag{1.55}
$$

where the matrix $\bar{\Upsilon}$ is equal to $-\gamma^5(1)$ for $\Upsilon = \gamma_5(1)$. In the first and second line, we have used

$$
\sum_s u^\alpha(\boldsymbol{p},s)\bar{u}^\beta(\boldsymbol{p},s) = (\not{p}+m)^{\alpha\beta}\ . \tag{1.56}
$$

From the third to the fourth line we have exploited that traces of an odd number of gamma matrices vanish. From the sixth to the seventh line we used $\mathrm{Tr}(\gamma^\rho\gamma^\sigma) = 4g^{\rho\sigma}$. In the third-to-last line we simplified the expression by taking advantage of the assumption that the initial particle was at rest, i.e., $k^\mu = (m_{B'},0,0,0)^T$.

Thus, Eq. (1.51) becomes

$$
\Gamma_{B'\to BP} = \frac{\lambda_{\mathrm{iso}}\,p_f}{4\pi m_{B'}}\left[g_{B'PB} - g_{B'\partial PB}\ \ (m_B \pm m_{B'})\right]^2(E_B \mp m_B)\ , \tag{1.57}
$$

where we have included the factor λ_{iso} by hand. It takes care for possible additional factors arising when isospin is regarded. [Once again, the upper and lower sign in Eq. (1.57) is valid if the parities of B' and B are equal or not, respectively. The momentum p_f is given in Eq. (1.48).]

1.2.3. Noether's theorem - continuous symmetries and conserved currents

This section is devoted to symmetries in physical laws, which will provide important guidelines for the construction of an effective Lagrangian of quantum chromodynamics and thus help to understand certain properties of the strong interaction.

Nature seems to prefer symmetric items, for instance Romanesco broccoli shows a fractal symmetry, water splashes are approximately radially symmetric or the Nautilus shell exhibits an interesting numerical symmetry known as the Fibonacci sequence.
However, besides these symmetries of objects, there exist symmetries of physical laws as investigated in this section.

First of all, we have to define what a symmetry actually is. In Ref. [37] Weyl defines it as an "invariance of a configuration of elements under a group of automorphic transformations", i.e., if we change something in a system it appears exactly the same after the operation. For example, if we take a sphere and rotate it about any of its center axes it still looks identical.

Symmetries in physical laws mean that we can change something in the physical situation of an experiment and the result is still the same. Thus, the law does not change upon some transformations.
As an example, if we perform an experiment in different places and the outcome is the same, the underlying physical phenomena exhibit a translation invariance. Besides this, we can also investigate other continuous transformations, such as the translation in time, rotations, or boosts, but also discrete symmetries like time reversal, space reflections (parity), or the symmetry under the exchange of matter and antimatter (charge conjugation) can be considered.

It appears that Nature shows a very fascinating feature about continuous symmetries. Namely, in every system that obeys a continuous symmetry we also have a conservation law (and a conserved charge). To name some examples:

- the invariance under spatial translations implies the conservation of linear momentum,
- the conservation of angular momentum follows from the invariance under rotations,
- the conservation of energy is related to the invariance under time translations.

This fact is formulated in Emmy Noether's theorem [38]. It reveals that every symmetry of the action functional upon a continuous transformation of the fields and/or the spacetime variables implies a conserved current density, which is given by [see Ref. [38] or App. A]

$$\mathcal{J}^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a - \theta^\mu{}_\nu \delta x^\nu \ , \tag{1.58}$$

with the energy-momentum tensor [34]

$$\theta^{\mu}{}_{\nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a + g^{\mu}{}_{\nu} \mathcal{L} \ . \tag{1.59}$$

This entails a conservation law of the Noether charge, which is given by [see App. A]

$$Q = \int_V \mathrm{d}^3\boldsymbol{r} \ \mathcal{J}^0(x) \ . \tag{1.60}$$

1.2.4. Spontaneous symmetry breaking

As Noether's theorem indicates, the role of symmetries can hardly be overestimated. Symmetries are the starting point to write down a Lagrangian. However, a totally symmetric world would be quite boring. Indeed, "real life" is not that simple. For instance, while the equations describing the atoms of a book are rotationally invariant, the book itself obviously has a definite orientation in space. The symmetry of the system is broken. More precisely, we can even say that the system itself broke the rotation symmetry (by forming a book).
This kind of symmetry breaking is called spontaneous symmetry breaking (SSB). In contrast to explicit symmetry breaking where the Lagrangian contains a term that is discernibly not invariant, the phenomenon of SSB can only be seen in the ground/vacuum state of the system.

As it is common in several lectures on SSB, we explain it by a simple example. Let us consider a theory including real scalar fields $\phi(x)$ and a quartic self-interaction,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \mathcal{V}(\phi) \ , \qquad \mathcal{V}(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \ , \tag{1.61}$$

which admits the global symmetry under

$$\phi(x) \to -\phi(x) \ . \tag{1.62}$$

Note, this Lagrangian is similar to the famous ϕ^4-theory, but with a swapped sign of the ϕ^2 term. The potential density is a so-called double-well potential as displayed in Fig. 1.7. Its two minima lie at

$$\phi = \pm\phi_0 = \pm(6m^2/\lambda)^{\frac{1}{2}} \ . \tag{1.63}$$

The usual way to study such interacting field theories is a perturbative approach, where the interaction terms can be assumed to be small perturbations of the free theory. In this framework, we expand the theory around a minimum, i.e., we study small oscillations around the ground state. These fluctuations correspond to physical excitations, i.e., particles.
We decompose the scalar field $\phi(x)$ by shifting it by one of its vacuum expectation values

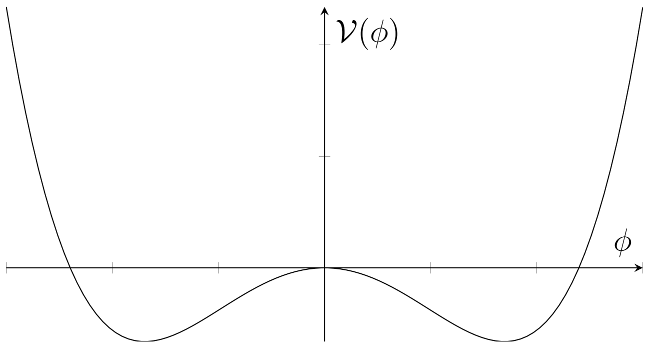


Figure 1.7. The double-well potential.

(VEV) in Eq. (1.63) and introducing a new dynamical variable $\sigma(x)$, which describes the fluctuations around that minimum,

$$\phi(x) = \sigma(x) \pm \phi_0 \ . \tag{1.64}$$

We write the Lagrangian (1.61) in terms of this fluctuating field,

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \sigma)^2 - \mathcal{V}(\sigma \pm \phi_0) \\
&= \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{m^2}{2}(\sigma^2 + \phi_0^2 \pm 2\sigma\phi_0) - \frac{\lambda}{4!}(\sigma^4 \pm 4\sigma^3\phi_0 + 6\sigma^2\phi_0^2 \pm 4\sigma\phi_0^3 + \phi_0^4) \\
&= \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(-m^2 + \frac{\lambda}{2}\phi_0^2)\sigma^2 \mp \frac{\lambda}{3!}\phi_0\sigma^3 - \frac{\lambda}{4!}\sigma^4 + \frac{m^2}{2}\phi_0^2 - \frac{\lambda}{4!}\phi_0^4 \pm (m^2 - \frac{\lambda}{3!}\phi_0^2)\phi_0\sigma \\
&= \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 \mp \frac{\lambda}{3!}\phi_0\sigma^3 - \frac{\lambda}{4!}\sigma^4 + \mathcal{V}(\phi_0) \ ,
\end{aligned} \tag{1.65}$$

where we have used that the last term linear in the σ field vanishes due to Eq. (1.63). The potential density evaluated at its minimum $\mathcal{V}(\phi_0)$ is an irrelevant constant term which will only shift the zero of the energy scale. We omit it in the following considerations. We determine the squared mass of the σ field, $m_\sigma^2 = -m^2 + \frac{\lambda}{2}\phi_0^2 = 2m^2$, which is the curvature of the potential density at its minimum. Moreover, we see that SSB generates a new cubic self-interaction term. Obviously, this term is not invariant under reflections of the fluctuating field $\sigma(x) \to -\sigma(x)$, and expresses the breaking of the initial reflection symmetry in the ground state. Note, the Lagrangian itself is still invariant under the initial reflection symmetry $\phi(x) \to -\phi(x)$, although it is now hidden. The translated reflection transformation of the initial field $\phi(x)$ in terms of the new fluctuation field $\sigma(x)$ reads $\sigma(x) \to -\sigma(x) \mp 2\phi_0$ and leaves the Lagrangian invariant. This shows that the usual saying of a spontaneously "broken" symmetry, actually might be misleading. Anyway, the term of "spontaneous symmetry breaking" has been adopted into the language and is frequently used in practice.

The toy model (1.61) is good as a first encounter of the effect of spontaneous breaking of a discrete symmetry. The situation gets much more interesting when we consider the

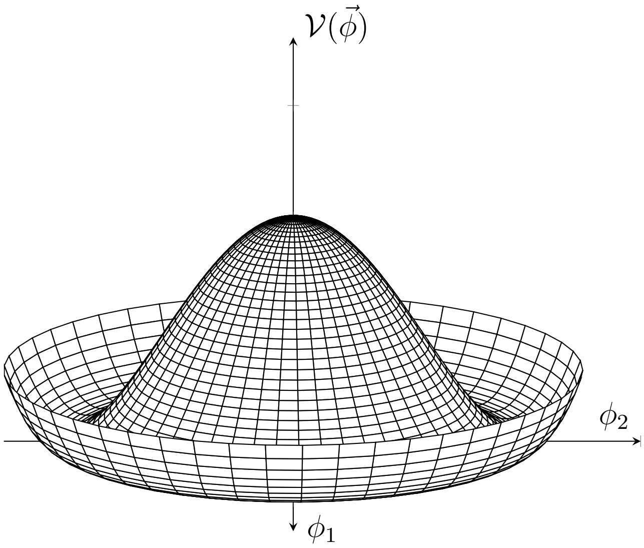


Figure 1.8. The Mexican hat potential of the theory (1.66).

spontaneous breaking of global continuous symmetries. We expand the previous example to include two real scalar fields which we combine into the vector $\vec{\phi}(x) = (\phi_1(x), \phi_2(x))^T$. The two-dimensional generalization of the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - \mathcal{V}(\vec{\phi}) \ , \qquad \mathcal{V}(\vec{\phi}) = -\frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{4!}\vec{\phi}^4 \ . \tag{1.66}$$

It has a *continuous* symmetry under global $O(2)$ rotations in the internal two-dimensional (ϕ_1, ϕ_2) space:

$$\phi_i(x) \to \phi'(x) = O_{ij}\phi_j(x) \ , \qquad (O_{ij}) \in O(2) \ , \tag{1.67}$$

where $i, j = 1, 2$. The matrices (O_{ij}) represent the elements of the $O(2)$ group, i.e., they fulfill the relation $O_{ij}O_{jk} = \delta_{ik}$. With this in mind, the $O(2)$ invariance of the Lagrangian (1.66) can be seen easily. The potential density has the form of a Mexican hat as depicted in Fig. 1.8. The minima of this potential density are determined by the condition

$$\frac{\partial \mathcal{V}}{\partial \phi_i} = \phi_i\big[-m^2 + \frac{\lambda}{6}(\phi_1^2 + \phi_2^2)\big] \stackrel{!}{=} 0 \ , \tag{1.68}$$

which is fulfilled for

$$\phi_{0,i} = 0, \qquad \phi_{0,1}^2 + \phi_{0,2}^2 = |\vec{\phi}_0|^2 \equiv \phi_0^2 = \frac{6m^2}{\lambda} \ . \tag{1.69}$$

As can be found by checking the eigenvalues of the Hessian matrix and investigating the vicinity of the extrema (or simply just by taking Fig. 1.8 into account), we observe that only the second condition corresponds to minima of the potential. The relation defines a

1-sphere, i.e., a circle in the (ϕ_1, ϕ_2) plane with radius ϕ_0. Therefore the potential features an infinite number of minima.
However, the smallest external perturbation will drive the system into one of these minima. Without loss of generality, we choose the coordinate system of the internal (ϕ_1, ϕ_2) space in such a way that the selected minimum is lying at

$$\vec{\phi}_0 = (0, \phi_0)^T \ . \tag{1.70}$$

In analogy to the one-dimensional example, we decompose the field variables by introducing new dynamical variables $\pi(x)$ and $\sigma(x)$, which describe the fluctuations around the ground state in ϕ_1 and ϕ_2 direction, respectively:

$$\vec{\phi}(x) = (\pi(x), \sigma(x) + \phi_0)^T \ . \tag{1.71}$$

In terms of these new field variables the Lagrangian (1.66) reads

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{m^2}{2}\left[\pi^2 + (\sigma + \phi_0)^2\right] - \frac{\lambda}{4!}\left[\pi^2 + (\sigma + \phi_0)^2\right]^2 \\
&= \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{m^2}{2}(\pi^2 + \sigma^2 + 2\phi_0\sigma + \phi_0^2) \\
&\quad - \frac{\lambda}{4!}\left[(\pi^2 + \sigma^2)^2 + 2\phi_0^2(\pi^2 + \sigma^2) + 4\phi_0\sigma(\pi^2 + \sigma^2) + 4\phi_0^2\sigma^2 + 4\phi_0^3\sigma + \phi_0^4\right] \\
&= \frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{2}(-m^2 + \frac{\lambda}{6}\phi_0^2)\pi^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(-m^2 + \frac{\lambda}{2}\phi_0^2)\sigma^2 \\
&\quad - \frac{\lambda}{4!}(\pi^2 + \sigma^2)^2 - \frac{\lambda}{3!}\phi_0\sigma(\pi^2 + \sigma^2) + \phi_0(m^2 - \frac{\lambda}{6}\phi_0^2)\sigma + \mathcal{V}(0, \phi_0) \\
&= \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{\lambda}{4!}(\pi^2 + \sigma^2)^2 - \frac{\lambda}{3!}\phi_0\sigma^3 - \frac{\lambda}{3!}\phi_0\sigma\pi^2 \ ,
\end{aligned} \tag{1.72}$$

where in the last step some terms vanish due to Eq. (1.69) and we dropped the constant term $\mathcal{V}_0(0, \phi_0)$, as it does not contribute to the dynamics of the system. Furthermore, we (again) identified $m_\sigma^2 = -m^2 + \frac{\lambda}{2}\phi_0^2 = 2m^2$.
As already expected, the Lagrangian shows the spontaneous breaking of the $O(2)$ symmetry, but also indicates a new important phenomenon: a massless tangential excitation, the π field, pops up (while the radial excitation, the σ field, remains massive). This can be understood as follows: excitations in the radial field direction ϕ_2 correspond to fluctuations that have to “climb the wall”, while the tangential field direction ϕ_1 corresponds to fluctuations in angular direction and therefore to a “rolling in the rim of the hat”, which costs no energy, see Fig. 1.8.

In general, the appearance of massless particles is an exact result of spontaneously broken global continuous symmetries. The phenomenon is described by Goldstone’s theorem [39], which predicts the occurrence of massless spin-0 bosons with negative parity for each spontaneously broken generator of the global continuous symmetry group. These massless particles are called (Nambu-)Goldstone bosons.

1.3. Quantum chromodynamics

The present knowledge of nature is based on four fundamental interactions. Two of them, gravitation and electromagnetism affect our macroscopic everyday experiences. However, the further we go to microscopic scales, of the order of the nucleon radius or smaller, the so-called weak and strong interactions become more and more important. The strong interaction ensures that elementary quarks are confined inside hadrons. Mathematically, this force is described by the QFT called quantum chromodynamics (QCD), which describes the interaction between quarks (as an exchange of gluons).

1.3.1. The QCD Lagrangian as a non-abelian color gauge theory

We develop a Lagrangian of the strong interaction. Starting with the construction of the quark sector, we regard that quarks come in six different flavors and carry spin 1/2. Thus, the Lagrangian describing the kinematics of free quarks is given by

$$\mathcal{L}_{\text{quarks}} = \sum_{f=u,d,\dots,b} \bar{q}_f(x)\big(i\gamma^\mu\partial_\mu - m_f\big)q_f(x) = \bar{q}(x)\big(i\gamma^\mu\partial_\mu - m\big)q(x) \ , \tag{1.73}$$

where the quarks $q_f(x)$ and anti-quarks $\bar{q}_f(x)$ are represented by Dirac spinors and Dirac-adjoint spinors respectively, i.e., they are four-component vectors in spinor space. For the sake of a better clarity, in the last equality, we introduced the quark field $q(x) = (q_u(x), q_d(x), \dots, q_b(x))^T$ and the mass matrix m as a diagonal matrix with m_f as diagonal elements, $m = \text{diag}(m_u, m_d, \dots, m_b)$.

Additionally, we know that each quark can carry three different colors. Consequently, a quark spinor $q_f(x)$ is also a three-component vector in color space:

$$q_f(x) = \begin{pmatrix} q_{f,r}(x) \\ q_{f,g}(x) \\ q_{f,b}(x) \end{pmatrix} \ . \tag{1.74}$$

Mathematically, this triplet is understood as the fundamental representation of the special unitary $SU(3)_c$ color group, i.e., $q_f(x)$ behaves as a vector under $SU(3)_c$ transformations:

$$q_f(x) \ \to \ q'_f(x) = U_c q_f(x) = e^{-i\theta^a T^a} q_f(x) \ , \tag{1.75}$$

$$\bar{q}_f(x) \ \to \ \bar{q}'_f(x) = \bar{q}_f(x) U_c^\dagger = \bar{q}_f(x) e^{i\theta^a T^a} \quad , \qquad \forall \ a \in \{1, 2, \dots, 8\} \ , \tag{1.76}$$

where

$$U_c = \exp(-i\theta^a T^a) \in SU(3)_c \tag{1.77}$$

with the group parameter θ^a. The eight 3×3 matrices T^a are the generators of $SU(3)$ and fulfill the Lie algebra $[T^a, T^b] = i f^{abc} T^c$ with the totally antisymmetric structure constants f^{abc}. The generators are defined as

$$T^a = \lambda^a / 2 \ , \tag{1.78}$$

where λ^a denote the usual Gell-Mann matrices. Their properties follow from the properties of the matrices $U_c \in SU(3)_c$. Namely, the unitarity of the matrices U_c ensures that the generators are hermitian and the condition $\det U_c = 1$ implies that they are traceless, $0 = \ln\det U_c = \operatorname{Tr}\ln U_c = -i\theta^a \operatorname{Tr} T^a$, where we have used that $\ln\det A = \operatorname{Tr}\ln A \;\forall A$ being a symmetric positive definite matrix. Furthermore, the generators are chosen such that they are orthogonal in the sense $\operatorname{Tr}(T^a T^b) = \delta^{ab}/2$.
In simple terms, this transformation (1.75) represents a rotation of the complex 3-component vector $q_f(x)$ in the complex color space. Since the transformation matrices depend neither on space nor on time variables, the transformation is called global.

The Lagrangian (1.73) is invariant under such global rotations in color space,

$$\begin{aligned}\mathcal{L}_{\text{quarks},f} \;\rightarrow\; \mathcal{L}'_{\text{quarks},f} &= \bar{q}'_f(x)(i\gamma^\mu\partial_\mu - m_f)q'_f(x)\\ &= \bar{q}_f(x)U_c^\dagger(i\gamma^\mu\partial_\mu - m_f)U_c q_f(x)\\ &= \bar{q}_f(x)(i\gamma^\mu\partial_\mu - m_f)U_c^\dagger U_c q_f(x)\\ &= \bar{q}_f(x)(i\gamma^\mu\partial_\mu - m_f)q_f(x)\\ &= \mathcal{L}_{\text{quarks},f}\;,\end{aligned} \tag{1.79}$$

where the Dirac matrices commute with the matrices $U_c \in SU(3)_c$, because they act in different spaces. Furthermore, we have used the hermeticity of the transformation matrices to show the invariance.

What happens if we consider a local $SU(3)_c$ color transformation? We modify the color transformation by making the group elements spacetime dependent, i.e., $U_c \rightarrow U_c(x)$. This entails that the parameters of the group become spacetime dependent,

$$\theta^a \rightarrow \theta^a(x)\;. \tag{1.80}$$

That is, we can pick different group parameters at different spacetime points. Then, the local color transformation of the quark field is given by

$$q_f(x) \;\rightarrow\; q'_f(x) = U_c(x)q_f(x) = e^{-i\theta^a(x)T^a}q_f(x)\;, \tag{1.81}$$

$$\bar{q}_f(x) \;\rightarrow\; \bar{q}'_f(x) = \bar{q}_f(x)U_c^\dagger(x) = \bar{q}_f(x)e^{i\theta^a(x)T^a}\;. \tag{1.82}$$

Under this local color transformation, however, the Lagrangian (1.73) is not invariant:

$$\begin{aligned}\mathcal{L}_{\text{quarks},f} \;\rightarrow\; \mathcal{L}'_{\text{quarks},f} &= \bar{q}_f(x)e^{i\theta^a(x)T^a}(i\gamma^\mu\partial_\mu - m_f)e^{-i\theta^a(x)T^a}q_f(x) =\\ &= \bar{q}_f(x)e^{i\theta^a(x)T^a}\left[i\gamma^\mu(\partial_\mu e^{-i\theta^a(x)T^a}) + e^{-i\theta^a(x)T^a}i\gamma^\mu\partial_\mu - m_f\right]q_f(x) =\\ &= \bar{q}_f(x)(i\gamma^\mu\partial_\mu - m_f)q_f(x) + \bar{q}_f(x)\gamma^\mu[\partial_\mu\theta^a(x)]T^a q_f(x) =\\ &= \mathcal{L}_{\text{quarks},f} + \Delta\mathcal{L}_{\text{quarks},f}\;,\end{aligned} \tag{1.83}$$

where the derivative in the Dirac operator, acting not only on the quark field but also on the transformation matrix, causes an additional term

$$\Delta\mathcal{L}_{\text{quarks},f} = \bar{q}_f(x)\gamma^\mu[\partial_\mu\theta^a(x)]T^a q_f(x) \tag{1.84}$$

We adjust the Lagrangian (1.73) such that it remains invariant under the local transformation (1.81). To this end, we introduce a so-called gauge field, denoted as

$$\mathcal{A}^\mu(x) \equiv A^{\mu,a}(x) T^a \ . \tag{1.85}$$

(As we will see, the eight $A^{\mu,a}$ fields $\forall\ a = 1, 2, \ldots 8$ can be associated to gluons.)
We incorporate the gauge field (1.85) into the Lagrangian (1.73) by so-called minimal coupling, which requires to replace the derivative by a covariant derivative which is defined as

$$D_\mu(x) = \partial_\mu - ig\mathcal{A}_\mu(x) \ , \tag{1.86}$$

where g is a coupling constant. The name "covariant" does not refer to the index position, but to the transformation behavior. Namely, the covariant derivative of a field should transform as the field itself,

$$D_\mu(x) q_f(x) \ \rightarrow \ D'_\mu(x) q'_f(x) = U_c(x) D_\mu(x) q_f(x) \ , \tag{1.87}$$

which is tantamount to

$$D_\mu(x) \ \rightarrow \ D'_\mu(x) = U_c(x) D_\mu(x) U_c^\dagger(x) \ . \tag{1.88}$$

Hence, the minimally coupled Lagrangian,

$$\mathcal{L}_{\text{quarks+gluon}} = \bar{q}(x) \big[i\gamma^\mu D_\mu(x) - m \big] q(x) \ , \tag{1.89}$$

is invariant under the local $SU(3)_c$ transformations (1.81):

$$\begin{aligned} \mathcal{L}_{\text{quarks+gluon},f} \ \rightarrow \ \mathcal{L}'_{\text{quarks+gluon},f} &= \bar{q}'_f(x) \big[i\gamma^\mu D'_\mu(x) - m_f \big] q'_f(x) \\ &= \bar{q}_f(x) U_c^\dagger(x) \big[i\gamma^\mu U_c(x) D_\mu(x) U_c^\dagger(x) - m_f \big] U_c(x) q_f(x) \\ &= \bar{q}_f(x) \big[i\gamma^\mu D_\mu(x) - m_f \big] q_f(x) \\ &= \mathcal{L}_{\text{quarks+gluon},f} \ . \end{aligned} \tag{1.90}$$

As the introduction of $\mathcal{A}^\mu(x)$ follows from the claim of locality, this gauge field can be understood as the "transmitter" of the strong interaction: the so-called gluon.

Note, ensuring Eq. (1.88) to be true, the gauge field $\mathcal{A}_\mu(x)$ has to transform under local $SU(3)_c$ transformations as

$$\mathcal{A}_\mu(x) \ \rightarrow \ \mathcal{A}'_\mu(x) = U_c(x) \mathcal{A}_\mu(x) U_c^\dagger(x) - \frac{i}{g} \big[\partial_\mu U_c(x) \big] U_c^\dagger(x) \ , \tag{1.91}$$

because then

$$\begin{aligned} D_\mu(x) \ \rightarrow \ D'_\mu(x) &= \partial_\mu - ig \Big\{ U_c(x) \mathcal{A}_\mu(x) U_c^\dagger(x) - \frac{i}{g} \big[\partial_\mu U_c(x) \big] U_c^\dagger(x) \Big\} \\ &= U_c(x) U_c^\dagger(x) \partial_\mu - ig U_c(x) \mathcal{A}_\mu(x) U_c^\dagger(x) \\ &\quad - \partial_\mu \big[U_c(x) U_c^\dagger(x) \big] + U_c(x) \big[\partial_\mu U_c^\dagger(x) \big] \\ &= U_c(x) \big(\partial_\mu - ig\mathcal{A}_\mu(x) \big) U_c^\dagger(x) \\ &= U_c(x) D_\mu(x) U_c^\dagger(x) \ , \end{aligned} \tag{1.92}$$

where we have applied the product rule for derivatives several times and $\partial_\mu[U_c(x)U_c^\dagger(x)] = \partial_\mu 1 = 0$.

We take a closer look at the Lagrangian (1.89):

$$\begin{aligned}\mathcal{L}_{\text{quarks+gluon}} &= \bar{q}\big(i\gamma^\mu D_\mu - m\big)q = \\ &= \bar{q}\big(i\gamma^\mu \partial_\mu - m\big)q + g\ \bar{q}\gamma^\mu \mathcal{A}_\mu q\ .\end{aligned} \tag{1.93}$$

The additional term introduced via minimal coupling corresponds to a coupling of the quark fields to an external gauge field,

$$g\ \bar{q}\gamma^\mu \mathcal{A}_\mu q\ . \tag{1.94}$$

This interaction vertex is depicted in Fig. 1.9a. The upper diagram represents the flavor flow and the lower diagram the color flow of quark-gluon interaction.
In the latter diagram the so-called double-line notation for the gluon is used, see Refs. [40, 41]: Considering the color structure of the gluon, we find that the "gluon field $A^i_{\mu j}$ has one upper [color] index i like the quark field q^i and one lower [color] index j like the antiquark field $\bar{q}_j$"[40]. Thus, for the purpose of tracing the color quantum numbers, we may understand the gluon as a quark-antiquark or color-anticolor object. In turn, this suggests to represent the gluon in a double-line notation as depicted in Fig. 1.10b.
Furthermore the flavor- and color-flow representation in Fig. 1.9a shows that QCD allows the gluon to change the color of the quark but not its flavor.

The Lagrangian (1.89) features the gluon fields only as external fields. As it is common in gauge theory, we introduce a kinetic term for the gluon fields as the square of the Yang-Mills field-strength tensor. In general, this field-strength tensor is defined as the commutator of two covariant derivatives. For QCD, it is given by

$$\begin{aligned}\mathcal{G}_{\mu\nu} &= \frac{i}{g}\big[D_\mu, D_\nu\big] = \frac{i}{g}\big[(\partial_\mu - ig\mathcal{A}_\mu)(\partial_\nu - ig\mathcal{A}_\nu) - (\partial_\nu - ig\mathcal{A}_\nu)(\partial_\mu - ig\mathcal{A}_\mu)\big] \\ &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big] \\ &= \partial_\mu A^a_\nu T^a - \partial_\nu A^a_\mu T^a - igA^b_\mu A^c_\nu\big[T^b, T^c\big] \\ &= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - igf^a{}_{bc}A^b_\mu A^c_\nu)T^a \equiv G^a_{\mu\nu}T^a\ ,\end{aligned} \tag{1.95}$$

where we have exploited the Lie algebra $[T^b, T^c] = if^{bca}T^a$ of the generators and used the total antisymmetry of the $SU(3)$ structure constants, $f^{bca} = -f^{bac} = f^{abc}$. This allows to express the tensor in terms of the field-strength tensor components $G^a_{\mu\nu}(x)$. Considering the color transformation of the covariant derivative (1.88), it is clear that the field-strength tensor of QCD transforms as

$$\begin{aligned}\mathcal{G}_{\mu\nu}(x) \to \mathcal{G}'_{\mu\nu}(x) &= \frac{i}{g}\big[D'_\mu(x), D'_\nu(x)\big] = \frac{i}{g}U_c(x)\big[D_\mu(x), D_\nu(x)\big]U_c^\dagger(x) \\ &= U_c(x)\mathcal{G}_{\mu\nu}(x)U_c^\dagger(x)\ .\end{aligned} \tag{1.96}$$

Taking into account the cyclic invariance of traces, a gauge [i.e., local $SU(3)_c$] invariant Lagrangian of gluon fields is given by

$$\mathcal{L}_{\text{gluons}} = -\frac{1}{2}\,\text{Tr}[\mathcal{G}_{\mu\nu}(x)\mathcal{G}^{\mu\nu}(x)] \;. \tag{1.97}$$

This Lagrangian does not only contain a kinetic part for the gluons, but also includes gluon self-interaction terms. To see this explicitly, we rewrite the Lagrangian (1.97) by using the component representation of the field-strength tensor:

$$\begin{aligned}
-\frac{1}{2}\,\text{Tr}(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}) &= -\frac{1}{2}G^a_{\mu\nu}G^{\mu\nu}_b\,\text{Tr}(T_aT^b) = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \\
&= -\frac{1}{2}\partial_\mu A^a_\nu(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) - \frac{g}{4}\big[f_a{}^{bc}(\partial_\mu A^a_\nu A^\mu_b A^\nu_c - \partial_\nu A^a_\mu A^\mu_b A^\nu_c) \\
&\quad + f^a{}_{bc}(A^b_\mu A^c_\nu \partial^\mu A^\nu_a - A^b_\mu A^c_\nu \partial^\nu A^\mu_a)\big] - \frac{g^2}{4}f^a{}_{bc}f_a{}^{de}A^b_\mu A^c_\nu A^\mu_d A^\nu_e \\
&= -\frac{1}{2}\partial_\mu A^a_\nu(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a) - g f_a{}^{bc}(\partial_\mu A^a_\nu)A^\mu_b A^\nu_c - \frac{g^2}{4}f^a{}_{bc}f_a{}^{de}A^b_\mu A^c_\nu A^\mu_d A^\nu_e \;.
\end{aligned} \tag{1.98}$$

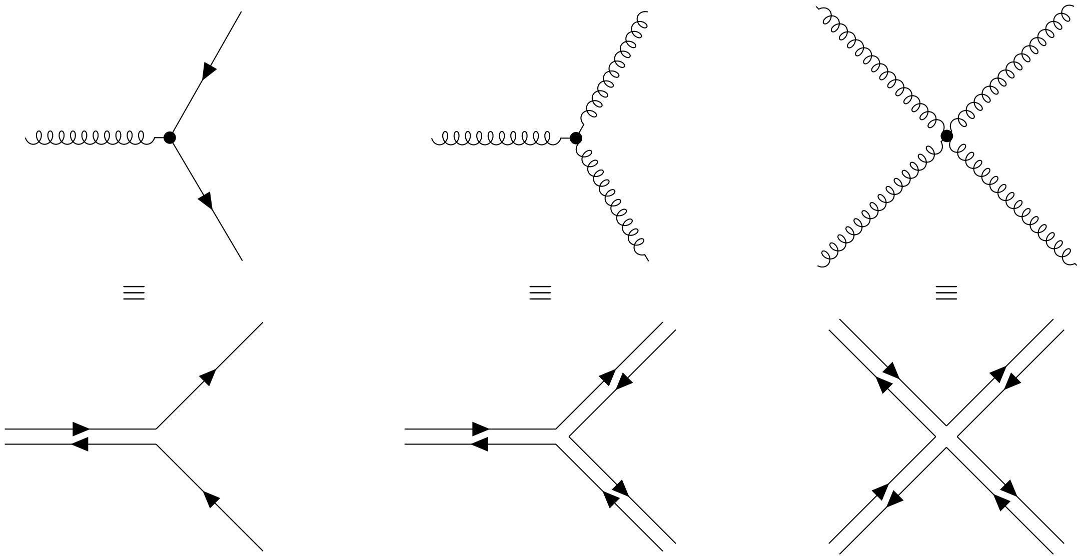
The quadratic term describes the free propagation of the gluon, and the cubic and quartic terms represent three- and four-gluon self-interaction[8] vertices, as depicted in Figs. 1.9b and 1.9c.

Finally, we combine Eq. (1.89) and (1.97) to obtain the QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = \bar{q}\big(i\gamma^\mu D_\mu - m\big)q - \frac{1}{2}\,\text{Tr}(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}) \;. \tag{1.99}$$

The quantisation of this Lagrangian posses some difficulties, which arise from the additional degree of freedom of gauge transformations. Since the Lagrangian is gauge invariant, two in general different fields A^a_μ and $A'^a_\mu = A^a_\mu + f^a{}_{bc}A^b_\mu\theta^c + \partial_\mu\theta^a/g$ [Eq. (1.91) up to first order in the infinitesimal parameter θ^a] are gauge equivalent, in the sense that they describe the same physics. This causes troubles, because due to Eq. (1.30), in QFT all possible field configurations contribute to the physical observables, i.e., we integrate over A^a_μ as well as over A'^a_μ. Thus, we sum up redundant information. In order to eliminate this additional degree of freedom, we have to include a term which puts a constraint on the gauge fields. This gauge-fixing term may be obtained by the method of Faddeev and Popov, see App. B, which is accompanied by the introduction of a new artificial field. The related particle excitations are called (Faddeev-Popov-)ghosts. Their "spooky" manifestation arises from the fact that they are described by anticommuting Grassmannian fields, but transform as scalar particles under Lorentz transformations. Depending on the chosen gauge, these ghosts may couple to gluons. In this case the resulting gluon-ghost vertex has to be taken into account in calculations. The quantum field theoretical QCD Lagrangian, including ghost fields, is given in Eq. (B.1) in the Appendix.

[8]These gluon self-interactions are the main reason for a phenomenon called asymptotic freedom, which states that the interaction of quarks and gluons at high energies or small distances is small.



(a) Quark-gluon interaction (b) Three-gluon self interaction (c) Four-gluon self interaction

Figure 1.9. QCD interaction vertices of quarks and gluons in the usual “Feynman” notation (above) and in double-line notation for gluons (below), resulting from Eq. (1.99).

(a) The usual “Feynman” notation for gluons. (b) The double-line notation for gluons.

Figure 1.10. Free propagation of the gluon described by the first term in Eq. (1.98) which corresponds to the inverse gluon propagator.

1.3.2. Lorentz invariance of the QCD Lagrangian

As all quantum field theories, QCD combines quantum mechanics with special relativity. Space and time are placed on an equal footing and the equations of motion have to be relativistically covariant. This means, if we have an equation of motion which is satisfied by a field $\phi(x)$ and perform a rotation or boost to a different frame of reference, then the transformed field, in the new reference frame, satisfies the same equation. Equivalently, the equation of motion remains true after adopting an "active" transformation where we rotate the fields.

Proving Lorentz invariance is especially easy in the Lagrangian formalism. As a consequence of Hamilton's principle of stationary action, an equation of motion is automatically Lorentz invariant, if it follows from a Lagrangian that is a Lorentz scalar. This is clear, because if the Lagrangian does not change under a boost or rotation, also the extremum in the action remains.

A Lorentz transformation is given by

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \;, \tag{1.100}$$

where Λ is a matrix fulfilling the constraint

$$\Lambda_\alpha{}^\mu g_{\mu\nu} \Lambda^\nu{}_\beta = g_{\alpha\beta} \;. \tag{1.101}$$

The corresponding transformation of a scalar field $\phi'(x') = \phi(x)$ is given by

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \;. \tag{1.102}$$

As stated, each Lagrangian should be a scalar object (eventual Lorentz indices are contracted): $\mathcal{L}'(x') = \mathcal{L}(x)$. Hence it should transform as

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x) \;. \tag{1.103}$$

In order to prove this condition for the QCD Lagrangian (1.99), we still need to give the transformations of the included fields, see e.g. Ref. [34]. Considering the transformation of a vector field, such as the gauge field $\mathcal{A}^\mu(x)$, we have to take into account that this quantity carries an orientation in spacetime, which must be rotated or boosted, when performing a transformation:

$$\mathcal{A}^\mu(x) \to \Lambda^\mu{}_\nu \, \mathcal{A}^\nu(\Lambda^{-1}x) \;. \tag{1.104}$$

The quark and the antiquark fields are represented by Dirac spinors, which transform under Lorentz transformations as

$$q(x) \to q'(x) = S(\Lambda) q(\Lambda^{-1}x) \;, \tag{1.105}$$
$$\bar{q}(x) \to \bar{q}'(x) = \bar{q}(\Lambda^{-1}x) S^{-1}(\Lambda) \;, \tag{1.106}$$

where $S(\Lambda)$ is the spinor representation of the Lorentz transformation matrix Λ:

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}} \quad \text{with} \quad \sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \ , \tag{1.107}$$

and $\omega_{\mu\nu}$ is an antisymmetric tensor, containing the group parameters.

We prove that QCD is invariant under the Lorentz transformations (1.104) to (1.106), i.e., that the Lagrangian (1.99) satisfies Eq. (1.103). The Dirac part of Lagrangian describing the kinematics of quarks transforms as

$$\begin{aligned} \bar{q}(x)(i\gamma^\mu \partial_\mu - m)q(x) \quad \to \quad & \bar{q}(\Lambda^{-1}x)S^{-1}(\Lambda)i\gamma^\mu(\Lambda^{-1})^\nu{}_\mu \partial_\nu S(\Lambda)q(\Lambda^{-1}x) \\ & - m\bar{q}(\Lambda^{-1}x)S^{-1}(\Lambda)S(\Lambda)q(\Lambda^{-1}x) \\ & = \bar{q}(\Lambda^{-1}x)(i\gamma^\mu\partial_\mu - m)q(\Lambda^{-1}x) \ , \end{aligned} \tag{1.108}$$

where we have used that the partial derivative transforms as

$$\partial_\mu = (\Lambda^{-1})^\nu{}_\mu \partial_\nu \ .$$

Furthermore, the property

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu \ ,$$

holds for all Dirac matrices, i.e., they are invariant under the simultaneous rotation of their vector and spinor indices. (Thus, we can take the vector index μ on γ^μ seriously and contracted with another vector index, as the index μ of the partial derivative ∂_μ, we obtain a scalar object.)

The transformation of the quark-gluon interaction term reads

$$\begin{aligned} -g\bar{q}(x)\gamma^\mu \mathcal{A}_\mu(x)q(x) \quad \to \quad & - g\bar{q}(\Lambda^{-1}x)S^{-1}(x)\gamma^\mu(\Lambda^{-1})^\nu{}_\mu \mathcal{A}_\nu(x)S(x)q(\Lambda^{-1}x) \\ & = -g\bar{q}(\Lambda^{-1}x)\gamma^\mu\mathcal{A}_\mu(x)q(\Lambda^{-1}x) \ . \end{aligned} \tag{1.109}$$

Similarly one observes, that the Yang-Mills term (1.97) transforms also as a scalar under Lorentz transformations. This can be seen in Eq. (1.98), where all free Lorentz indices are contracted.

Thus, we have shown that the QCD Lagrangian (1.99) transforms as a scalar,

$$\mathcal{L}_{\text{QCD}}(x) \to \mathcal{L}_{\text{QCD}}(\Lambda^{-1}x) \ . \tag{1.110}$$

That is QCD is indeed a Lorentz invariant theory.

1.3.3. Parity and charge-conjugation symmetry

Besides the already discussed continuous symmetry transformations, QCD possesses also discrete symmetries. There are two spacetime operations: space reflections, which are denoted as parity transformations P, which send $(t, \boldsymbol{r}) \to (t, -\boldsymbol{r})$, and time reversal transformations T, which send $(t, \boldsymbol{r}) \to (-t, \boldsymbol{r})$, i.e., they interchange the forward and backward light cones. A third discrete operation is the charge conjugation C, which interchanges particles and antiparticles.

The parity transformation of Dirac spinors, and thus also the quark and antiquark fields, is given by [34]

$$q(x) \xrightarrow{P} q'(x') = \gamma^0 q(t, -\boldsymbol{r}) \; , \tag{1.111}$$
$$\bar{q}(x) \xrightarrow{P} \bar{q}'(x') = \bar{q}(t, -\boldsymbol{r})\gamma^0 \; , \tag{1.112}$$

while their charge-conjugation transformation reads [34]

$$q(x) \xrightarrow{C} q'(x') = C\bar{q}^T(x) \; , \tag{1.113}$$
$$\bar{q}(x) \xrightarrow{C} \bar{q}'(x') = q^T(x)C \; . \tag{1.114}$$

Here the transposition acts in spinor space and C is the charge-conjugation operator which is determined such that it fulfills the relations

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \qquad \text{and} \qquad C^{-1} = C^\dagger = C^T = -C \; . \tag{1.115}$$

In the Dirac representation this operator is defined as $C = i\gamma^2\gamma^0$. The charge-conjugated quark corresponds then to the antiquark, and vice versa.

The parity transformation of the gauge field is given by

$$\mathcal{A}^\mu(x) \xrightarrow{P} \mathcal{A}^{\mu\prime}(x') = \mathcal{A}_\mu(t, -\boldsymbol{r}) \; , \tag{1.116}$$

and its charge conjugation reads

$$\mathcal{A}^\mu(x) \xrightarrow{C} \mathcal{A}^{\mu\prime}(x') = -\mathcal{A}^{\mu T}(x) \; . \tag{1.117}$$

We prove that the QCD Lagrangian (1.99)is invariant under these parity and charge-conjugation transformations.

The QCD gauge part (1.97) is clearly invariant under parity and charge-conjugation transformations using Eqs. (1.117) and (1.116).
The parity invariance of the Dirac part can be seen as follows:

$$\begin{aligned}\bar{q}(x)(i\gamma^\mu D_\mu - m)q(x) \quad \xrightarrow{P} \quad & \bar{q}(t, -\boldsymbol{r})\gamma^0 \left[i\left(\gamma^0 D_0 - \gamma^i D_i\right) - m\right]\gamma^0 q(t, -\boldsymbol{r}) \\ & = \bar{q}(t, -\boldsymbol{r})\left(i\gamma^\mu D_\mu - m\right)q(t, -\boldsymbol{r}) \; ,\end{aligned} \tag{1.118}$$

where we have used that

$$D_\mu = \partial_\mu - ig\mathcal{A}_\mu \xrightarrow{P} \partial_0 - ig\mathcal{A}_0 - \partial_i + ig\mathcal{A}_\mu = D_0 - D_i \ . \tag{1.119}$$

The charge-conjugation invariance of the QCD Dirac part is checked by calculating

$$\begin{aligned} \bar{q}(i\gamma^\mu D_\mu - m)q &= \bar{q}\left[i\gamma^\mu(\partial_\mu - ig\mathcal{A}_\mu) - m\right] q \\ &\xrightarrow{C} \quad q^T C \left[i\gamma^\mu(\partial_\mu + ig\mathcal{A}_\mu^T) - m\right] C\bar{q}^T \\ &= q^T \left[i\gamma^{\mu T}(\partial_\mu + ig\mathcal{A}_\mu^T) + m\right] \bar{q}^T \\ &= q^T i\gamma^{\mu T}\partial_\mu \bar{q}^T + q^T i\gamma^{\mu T} ig\mathcal{A}_\mu^T \bar{q}^T + q^T m\bar{q}^T \\ &= -(\partial_\mu \bar{q}) i\gamma^\mu q - \bar{q} i\gamma^\mu ig\mathcal{A}_\mu q - \bar{q}mq \\ &= \bar{q} i\gamma^\mu \partial_\mu q - \bar{q} i\gamma^\mu ig\mathcal{A}_\mu q - \bar{q}mq \\ &= \bar{q} i\gamma^\mu D_\mu q - \bar{q}mq \end{aligned} \tag{1.120}$$

where from the first to the second line, we have used Eq. (1.115) such that

$$C\gamma^\mu = -C^{-1}\gamma^\mu C C^{-1} = \gamma^{\mu T} C^{-1} \ . \tag{1.121}$$

The step from the third to the fourth line gets clear by explicitly writing out all indices, e.g.,

$$q_{i,\alpha}(i\gamma^{\mu T})_{\alpha\beta} ig(\mathcal{A}_\mu^T)_{ij}\bar{q}_{j,\beta} = -\bar{q}_{j,\beta} i\gamma^\mu_{\beta\alpha} ig\mathcal{A}_{\mu,ji} q_{i,\alpha} \ , \tag{1.122}$$

where α, β are Dirac indices and i, j run from 1 to 3 such that $\mathcal{A}_{\mu,ji}$ is the ji-th entry in $\mathcal{A}_\mu$ in the fundamental representation of $SU(3)$. The minus sign appears because quarks are described by anticommuting Grassmannian fields, i.e.,

$$q^T \bar{q}^T = -\bar{q}q \ . \tag{1.123}$$

Finally in the penultimate line in Eq. (1.120), an integration by parts took place.

Thus, the QCD Lagrangian obeys a symmetry under parity as well as charge-conjugation transformations, q.e.d.

Note, due to the so-called CPT theorem (according to which any Lorentz-invariant local QFT described by a hermitian Hamiltonian obeys a combined CPT symmetry), the QCD Lagrangian (1.99) is even invariant under the combined CPT transformation.
Consequently, due to its separately valid parity and charge-conjugation invariance the QCD Lagrangian is also invariant under time reversal.

1.3.4. Chiral symmetry

The QCD Lagrangian is (approximately) invariant under global $U(3)_L \times U(3)_R$ chiral transformations. This symmetry is only exact in the limit of massless quarks, as shown in the following discussion. However, the lightest quark masses are small compared to the typical hadronic mass of around 1 GeV, see Tab. 1.1. Indeed, the study of the chiral symmetry turns out to be especially important in the low-energy effective approaches where only color-confined particle states can be observed (i.e., the basic gauge symmetry under color transformations is fulfilled by construction and thus cannot be used to restrict the number of possible terms in the Lagrangian).

The British mathematical physicist and engineer Lord Kelvin was the first who used the term "chirality" in a lecture about molecular tactics of a crystal in 1893 [42]. He defines an object as chiral if "its image in a plane mirror, ideally realized, cannot be brought to coincide with itself". The most common example of an object with chirality are human hands. Although the left hand is the mirror image of the right hand, they are fundamentally different from each other.[9].

In physics, the definition of chirality is more abstract. In fact, in a physical sense the definition of Lord Kelvin refers to something we call handedness and a physical quantity which is more similar to the given explanation is the notion of helicity. However, it is closely related to (the physical definition of) chirality and thus it is better to study helicity first.

Every fermionic particle carries the intrinsic quantum number of spin, which is the quantum description of its eigenrotation. If we assume that the particle is moving in a certain direction, the particle may spin clockwise or anticlockwise around that direction of motion. The comparison of these orientations define a handedness: a particle is called left-(right-)handed if the orientations are of a left(right) hand where the thumb points in the direction of motion and the fingers wrap in spinning direction. Physically, we call this handedness the helicity of a particle. It is defined through the eigenvalues $\pm\frac{1}{2}$ of the so-called helicity operator, which projects the spin $\boldsymbol{S}$ onto the momentum direction,

$$h = \frac{\boldsymbol{S}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} \; . \tag{1.124}$$

According to Lord Kelvin's definition, the term handedness indicates that the mirrored particle does not coincide with itself. In fact, in the context of helicity, if we mirror a left-handed particle it becomes right-handed and vice versa, because the direction of motion is reversed while the orientation of the spin remains unchanged.
Interestingly, besides mirroring, there exists another simple possibility to flip this handedness of a particle - at least in the case of massive particles. If a particle travels with less than the speed of light (hence it has to be massive), then it is possible to move faster than the particle. The transition to this moving reference frame reverses the direction of motion of the particle (while the spin remains the same) and therefore changes its handedness. Consequently, it depends on the reference frame whether a massive particle

[9] That is immediately getting clear if we try to put a left-handed glove on a right hand (where the palm and back sides of the gloves have to be distinguishable).

is left- or right-handed. On the other hand, since we can never catch up massless particles that are traveling with the speed of light, in the massless limit the definition of helicity is “intrinsic”, i.e., it has a fixed value in all reference frames.

A related concept is chirality. While for massless particles, it is the same as helicity (up to a factor of 1/2), the important difference is that the observers in several valid reference frames measure the same chirality for all particles (massless or not). Chirality is Lorentz-invariant.

The definition of chirality is mathematically abstract. Just as we say that a particle can have left- or right-handed helicity, we also say that a particle can have left- or right-handed chirality.

The limit of massless particles is a good starting point: we rewrite the spin operator[10] for Dirac particles [43],

$$\boldsymbol{S} = \frac{\boldsymbol{\Sigma}}{2} \; , \tag{1.125}$$

as follows

$$\begin{aligned}\Sigma^i \equiv \frac{1}{2}\epsilon^i{}_{jk}\sigma^{jk} &= \frac{1}{2}\epsilon^i{}_{jk}\frac{i}{2}[\gamma^j,\gamma^k] = \frac{i}{4}\epsilon^i{}_{jk}(2\gamma^j\gamma^k - \{\gamma^j,\gamma^k\}) \\ &= \frac{i}{2}\epsilon^i{}_{jk}(\gamma^j\gamma^k - g^{jk}) = \frac{i}{2}\epsilon^i{}_{jk}\gamma^j\gamma^k \\ &= -i\gamma^1\gamma^2\gamma^3\gamma^i = -\gamma^0\gamma_5\gamma^i \\ &= \gamma_5\gamma^0\gamma^i \; ,\end{aligned} \tag{1.126}$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu]$. In the first line, we have used $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ in order to rewrite the commutator as $[\gamma^j,\gamma^k] = 2\gamma^\mu\gamma^\nu - \{\gamma^\mu,\gamma^\nu\} = 2(\gamma^\mu\gamma^\nu - g^{\mu\nu})$. The fourth line is obtained considering certain choices of i, e.g. for $i=1$ we have $\epsilon^1_{jk}\gamma^j\gamma^k = \gamma^2\gamma^3 - \gamma^3\gamma^2 = 2\gamma^2\gamma^3 = -2\gamma^1\gamma^1\gamma^2\gamma^3 = -2\gamma^1\gamma^2\gamma^3\gamma^1$, where we have used that $\gamma^1\gamma^1 = g^{11} = -1$ (as it follows from the commutator of two Dirac matrices).

With $\boldsymbol{S} = \boldsymbol{\Sigma}/2$ and Eq. (1.126) we can express the helicity operator (1.124) as

$$h = \frac{\boldsymbol{S}\cdot\boldsymbol{p}}{|\boldsymbol{p}|} = \frac{1}{2|\boldsymbol{p}|}\gamma_5\gamma^0\gamma^i p_i = \frac{1}{2|\boldsymbol{p}|}\gamma_5\gamma^0(-\gamma^\mu p_\mu + \gamma^0 p_0) \; . \tag{1.127}$$

[10] Actually, this is the spin operator for *massive* Dirac particles. An explicit derivation can be found in Ref. [43]. Therefore the following presentation to introduce the chirality operator looks simple but cheats a little bit. However, it yields the desired result of giving a feeling of what chirality is.

We calculate its action onto a solution of the Dirac equation for massless fermions:

$$
\begin{aligned}
h\ \psi_{m=0}(x) &= \frac{1}{2|\boldsymbol{p}|}\gamma_5\gamma^0(-\gamma^\mu p_\mu + \gamma^0 p_0)\ \psi_{m=0}(x) \\
&= \frac{1}{2|\boldsymbol{p}|}\gamma_5\gamma^0\gamma^0 p_0\ \psi_{m=0}(x) = \frac{1}{2p_0}\gamma_5 p_0\ \psi_{m=0}(x) \\
&= \frac{1}{2}\gamma_5\ \psi_{m=0}(x)\ ,
\end{aligned}
\tag{1.128}
$$

where we have used the momentum-space Dirac equation for massless fermions, $\gamma^\mu p_\mu \psi_{m=0}(x) = 0$, and that $|\boldsymbol{p}| = p^0$ for massless particles. The result shows that for massless fermions the action of the helicity operator is identical to the action of γ_5 (up to a factor of $\frac{1}{2}$). Since we already know that helicity and chirality in the limit of massless particles should be the same, we define γ_5 to be the so-called chirality operator. Now, in the same way as we use the eigenvalues $-\frac{1}{2}$ and $+\frac{1}{2}$ of the helicity operator to define the left- and right-handedness of a particle, we call a fermion either left- or right-handed in the sense of chirality, according to whether it is an eigenstate of γ_5 with eigenvalue -1 or $+1$, respectively.

With this definition of chirality, we can generalize the discussion to fermions with arbitrary mass. We use so-called chiral projection operators in order to project out the spinor components with left- and right-handed chirality. They are defined as

$$
\mathcal{P}_{R/L} = \frac{1 \pm \gamma_5}{2}\ , \tag{1.129}
$$

and, as it is commonly required for projection operators, they are orthogonal, idempotent, and complete,

$$
\mathcal{P}_L\mathcal{P}_R = 0 = \mathcal{P}_R\mathcal{P}_L\ , \quad \mathcal{P}_{R/L}^2 = \mathcal{P}_{R/L}\ , \quad \mathcal{P}_L + \mathcal{P}_R = 1\ . \tag{1.130}
$$

As desired, these projection operators are chosen such that, when acting on fermion fields, they decompose them into their left- and right-handed components. For example, the quark and anti-quark fields $q(x)$ and $\bar{q}(x)$ decompose as

$$
q = (\mathcal{P}_L + \mathcal{P}_R)q = q_L + q_R\ , \quad \bar{q} = \bar{q}(\mathcal{P}_L + \mathcal{P}_R) = \bar{q}_R + \bar{q}_L\ , \tag{1.131}
$$

where the components indicated with $_L$ and $_R$ correspond to left- and right-handed chirality, as can be seen by considering the action of the chirality operator γ_5 onto these fields,

$$
\gamma_5 q_{R/L} = \gamma_5\mathcal{P}_{R/L}\ q = \gamma_5\frac{1 \pm \gamma_5}{2}q = \frac{\gamma_5 \pm 1}{2}q = \pm\frac{1 \pm \gamma_5}{2}q = \pm q_{R/L}\ . \tag{1.132}
$$

In the outlined context, we are able to define a so-called chiral symmetry. Namely, a Lagrangian is said to be chirally symmetric, if it consists of two distinct Lagrangians for left- and right-handed fermionic fields, respectively. This means that we can rotate the left- and right-handed components separately in flavor space, implying that the chiral symmetry group of the Lagrangian is given by

$$U(N_f)_L \times U(N_f)_R \ . \tag{1.133}$$

Since the group of unitary matrices $U(N)$ can be decomposed into a direct product of the subgroup of unitary matrices with unit determinant, $SU(N)$, and the subgroup of complex phase transformations, $U(1)$, the chiral symmetry group is isomorphic to

$$SU(N_f)_L \times U(1)_L \times SU(N_f)_R \times U(1)_R \ . \tag{1.134}$$

Its elements are given by

$$U_{L/R} = e^{-i\theta^a_{L/R}T^a} \ , \qquad a = 0, 1, \ldots, N_f^2 - 1 \ , \tag{1.135}$$

where θ^a_L and θ^a_R are the parameters of the transformations.

The chiral transformations of the left- and right-handed quark fields are then given by

$$\begin{aligned} q_{L/R} \to q'_{L/R} &= U_{L/R} q_{L/R} = e^{-i\theta^a_{L/R}T^a} q_{L/R} \\ &\simeq q_{L/R} - i\theta^a_{L/R} T^a q_{L/R} = q_{L/R} + \delta q_{L/R} \ , \end{aligned} \tag{1.136}$$

and the ones of the Dirac-adjoint antiquark fields read

$$\begin{aligned} \bar{q}_{L/R} \to \bar{q}'_{L/R} &= \bar{q}_{L/R} U^\dagger_{L/R} = \bar{q}_{L/R} e^{i\theta^a_{L/R}T^a} \\ &\simeq \bar{q}_{L/R} + \bar{q}_{L/R} i\theta^a_{L/R} T^a = \bar{q}_{L/R} + \delta\bar{q}_{L/R} \ . \end{aligned} \tag{1.137}$$

Each second line in the Eqs.(1.136) and (1.137) shows the infinitesimal form of the transformation obtained from a Taylor expansion up to first order in the transformation parameters. For later calculations it will be sufficient to restrict to these infinitesimal transformations, because in the case of continuously connected groups every transformation can be expressed as an (infinite) sequence of infinitesimal transformations.

A concrete study of chiral symmetry is possible by considering the terms (1.89) of the QCD Lagrangian in the limit of vanishing quark masses. After the decomposition into left- and right-handed quark fields, the Lagrangian (1.89) for $m = 0$ reads

$$\begin{aligned} \mathcal{L}_{\text{quarks+gluons},m=0} &= \bar{q} i\gamma^\mu D_\mu q \\ &= \bar{q}(\mathcal{P}_L + \mathcal{P}_R) i\gamma^\mu D_\mu (\mathcal{P}_L + \mathcal{P}_R) q \\ &= \bar{q}\mathcal{P}_R i\gamma^\mu D_\mu \mathcal{P}_L q + \bar{q}\mathcal{P}_L i\gamma^\mu D_\mu \mathcal{P}_R q \\ &\quad + \bar{q}\mathcal{P}_L i\gamma^\mu D_\mu \mathcal{P}_L q + \bar{q}\mathcal{P}_R i\gamma^\mu D_\mu \mathcal{P}_R q \\ &= \bar{q}\mathcal{P}_R i\gamma^\mu D_\mu \mathcal{P}_L q + \bar{q}\mathcal{P}_L i\gamma^\mu D_\mu \mathcal{P}_R q \\ &\quad + \bar{q}\mathcal{P}_L \mathcal{P}_R i\gamma^\mu D_\mu q + \bar{q}\mathcal{P}_R \mathcal{P}_L i\gamma^\mu D_\mu q \\ &= \bar{q}_L i\gamma^\mu D_\mu q_L + \bar{q}_R i\gamma^\mu D_\mu q_R \ , \end{aligned} \tag{1.138}$$

where we have used the orthogonality of the projection operators (1.130) in the last line. The Lagrangian decomposes into two parts, one for each handedness of the quark fields. Since the Yang-Mills term (1.97) does not contain any fermions (and therefore is not affected by the chiral decomposition), we conclude that the QCD Lagrangian for vanishing quark masses [Eq. (1.89) for $m = 0$] is invariant under the chiral transformations (1.136) and (1.137), and therefore possesses a chiral symmetry.

As we know from Noether's theorem [38], each continuous symmetry implies a conserved current (1.58). In the case of chiral symmetry, there are two conserved currents arising from the symmetry under complex phase transformations $U(1)_L \times U(1)_R$ and $2(N_f^2 - 1)$ conserved currents associated with the $SU(N_f)_L \times SU(N_f)_R$ symmetry. In total, a symmetry under transformations of the chiral $U(N_f)_L \times U(N_f)_R$ group leads to $2N_f^2$ conserved currents.
According to Eq. (1.58) we can compute the N_f^2 left- and N_f^2 right-handed currents associated to the massless QCD Lagrangian [Eq. (1.89) for $m = 0$]:

$$\begin{aligned} J^\mu_{L/R} &= \frac{\partial \mathcal{L}_{\mathrm{QCD},m=0}}{\partial(\partial_\mu q_{L/R})}\delta q_{L/R} + \delta\bar{q}_{L/R}\frac{\partial \mathcal{L}_{\mathrm{QCD},m=0}}{\partial(\partial_\mu \bar{q}_{L/R})} \\ &= \bar{q}_{L/R} i\gamma^\nu \frac{\partial(\partial_\nu q_{L/R})}{\partial(\partial_\mu q_{L/R})}(-i\theta^a_{L/R} T_a q_{L/R}) \\ &= \theta^a_{L/R}\bar{q}_{L/R}\gamma^\mu T_a q_{L/R} \\ &\equiv \theta^a_{L/R} J^\mu_{a,L/R} \ , \end{aligned} \tag{1.139}$$

where we used that the spacetime variable remains unchanged, i.e., $\delta x^\mu = 0$ in Eq. (1.58). The currents $J^\mu_{a,L}$ and $J^\mu_{a,R}$ denote the N_f^2 currents assigned to the left- and right-handed transformations, respectively.

However, commonly one does not work with these left- and right-handed currents, but considers linear combinations of them, which have definite parity transformations. On the one hand the so-called vector current is defined as the sum of the left- and the right-handed currents,

$$\begin{aligned} J^\mu_{a,V} = J^\mu_{a,L} + J^\mu_{a,R} &= \bar{q}_L\gamma^\mu T_a q_L + \bar{q}_R\gamma^\mu T_a q_R \\ &= \bar{q}\mathcal{P}_R\gamma^\mu T_a\mathcal{P}_L q + \bar{q}\mathcal{P}_L\gamma^\mu T_a\mathcal{P}_R q \\ &= \bar{q}\gamma^\mu T_a\mathcal{P}_L q + \bar{q}\gamma^\mu T_a\mathcal{P}_R q \\ &= \bar{q}\gamma^\mu T_a(\mathcal{P}_L + \mathcal{P}_R) q \\ &= \bar{q}\gamma^\mu T_a q \ , \end{aligned} \tag{1.140}$$

where we have exploited that $\gamma^\mu\mathcal{P}_{L/R} = \mathcal{P}_{R/L}\gamma^\mu$ holds, since $\{\gamma_5, \gamma^\mu\} = 0$. Furthermore, we used that the projection operators are idempotent and complete see Eq. (1.130). As the name "vector current" indicates, this current transforms as a vector under Lorentz (parity) transformations. On the other hand the difference of right- and left-handed currents is referred to as axial-vector current,

$$\begin{aligned} J^{\mu}_{a,A} = J^{\mu}_{a,R} - J^{\mu}_{a,L} &= \bar{q}_R\gamma^{\mu}T_a q_R - \bar{q}_L\gamma^{\mu}T_a q_L \\ &= \bar{q}\mathcal{P}_L\gamma^{\mu}T_a\mathcal{P}_R q - \bar{q}\mathcal{P}_R\gamma^{\mu}T_a\mathcal{P}_L q \\ &= \bar{q}\gamma^{\mu}T_a\mathcal{P}_R q + \bar{q}\gamma^{\mu}T_a\mathcal{P}_L q \\ &= \bar{q}\gamma^{\mu}T_a(\mathcal{P}_R - \mathcal{P}_L)q \\ &= \bar{q}\gamma^{\mu}T_a(\frac{1+\gamma_5}{2} - \frac{1-\gamma_5}{2})q \\ &= \bar{q}\gamma^{\mu}\gamma_5 T_a q \ , \end{aligned} \tag{1.141}$$

which transforms as an axial-vector under Lorentz (parity) transformations.

This complete set of vector and axial-vector currents arise from the invariance under the symmetry group

$$U(N_f)_V \times U(N_f)_A \equiv U(1)_V \times SU(N_f)_V \times U(1)_A \times SU(N_f)_A \ . \tag{1.142}$$

(Note, the terminology is commonly used, but the axial transformations do not form a group since they are not closed.). $U(N_f)_V \times U(N_f)A$ is isomorphic to the chiral group, if the elements are given by

$$\begin{aligned} U_V &= e^{-i\theta^a_V T^a_V} \ , \\ U_A &= e^{-i\theta^a_A T^a_A} \ , \qquad \forall\ a \in \{0, 1, \ldots, N_f^2 - 1\} \end{aligned} \tag{1.143}$$

where the parameters are defined as

$$\theta^a_V = \frac{\theta^a_L + \theta^a_R}{2} \ , \quad \theta^a_A = \frac{\theta^a_R - \theta^a_L}{2} \ , \tag{1.144}$$

and the generators are given by

$$T^a_V = T^a \ , \quad T^a_A = \gamma_5 T^a \ . \tag{1.145}$$

Then, the chiral transformation of the quark fields can be expressed as vector and axial-vector transformations,

$$q(x) \xrightarrow{V} q'(x) = U_V q(x) = e^{-i\theta^a_V T^a} q(x) \simeq q(x) - i\theta^a_V T^a q(x) \ , \tag{1.146}$$

$$q(x) \xrightarrow{A} q'(x) = U_A q(x) = e^{-i\theta^a_A \gamma_5 T^a} q(x) \simeq q(x) - i\theta^a_A \gamma_5 T^a q(x) \ . \tag{1.147}$$

Equivalently, for the Dirac adjoint fields the transformations read

$$\bar{q}(x) \xrightarrow{V} \bar{q}'(x) = \bar{q}(x)U_V^{\dagger} = \bar{q}(x)e^{i\theta^a_V T^a} \simeq \bar{q}(x) + \bar{q}(x)i\theta^a_V T^a \ , \tag{1.148}$$

$$\bar{q}(x) \xrightarrow{A} \bar{q}'(x) = \left[U_A q(x)\right]^{\dagger}\gamma_0 = q^{\dagger}(x)e^{i\theta^a_A\gamma_5 T^a}\gamma_0 \simeq \bar{q}(x) - \bar{q}(x)i\theta^a_A\gamma_5 T^a \ , \tag{1.149}$$

where we used the anticommutation relation between γ_0 and γ_5 in the last line.

Up to now we considered the QCD Lagrangian only for vanishing quark masses. For a realistic description of Nature we decompose the QCD Lagrangian with arbitrary quark masses into left- and right-handed quark fields. Again, since the Yang-Mills term (1.97) does not contain quarks, it is sufficient to consider the decomposition of the term (1.89):

$$\begin{aligned}\mathcal{L}_{\text{quarks+gluon}} &= \bar{q}(x)\big(i\gamma^\mu D_\mu - m\big)q(x)\\ &= \bar{q}_L\big(i\gamma^\mu D_\mu - m\big)q_R + \bar{q}_R\big(i\gamma^\mu D_\mu - m\big)q_L\\ &+ \bar{q}_L\big(i\gamma^\mu D_\mu - m\big)q_L + \bar{q}_R\big(i\gamma^\mu D_\mu - m\big)q_R\\ &= \bar{q}_L i\gamma^\mu D_\mu q_L + \bar{q}_R i\gamma^\mu D_\mu q_R + \bar{q}_L m q_R + \bar{q}_R m q_L \ ,\end{aligned} \tag{1.150}$$

where we used the orthogonality of the projection operators. The QCD Lagrangian (with $m \neq 0$) does not decouple into two separate Lagrangians for left- and right-handed quark fields, because of the mass term. Thus it is not chirally invariant.

Even if some of the currents will no longer be conserved, the inclusion of the mass term does not change the calculations leading to Eq. (1.139). We can study the contribution of the mass term to the divergence of the left- and right-handed currents explicitly. The divergence of the left-handed currents read

$$\begin{aligned}\partial_\mu J^\mu_L &= \partial_\mu\Big[\frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu q_L)}\delta q_L + \delta\bar{q}_L\frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu \bar{q}_L)}\Big]\\ &= \frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial q_L}\delta q_L + \frac{\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu q_L)}(\partial_\mu\delta q_L) + (\partial_\mu\delta\bar{q}_L)\frac{\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu \bar{q}_L)}\\ &\quad + \delta\bar{q}_L\frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial\bar{q}_L}\\ &= \big(\bar{q}_L i\gamma^\mu D_\mu - \bar{q}_R m\big)(-i)\theta^a_L T_a q_L + \bar{q}_L i\gamma^\mu(-i)\theta^a_L T_a\partial_\mu q_L + \bar{q}_L i\theta^a_L T_a\big(i\gamma^\mu D_\mu q_L - m q_R\big)\\ &= \theta^a_L\ i(\bar{q}_R m T_a q_L - \bar{q}_L T_a m q_R)\\ &\equiv \theta^a_L\ \partial_\mu J^\mu_{a,L}\ ,\end{aligned} \tag{1.151}$$

where we have used Eq. (1.58) with $\delta\bar{q}_L$ and δq_L defined in Eqs. (1.136) and (1.137). In the second line, we exploited the Euler-Lagrange equation (1.28) for the left-handed quark and antiquark fields. Similarly, we obtain the divergence of the right-handed current,

$$\begin{aligned}\partial_\mu J^\mu_R &= \partial_\mu\Big[\frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu q_R)}\delta q_R + \delta\bar{q}_R\frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu \bar{q}_R)}\Big]\\ &= \frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial q_R}\delta q_R + \frac{\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu q_R)}(\partial_\mu\delta q_R) + (\partial_\mu\delta\bar{q}_R)\frac{\mathcal{L}_{\text{quarks+gluon}}}{\partial(\partial_\mu \bar{q}_R)}\\ &\quad + \delta\bar{q}_R\frac{\partial\mathcal{L}_{\text{quarks+gluon}}}{\partial\bar{q}_R}\\ &= \big(\bar{q}_R i\gamma^\mu D_\mu - \bar{q}_L m\big)(-i)\theta^a_R T_a q_R + \bar{q}_R i\gamma^\mu(-i)\theta^a_R T_a\partial_\mu q_R + \bar{q}_R i\theta^a_R T_a\big(i\gamma^\mu D_\mu q_R - m q_L\big)\\ &= \theta^a_R\ i(\bar{q}_L m T_a q_R - \bar{q}_R T_a m q_L)\\ &\equiv \theta^a_R\ \partial_\mu J^\mu_{a,R}\ .\end{aligned} \tag{1.152}$$

The divergences are proportional either to the left- or the right-handed group parameters. According to Eq. (1.144), we can compute the divergence of the vector current (1.140) only if we require

$$\theta_L^a = \theta_R^a = \theta_V^a \ , \tag{1.153}$$

and the axial-vector current (1.141) if we take

$$\theta_L^a = -\theta_R^a = \theta_A^a \ . \tag{1.154}$$

Then we calculate

$$\begin{aligned}
\partial_\mu J^\mu_{a,V} &= \partial_\mu J^\mu_{a,L} + \partial_\mu J^\mu_{a,R} \\
&= i(\bar{q}_R m T_a q_L - \bar{q}_L T_a m q_R) + i(\bar{q}_L m T_a q_R - \bar{q}_R T_a m q_L) \\
&= i\bar{q}_L [m, T_a] q_R + i\bar{q}_R [m, T_a] q_L \\
&= i\bar{q}[m, T_a](\mathcal{P}_L + \mathcal{P}_R) q \\
&= i\bar{q}[m, T_a] q \ ,
\end{aligned} \tag{1.155}$$

and

$$\begin{aligned}
\partial_\mu J^\mu_{a,A} &= \partial_\mu J^\mu_{a,R} - \partial_\mu J^\mu_{a,L} \\
&= i(\bar{q}_L m T_a q_R - \bar{q}_R T_a m q_L) - i(\bar{q}_R m T_a q_L - \bar{q}_L T_a m q_R) \\
&= i\bar{q}_L \{m, T_a\} q_R + i\bar{q}_R \{m, T_a\} q_L \\
&= i\bar{q}\{m, T_a\}(\mathcal{P}_R - \mathcal{P}_L) q \\
&= i\bar{q}\{m, T_a\}\Big(\frac{1+\gamma_5}{2} - \frac{1-\gamma_5}{2}\Big) q \\
&= i\bar{q}\{m, T_a\}\gamma_5 q \ .
\end{aligned} \tag{1.156}$$

This shows that the axial-vector current is conserved, if the anticommutator $\{m, T_a\}$ vanishes, which is the case only if the quark mass matrix m is zero. Consequently, non-vanishing quark masses break the symmetry under axial-vector transformations explicitly. The vector current is conserved if the commutator $[m, T_a]$ in Eq. (1.155) vanishes. We can think of three scenarios where this is the case:
(i) all quark masses are zero, i.e., $m = 0$,
(ii) the generators are proportional to the unit matrix or
(iii) the mass matrix m is proportional to the unit matrix.
Case (ii) is only valid for the generator T_0. While the remaining generators are proportional to the generalized Gell-Mann matrices, T_0 is by definition proportional to the unit matrix. As a consequence, the vector current which originates from the $U(1)_V$ symmetry,

$$J^\mu_{0,V} = \bar{q}(x)\gamma^\mu q(x) \ , \tag{1.157}$$

is always conserved. The corresponding conserved charge $Q_{0,V}$ is the baryon number.
Case (iii) requires all quark masses to be equal. In nature this is only approximately valid

in the light quark sector, see Tab. 1.1. The mass difference of the two lightest quarks, u and d is about $\Delta m = m_d - m_u \simeq 2.5$ MeV, which is is small compared to the typical hadronic mass scale of $m_h \simeq 1$ GeV. Thus, for $N_f = 2$ the assumption of equal quark masses is satisfied to a very good approximation, leading to the symmetry under the $SU(2)_V$, the so-called isospin symmetry. The s quark is heavier than the u and d quarks, but the mass differences are also only of the order 10% of the typical hadronic mass scale. Thus, we assume the $SU(3)_V$ symmetry to be fulfilled to a good approximation.

Note that we restricted our conclusions to *classically* conserved currents. The reason for this is a phenomenon called $U(1)_A$ anomaly. The term anomaly denotes a symmetry that is present on the classical level but gets broken at the quantum level. The statement "the axial current is conserved for vanishing quark masses" is valid on the classical level only, while at the quantum level, the $U(1)_A$ symmetry is explicitly broken by quantum fluctuations [28, 44]. Consequently, even in the chiral limit ($m_f \to 0$) the full chiral symmetry group of the QCD Lagrangian at the quantum level is reduced to $SU(N_f)_V \times U(1)_V \times SU(N_f)_A$.

Finally, there is also SSB in QCD, see Ref. [45]. We consider $N_f = 3$ and assume to be in the chiral limit, i.e., $m_f \to 0$. Then the QCD Lagrangian is approximately symmetric under $SU(3)_V \times U(1)_V \times SU(3)_A$ transformations. In turn, also the QCD Hamiltonian exhibits a $SU(3)_V \times U(1)_V \times SU(3)_A$ symmetry, i.e., H_{QCD} commutes with the charges $Q_{0,V}$, $Q_{a,V}$, and $Q_{a,A}$ $\forall\ a \in \{1, 2, \ldots, 8\}$. This implies the existence of degenerate states with positive and negative parity (because $Q_{0,V}$ and $Q_{a,V}$ have opposite parity) but with equal quantum numbers (baryon number, strangeness, and spin). However, in Nature, such parity doublets are not observed, which indicates that the axial-vector symmetry under $SU(N_f)_A$ is spontaneously broken.

In summary, we have seen that the chiral $U(N_f)_V \times U(N_f)_A$ symmetry of the QCD Lagrangian is broken by various mechanisms:

- The symmetry under $U(1)_A$ is anomalously broken.
- For $m_f \to 0$ chiral symmetry is reduced to
$$SU(N_f)_V \times U(1)_V \times SU(N_f)_A \ .$$
- For $m_f \neq 0$ and $m_u = m_d = \cdots = m_t$ chiral symmetry is restricted to
$$SU(N_f)_V \times U(1)_V \ .$$
- For $m_f \neq 0$ and $m_u \neq m_d \neq \cdots \neq m_t$ only the symmetry under $U(1)_V$ remains.
- The symmetry under $SU(N_f)_A$ is also spontaneously broken.

1.3.5. Running coupling constant

This section discusses the so-called running coupling constant of QCD. In the following (very comprised) discourse we try to emphasize the conceptual rather than the computational aspects of it.

Interacting quantum field theories have to be treated with care, because computations of several quantities seem to deliver divergent results (at first).
For example, if we consider ϕ^4 theory [which is a simple toy model, similar to the one studied in Eq. (1.61), but with a switched sign in front of the mass term] and compute the scattering amplitude of two scalar fields, we will find that the one-loop correction term diverges logarithmically for large values of momentum [see e.g. Ref. [36]].
Such divergences do not only arise in the ϕ^4 theory, but the appearance of such infinities is rather a common feature to all interacting field theories.

The usual approach to deal with issue makes use of a "regularization" and "renormalization". The idea is that the theoretical "bare" parameters present in the Lagrangian (such as the coupling constant or the mass) cannot be interpreted as experimentally measurable. Instead, it turns out that the bare parameters are also divergent, exactly in such a way that they cancel out the divergences of the physical observables.

In general, regularization means that a maximal energy Λ is introduced as a parameter with dimension energy. As a consequence, the considered quantities are no longer divergent, but depend on this chosen regulation parameter Λ. However, since physical observables shall not depend on such a quantity, we use renormalization which hides this dependence in the bare parameters of the theory.

The bare parameters can be related to the physical quantities. This allows to rewrite the computed expressions of observable quantities in terms of finite physical parameters. These physical parameters are often referred to as renormalized parameters, labeled with an index r.

In the case of QCD, the procedure of renormalization yields the renormalized coupling constant g_r depending on the momentum scale μ:

$$g_{\text{bare}} \to g_r = g_r(\mu) \ . \tag{1.158}$$

Perturbative calculations up to one-loop order reveal the following dependence of the strong coupling $\alpha_s(\mu) = g_r^2(\mu)/4\pi$ on the momentum scale [36]:

$$\alpha_s(\mu) = \frac{4\pi}{(11 - 2N_f/3)\ln(\mu^2/\mu_0^2)} \ , \tag{1.159}$$

where $\mu_0 \simeq 200\text{MeV}$. This perturbative one-loop result is displayed in Fig. 1.11 indicating that the strong coupling remains small for large energies, but gets large for small energies.

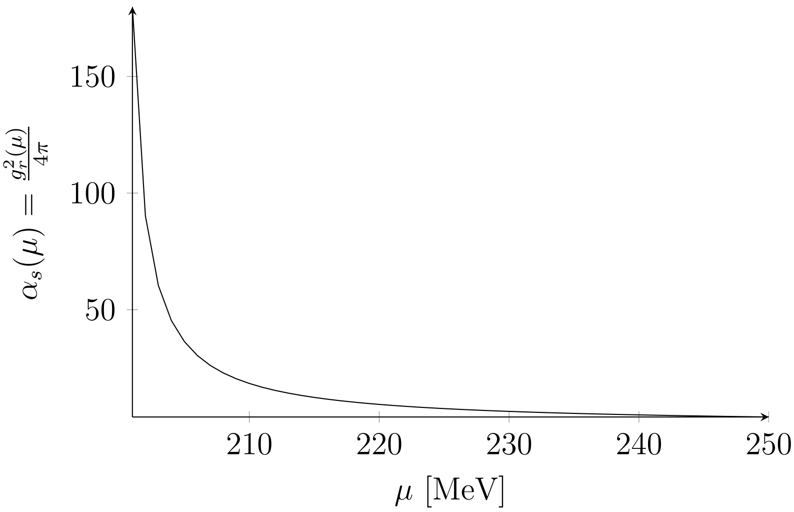


Figure 1.11. Momentum scale dependence of α_S resulting from perturbative calculations up to one-loop order.

Thus, for high energies one can study QCD by a perturbative expansion in the respective coupling constant $\alpha_S(\mu)$. However, since this approach is only valid as long as the coupling remains small, the large values of the strong coupling for low energies prohibits to use a perturbative approach in this regime.

1.3.6. Dilatation symmetry

This section is dedicated to the so-called dilatation symmetry, which allows to restrict the number of possible terms in the effective Lagrangian as presented in the following chapters. Note that renormalizability is not required in an effective approach, because such a model is not valid up to arbitrarily large energy but only to the energy scale of the heaviest considered resonance.

A dilatation transformation rescales the spacetime variable as follows:

$$x^\mu \to \lambda^{-1} x^\mu \ , \tag{1.160}$$

where λ is the scale parameter. A theory posses a symmetry under such transformations, if the corresponding Lagrangian transforms as

$$\mathcal{L} \to \lambda^4 \mathcal{L} \ , \tag{1.161}$$

because then the action (1.26) is invariant.

In the chiral limit $m_f \to 0$, we find that the QCD Lagrangian (1.99) behaves as in Eq. (1.161) under a scale transformation, if the quark and the scalar gluon field transform as

$$q_f(x) \to \lambda^{\frac{3}{2}} q_f(x) \qquad \text{and} \qquad A^a_\mu(x) \to \lambda A^a_\mu(x) \ . \tag{1.162}$$

This meets expectations from dimension analysis where [fermionic field] = [energy$^{3/2}$] and [scalar field] = [energy].

In full QCD, however, the dilatation symmetry is explicitly broken due to non-vanishing quark masses, but also at the quantum level ["trace anomaly", see e.g. Refs. [46, 47]].

In the effective QCD approach which we investigate in the following chapters (the extended linear sigma model), this phenomenon can be parametrized (at a composite level) by introducing a so-called dilaton field G which is described by

$$\mathcal{L}_{\text{dil}} = \frac{1}{2}\left(\partial_\mu G\right)^2 - \frac{1}{4}\frac{m_G^2}{\Lambda^2}\left(\frac{1}{2}G^4 \ln\frac{G^2}{\Lambda^2} - \frac{G^2}{4}\right) , \tag{1.163}$$

where m_G is the mass G. This particle is interpreted as the scalar glueball, see Ref. [2, 3]. The momentum scale Λ under the logarithm explicitly breaks the symmetry under dilatation transformations.
As a consequence in the framework of the extended linear sigma model, if we assume that the dilatation symmetry is eylpicitly broken only by the potential density in Eq. (1.163), all other terms in the effective Lagrangian have to be parametrized by dimensionless coupling constants "in order that, in the chiral limit, the dilatation anomaly in the model is generated in the same manner as in the QCD Lagrangian" [3]. This restricts the number of possible terms in the chiral effective Lagrangian which we investigate in the following chapters.

1.4. Effective approach to the low-energy regime of QCD

Computing hadronic bound states as analytic solutions of QCD is very difficult (up to now even impossible[11]). In order to find at least some perturbative results, we need to have a small parameter to expand in. The QCD Lagrangian (1.99), however, does not provide such a parameter (for all energy regimes), because in the framework of renormalization, it turns out that in the low-energy regime the strong coupling constant is too large for perturbation theory to be of any use, see Sec. 1.3.5.
However, as the appearance of massless Nambu-Goldstone bosons is explained by SSB [Sec. 1.2.4], some QCD properties follow from general considerations without directly using quantum field theory. Thus, as A. Zee writes [32] a new realization leaks trough:

> *"at the very least, the Lagrangian provides a mnemonic for any underlying QFT[, because] any Lagrangian incorporating these general properties had to produce the same results."*

[11] In fact, presenting an analytic solution of QCD would immediately turn you into a celebrity among physicists. By the way, a title that would make you one million dollars richer, because it is one of the millennium problems listed on the web page of the Clay Mathematics Institute, "Yang-Mills and mass gap": http://www.claymath.org/millennium-problems/yang-mills-and-mass-gap.

Instead of solving QCD directly, we develop low-energy (chiral) effective approaches whose degrees of freedom are hadrons, describing how the strong interaction determines their masses and interactions at low energies [48]. We distinguish two groups of such models: some are based on a non-linear realization of chiral symmetry [49] [so-called chiral perturbation theory [50]], others are linear sigma models which are based on the linear realization of chiral symmetry [3, 10, 12, 51, 52, 53].

In this work, we utilize the latter approach considering the so-called extended linear sigma model (eLSM) [1, 2, 3, 4, 9, 12, 13] which (additionally to the linearly realized chiral symmetry) is also based on dilatation invariance.

Chapter 2.

The extended linear sigma model describing mesons for $N_f = 3$ and baryons for $N_f = 2$

This chapter introduces the extended linear sigma model (eLSM). In Sec. 2.1 we briefly investigate the original sigma model as studied in Ref. [54]. Then, in Sec. 2.2 we present the mesonic sector of the eLSM for $N_f = 3$ as it has been investigated in Refs. [2, 3]. Finally, in Sec. 2.3, we turn to the baryonic sector describing the nucleon and its chiral partner, as it has been considered in Refs. [12, 13].

2.1. The linear σ model

The roots of the (e)LSM date back to the year 1960, when Gell-Mann and Lévy published a paper [54] in which they studied pion-nucleon interactions in the context of chiral symmetry.

At low energies, nucleons and pions are the dominant degrees of the freedom. The proton and neutron are described by the Dirac spinor

$$N \equiv (p, n)^T \ , \tag{2.1}$$

whose kinetic energy term

$$\bar{N} i\gamma^\mu \partial_\mu N = \bar{N}_L i\gamma^\mu \partial_\mu N_L + \bar{N}_R i\gamma^\mu \partial_\mu N_R \tag{2.2}$$

exhibits an $SU(2)_L \times SU(2)_R$ chiral symmetry, where $N_L = \mathcal{P}_L N$ and $N_R = \mathcal{P}_R N$ are the left- and right-handed fields transforming as a doublet under $SU(2)_L$ and $SU(2)_R$, respectively,

$$N_L \to U_L N_L \ , \qquad N_R \to U_R N_R \ . \tag{2.3}$$

However, the mass term

$$m\bar{N}N = m\left(\bar{N}_L N_R + \bar{N}_R N_L\right) \ , \tag{2.4}$$

explicitly breaks the chiral symmetry, since $\bar{N}_L N_R$ mixes right- and left-handed nucleons and transforms as

$$\bar{N}_L N_R \to U_R \bar{N}_L N_R U_L^\dagger \ . \tag{2.5}$$

Thus, insisting on an unbroken chiral symmetry a mass term for the nucleon is not possible.

Gell-Mann and Lévy circumvented the problem by including a new scalar-isoscalar meson σ, such that $(\sigma, \boldsymbol{\pi})$ constitutes a four-dimensional representation of the chiral group:

$$\phi = \sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi} \ , \tag{2.6}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)^T$ represents the pions and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)^T$ is the vector of Pauli matrices. This object transforms under $SU(2)_L \times SU(2)_R$ as

$$\phi \to U_L \phi U_R^\dagger \ . \tag{2.7}$$

Upon this introduction, it is possible to construct the following (approximately) chirally invariant Lagrangian representing the so-called linear[1] sigma model (LSM):

$$\mathcal{L}_{\text{LSM}} = \bar{N}\left[\gamma^\mu \partial_\mu + g(\sigma + i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5)\right]N + \mathcal{L}(\sigma, \boldsymbol{\pi}) \ , \tag{2.8}$$

where the terms not involving the nucleons are given by

$$\mathcal{L}(\sigma, \boldsymbol{\pi}) = \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \boldsymbol{\pi})^2\right] + \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 + \epsilon\sigma \ . \tag{2.9}$$

The model involves the nucleon, as well as the pseudoscalar pion and scalar σ meson which is the chiral partner of the pion. At the time of 1960, a particle like the σ meson was not known, but they argued that "it has strong interactions, and might easily have escaped observation if it is much heavier than π, so that it would disintegrate immediately into two pions. It would appear experimentally as a resonant state of two pions" [54].

Upon setting $\epsilon = 0$, we recognize that Eq. (2.9) has the same form as (the four-dimensional generalization of) the Lagrangian (1.66), and thus is featuring SSB. As we show subsequently, this will generate the nucleon mass. However, in the case $\epsilon = 0$ the pions $\boldsymbol{\pi}$ emerge as the massless Nambu-Goldstone bosons [39], which is not in accordance with experimental observations showing that the pion has a small but non-vanishing mass [18]. This problem will be resolved considering $\epsilon \neq 0$, which includes an explicit symmetry breaking term. In the following we discuss the two cases ($\epsilon = 0$ and $\epsilon \neq 0$) separately.

2.1.1. Case 1: LSM without explicit symmetry breaking

We investigate the potential of the LSM (2.8) for $\epsilon = 0$:

$$\mathcal{V}_{\epsilon=0} = -\frac{\mu^2}{2}(\pi^2 + \sigma^2) + \frac{\lambda}{4}(\pi^2 + \sigma^2)^2 - g\bar{N}(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5 + \sigma)N \ . \tag{2.10}$$

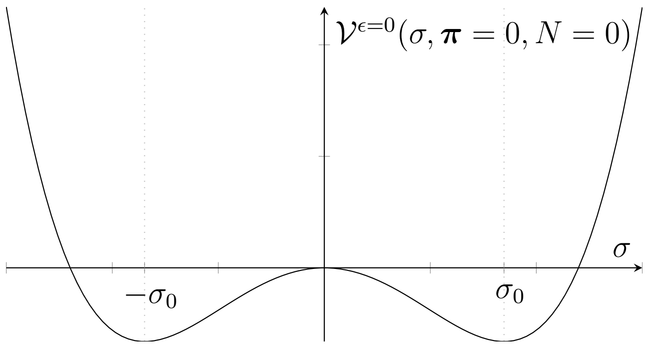


Figure 2.1. Case 1: potential density of the LSM without explicit symmetry breaking ($\epsilon = 0$).

Its shape for $\boldsymbol{\pi} = 0$ and $N = 0$ is displayed in Fig. 2.1. The isoscalar-scalar meson σ has two non-zero VEVs. Without loss of generality, we assume that the following VEV is realized:

$$\langle\sigma\rangle_{\epsilon=0} \equiv \sigma_0 = \sqrt{\frac{\mu^2}{\lambda}} \; . \tag{2.11}$$

Note that $\langle\boldsymbol{\pi}\rangle = 0$ and $\langle N\rangle = 0$, since non-zero VEVs of these fields would break the isospin and the parity symmetry of the vacuum.

We consider small fluctuations around this minimum by shifting the σ field by its VEV (2.11):

$$\sigma \to \sigma + \sigma_0 \; . \tag{2.12}$$

Substituting this shift into Eq. (2.10), (inter alia) the following mass term for the nucleon emerges:

$$g\sigma_0 \bar{N} N \; . \tag{2.13}$$

Thus in the LSM, the mechanism of SSB generates the nucleon mass,

$$m_N = g\sigma_0 \neq 0 \; . \tag{2.14}$$

Considering the masses of $\boldsymbol{\pi}$ and σ, the LSM Lagrangian (2.10) already contains explicit mass terms, but there arise also contributions from SSB. Substituting the shift (2.12) into Eq. (2.10), we find that the mass of the σ meson is given by

$$m_\sigma^2 = -\mu^2 + 3\lambda\sigma_0^2 = 2\mu^2 \; , \tag{2.15}$$

while the pion comes out massless:

$$m_\pi^2 = -\mu^2 + \lambda\sigma_0^2 = 0 \; . \tag{2.16}$$

[1]Due to the linear realization of chiral symmetry; in contrast to a non-linear realization, where the σ meson is removed as dynamical field, see Ref. [49].

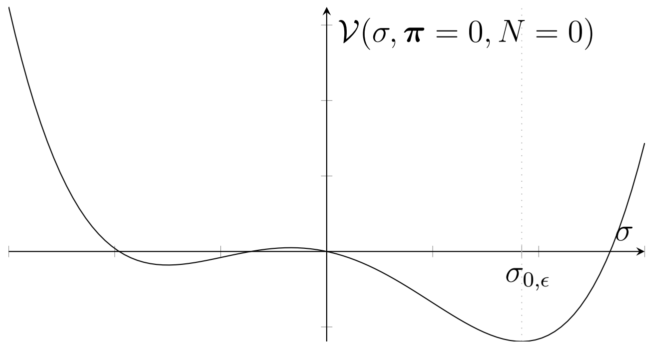


Figure 2.2. Case 2: potential density of the LSM with ESB ($\epsilon \neq 0$).

This reveals that the pions π^+, π^-, and π^0 embody the three massless Nambu-Goldstone bosons arising from SSB of $SU(2)_A$ (three broken generators) [39]. This contradicts the experimentally observed $m_\pi^{\rm exp} \simeq 139$ MeV, which is small but non-zero. The discrepancy is resolved by including an explicit symmetry breaking term ($\epsilon \neq 0$ in the LSM), which will generate further contributions to the masses.

2.1.2. Case 2: LSM with explicitly broken chiral symmetry

The potential of the LSM Lagrangian (2.8) (for $\epsilon \neq 0$) is given by

$$\mathcal{V} = -\frac{\mu^2}{2}(\pi^2 + \sigma^2) + \frac{\lambda}{4}(\pi^2 + \sigma^2)^2 - g\bar{N}(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}\gamma_5 + \sigma)N - \epsilon\sigma \ , \tag{2.17}$$

where the last term explicitly breaks the chiral symmetry. It entails in a tilt of the potential density in σ direction, see Fig. 2.2. In turn, the VEV of the σ field becomes unique:

$$\langle\sigma\rangle_{\epsilon\neq 0} \equiv \sigma_{0,\epsilon} = \sqrt{\frac{\mu^2}{\lambda}\left(1 + \frac{\epsilon}{\mu^2\sigma_{0,\epsilon}}\right)} = \sigma_0 + \frac{\epsilon}{2\lambda\sigma_0^2} + \mathcal{O}(\epsilon^2) \ , \tag{2.18}$$

where we have assumed that the ϵ parameter is small and expanded the expression in a Taylor series. Furthermore, we have used Eq. (2.11) to identify σ_0.
Investigating fluctuations around this VEV [which is slightly larger when compared to Eq. (2.11)] by substituting the shift

$$\sigma \to \sigma + \sigma_{0,\epsilon} \tag{2.19}$$

into Eq. (2.17) reveals the following masses:

$$m_N^2 = g\sigma_{0,\epsilon} \ , \tag{2.20}$$

$$m_\pi^2 = \frac{\epsilon}{\sigma_{0,\epsilon}} \ , \tag{2.21}$$

$$m_\sigma^2 = 2\lambda\sigma_{0,\epsilon} + \frac{\epsilon}{\sigma_{0,\epsilon}} = 2\lambda\sigma_{0,\epsilon} + m_\pi^2 \ . \tag{2.22}$$

Thus, in the case $\epsilon \neq 0$, the pion acquires a small but finite mass, as desired. Due to SSB (of the approximately realized chiral symmetry), the pions still remain much lighter than the sigma field. Thus, they are often referred to as Pseudo-Nambu-Goldstone bosons.

2.2. Meson vacuum phenomenology in the three-flavor extended linear sigma model

The extended LSM (eLSM) has been first constructed in Ref. [1] for $N_f = 2$. In contrast to the LSM it contains not only (pseudo)scalar degrees of freedom, but from the very beginning it incorporates also (axial-)vector mesons and glueballs. Besides explicit and spontaneous symmetry breaking terms, it contains also $U(1)_A$ anomalous terms to reproduce known features of the strong interaction. The first two-flavor eLSM approach (which does not yet consider baryons) [1, 2] has been further extended to $N_f = 3$ [3, 4, 6, 5, 7] and $N_f = 4$ [9].
Baryons have been included for the case $N_f = 2$ [12, 13, 14], see Sec. 2.3. The extension of the baryonic sector to $N_f = 3$ is the objective of the present research, see Chap. 4.

In this section, we present the mesonic sector of the eLSM for the case $N_f = 3$, as investigated in Ref. [3]. First, we give an overview of the described particles and resonances, see Sec. 2.2.1. Their mathematical representation within the eLSM is discussed in Sec. 2.2.2. The Lagrangian is presented in Sec. 2.2.3 and the mechanism of SSB within the eLSM is explained in Sec. 2.2.4. Finally, in Sec. 2.2.5 we present the results and the parameter values as obtained from a fit to experimental data.

2.2.1. Content of physical particles and resonances

We investigate the (pseudo)scalar and (axial-)vector degrees of freedom of the eLSM, as well as the assignment of experimentally observed particles and resonances.

Scalar mesons The eLSM contains the isoscalar-scalar fields

$$\sigma_N \quad \text{and} \quad \sigma_S \ , \tag{2.23}$$

and the isovector-scalar fields

$$\boldsymbol{a}_0 = (a_{0,1}, a_{0,2}, a_{0,3})^T \ , \quad \boldsymbol{K}_0^{*\pm} = (K_{0,1}^{*\pm}, K_{0,2}^{*\pm})^T \ , \quad \boldsymbol{K}_0^{*0} = (K_{0,1}^{*0}, K_{0,2}^{*0})^T \ . \tag{2.24}$$

Since the σ_N and σ_S fields have the same quantum numbers as the vacuum, they are allowed to develop a non-vanishing VEV, which leads to SSB.

The assignment of the scalar mesons is difficult, since some the scalar resonances have large decay widths causing a strong overlap between resonance and background in experimental resolutions. In addition, also non-$q\bar{q}$ scalar objects are expected in the low-mass regime < 2 GeV, such as glueballs, multiquark states, or meson-meson bound states, see Ref. [55]. Hence, the assignment in the isoscalar $I(J^{PC}) = 0(0^{++})$ sector is probably the most complex one. The eLSM prefers the following assignment, see Refs. [2, 3, 4]:

$$J^{PC} = 0^{++}: \qquad \{a_0(1450), K_0^*(1430), f_0(1370), f_0(1500)\}\ . \tag{2.25}$$

The possibility $\{\sigma, \boldsymbol{a}_0\} \hat{=} \{f_0(500), a_0(980)\}$ has to be excluded, because then the eLSM cannot describe the scattering lengths and the decay $\sigma \to \pi\pi$ at the same time, for more details see Ref. [3].

Pseudoscalar mesons The eLSM contains the isoscalar-pseudoscalar fields

$$\eta_N \quad \text{and} \quad \eta_S \tag{2.26}$$

and the isovector-pseudoscalar fields

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)^T\ , \quad \boldsymbol{K}^{\pm} = (K_1^{\pm}, K_2^{\pm})^T\ , \quad \boldsymbol{K^0} = (K_1^0, K_2^0)^T\ . \tag{2.27}$$

The eLSM prefers the following assignment of the pseudoscalar fields [3]:

$$J^{PC} = 0^{-+}: \qquad \{\pi^{[0,\pm]}, K^{[0,\pm]}, \eta = \eta(547), \eta' = \eta(958)\}\ . \tag{2.28}$$

The mixing between the strange and non-strange contributions, η_N and η_S, is large [56, 57]:

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta_P & \sin\theta_P \\ -\sin\theta_P & \cos\theta_P \end{pmatrix} \begin{pmatrix} \eta_N = (\bar{u}u + \bar{d}d)/\sqrt{2} \\ \eta_S = \bar{s}s \end{pmatrix}\ , \tag{2.29}$$

where $\theta_P = -44.6°$ [3] is the mixing angle.

Vector mesons The eLSM contains the isoscalar-vector fields

$$\omega_N^\mu \quad \text{and} \quad \omega_S^\mu \tag{2.30}$$

and the $J^{PC} = 1^{--}$ isovector-vector fields

$$\boldsymbol{\rho}^\mu = (\rho_1^\mu, \rho_2^\mu, \rho_3^\mu)^T\ , \quad \boldsymbol{K}^{*\mu} = (K_1^{*\mu}, K_2^{*\mu})^T\ , \quad \bar{\boldsymbol{K}}^{*\mu} = (\bar{K}_1^{*\mu}, \bar{K}_2^{*\mu})^T\ , \tag{2.31}$$

where the strange/non-strange isosinglet mixing is small and can be neglected. The eLSM prefers the following assignment [3]:

$$J^{PC} = 1^{--}: \qquad \{\rho(770), K^*(892), \omega(782), \phi(1020)\}\ . \tag{2.32}$$

Axial-vector mesons The eLSM contains the isoscalar-axial-vector fields

$$f_{1N}^\mu \quad \text{and} \quad f_{1S}^\mu \tag{2.33}$$

and the $J^{PC} = 1^{++}$ isovector-axial-vector fields

$$\boldsymbol{a}_1^\mu = (a_{1,1}^\mu, a_{1,2}^\mu, a_{1,3}^\mu)^T\ , \quad \boldsymbol{K}_1^\mu = (K_{1,1}^\mu, K_{1,2}^\mu)^T\ , \quad \bar{\boldsymbol{K}}^{\boldsymbol{\mu}}{}_1 = (\bar{K}_{1,1}^\mu, \bar{K}_{1,2}^\mu)^T\ , \tag{2.34}$$

where the strange/non-strange isoscalar mixing is again small and can be neglected. The eLSM prefers the following assignment [3, 5]:

$$J^{PC} = 1^{++}: \qquad \{a_1(1260), K_{1,A} \equiv K_1(1270), f_1(1285), f_1(1420)\}\ , \tag{2.35}$$

Table 2.1 summarizes the (predominant) assignment of the mesonic fields in the eLSM.

Field in eLSM	Assignment (predom.) [18]	Flavor content	I	J^{PC}
$\boldsymbol{a_0}$	$a_0(1450)$	$u\bar{d},\ (u\bar{u}-d\bar{d})/\sqrt{2},\ d\bar{u}$	1	
$\boldsymbol{K_0^{*[\pm,0]}}$	$K_0^*(1430)$	$u\bar{s},\ d\bar{s},\ \bar{d}s,\ \bar{u}s$	$\frac{1}{2}$	0^{++}
$\sigma_N,\ \sigma_S$	$f_0(1370),\ f_0(1500)$	$c_1(u\bar{u}+d\bar{d})+c_2(s\bar{s})$	0	
$\boldsymbol{\pi}$	$\{\pi^0,\pi^\pm\}$	$u\bar{d},\ (u\bar{u}-d\bar{d})/\sqrt{2},\ d\bar{u}$	1	
$\boldsymbol{K^{[\pm,0]}}$	$K^{[0,\pm]}$, $K(1460)$, $K(1630)$, $K(1830)$	$u\bar{s},\ d\bar{s},\ \bar{d}s,\ \bar{u}s$	$\frac{1}{2}$	0^{-+}
$\eta_N,\ \eta_S$	$\eta(547),\ \eta'(958)$, $\eta(1295)$, $\eta(1405)$, $\eta(1475)$	$c_1(u\bar{u}+d\bar{d})+c_2(s\bar{s})$	0	
$\boldsymbol{\rho^\mu}$	$\rho(770)$	$u\bar{d},\ (u\bar{u}-d\bar{d})/\sqrt{2},\ d\bar{u}$	1	
$\boldsymbol{K^{*\mu}},\ \boldsymbol{\bar{K}^{*\mu}}$	$K^*(892)$	$u\bar{s},\ d\bar{s},\ \bar{d}s,\ \bar{u}s$	$\frac{1}{2}$	0^{-+}
$\omega_N^\mu,\ \omega_S^\mu$ (small mixing angle)	$\omega(782),\ \phi(1020)$	$c_1(u\bar{u}+d\bar{d})+c_2(s\bar{s})$	0	
$\boldsymbol{a_1^\mu}$	$a_1(1260)$	$u\bar{d},\ (u\bar{u}-d\bar{d})/\sqrt{2},\ d\bar{u}$	1	
$\boldsymbol{K_1^\mu},\ \boldsymbol{\bar{K}_1^\mu}$	$K_{1,A}\equiv K_1(1270)$, $K_1(1400)$	$u\bar{s},\ d\bar{s},\ \bar{d}s,\ \bar{u}s$	$\frac{1}{2}$	0^{-+}
$f_{1N}^\mu,\ f_{1S}^\mu$ (small mixing angle)	$f_1(1285),\ f_1(1420)$	$c_1(u\bar{u}+d\bar{d})+c_2(s\bar{s})$	0	

Table 2.1. Physical meson content of the eLSM for the case $N_f = 3$ flavors.

2.2.2. Mathematical representation of (pseudo)scalar and (axial-)vector fields

1) (Pseudo)scalar and (axial-)vector bilinear forms

In the framework of the eLSM all mesonic fields are assumed to be predominantly quark-antiquark states, such that they can be assigned to quark-antiquark bilinear forms. For scalar and pseudoscalar fields these quadratic forms are given by

$$S^a(x) \equiv \sqrt{2}\,\bar{q}(x)\,T^a\,q(x) \quad \text{and} \quad P^a(x) \equiv \sqrt{2}\,\bar{q}(x)\,i\gamma_5 T^a\,q(x)\ , \tag{2.36}$$

while for the vector and axial-vector fields they read

$$V^{\mu a}(x) \equiv \sqrt{2}\,\bar{q}(x)\,\gamma^\mu T^a\,q(x) \quad \text{and} \quad A^{\mu a}(x) \equiv \sqrt{2}\,\bar{q}(x)\,\gamma^\mu\gamma_5 T^a\,q(x)\ , \tag{2.37}$$

where $a \in \{0, 1, \ldots, N_f^2 - 1\}$. Note, the equivalent sign is used because the left- and right-hand sides in Eqs. (2.36) and (2.37) are not equal due to not matching dimensions.

We study their transformation properties with respect to Lorentz transformations, charge-conjugation, and parity transformations using the behavior of the quark and antiquark fields. With Eqs. (1.105) and (1.106), we obtain the following Lorentz transformations of the fields (2.36) and (2.37):

$$S^a(x) \to S^{a\prime}(x') \equiv \sqrt{2}\bar{q}(\Lambda^{-1}x)S^{-1}(\Lambda)T^a S(\Lambda)q(\Lambda^{-1}x) \equiv S^a(\Lambda^{-1}x) \,, \tag{2.38}$$

$$P^a(x) \to P^{a\prime}(x') \equiv \sqrt{2}\bar{q}(\Lambda^{-1}x)S^{-1}(\Lambda)i\gamma_5 T^a S(\Lambda)q(\Lambda^{-1}x) \equiv P^a(\Lambda^{-1}x) \,, \tag{2.39}$$

$$V^{\mu a}(x) \to V^{\mu a\prime}(x') \equiv \sqrt{2}\bar{q}(\Lambda^{-1}x)S^{-1}(\Lambda)\gamma^\mu T^a S(\Lambda)q(\Lambda^{-1}x) \equiv \Lambda^\mu{}_\nu V^{\nu a}(\Lambda^{-1}x) \,, \tag{2.40}$$

$$A^{\mu a}(x) \to A^{\mu a\prime}(x') \equiv \sqrt{2}\bar{q}(\Lambda^{-1}x)S^{-1}(\Lambda)\gamma^\mu\gamma_5 T^a S(\Lambda)q(\Lambda^{-1}x) \equiv \Lambda^\mu{}_\nu A^{\nu a}(\Lambda^{-1}x) \,, \tag{2.41}$$

where we have used $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$ to compute the transformations of the vector and axial-vector bilinear forms. The results show that the bilinear forms of Eqs. (2.36) and (2.37) indeed transform as scalar objects and Lorentz vectors, respectively.
In order to obtain the discrete symmetry operations of parity and charge-conjugation transformations, we use Eqs. (1.111) – (1.114). For the scalar bilinear form we find

$$S^a(x) \xrightarrow{P} S^{a\prime}(x') \equiv \sqrt{2}\bar{q}(t,-\boldsymbol{r})\gamma^0 T^a\gamma^0 q(t,-\boldsymbol{r}) \equiv S^a(t,-\boldsymbol{r}) \,, \tag{2.42}$$

$$S^a(x) \xrightarrow{C} S^{a\prime}(x') \equiv \sqrt{2}q^T(x)CT^aC\bar{q}^T(x) = \sqrt{2}\bar{q}(x)T^{aT}q(x) \equiv S^{aT}(x) \,, \tag{2.43}$$

where the transposition in Eq. (2.43) acts in spinor space and two sign changes took place: one arises from $CC = -1$ and the second one originates from the interchange of two fermionic fields during the transposition, compare discussion after Eq. (1.120). In combination with the Lorentz transformation (2.38) these results prove that $S^a(x)$ really describes a scalar $J^{PC} = 0^{++}$ object.
In a similar way, we obtain the behavior of the pseudoscalar bilinear form under discrete symmetry transformations :

$$P^a(x) \xrightarrow{P} P^{a\prime}(x') \equiv \sqrt{2}\bar{q}(t,-\boldsymbol{r})\gamma^0 i\gamma_5 T^a\gamma^0 q(t,-\boldsymbol{r}) \equiv -P^a(t,-\boldsymbol{r}) \,, \tag{2.44}$$

$$P^a(x) \xrightarrow{C} P^{a\prime}(x') \equiv \sqrt{2}q^T(x)Ci\gamma_5 T^aC\bar{q}^T(x) = \sqrt{2}\bar{q}(x)i\gamma_5 T^{aT}q(x) \equiv P^{aT}(x) \,, \tag{2.45}$$

proving that $P^a(x)$ really can be assigned to a pseudoscalar $J^{PC} = 0^{-+}$ object.
With Eq. (1.115) we compute the discrete transformations of the vector bilinear form using similar manipulations as before:

$$V^{\mu a}(x) \xrightarrow{P} V^{\mu a\prime}(x') \equiv \sqrt{2}\bar{q}(t,-\boldsymbol{r})\gamma^0\gamma^\mu T^a\gamma^0 q(t,-\boldsymbol{r}) \equiv V^a_\mu(t,-\boldsymbol{r}) \,, \tag{2.46}$$

$$\begin{aligned} V^{\mu a}(x) \xrightarrow{C} V^{\mu a\prime}(x') &\equiv \sqrt{2}q^T(x)C\gamma^\mu T^aC\bar{q}^T(x) \\ &= \sqrt{2}q^T(x)\gamma^{\mu T}T^a\bar{q}^T(x) \\ &= -\sqrt{2}\bar{q}(x)\gamma^\mu T^{aT}q(x) \\ &\equiv -V^{\mu aT}(x) \,, \end{aligned} \tag{2.47}$$

where the covariant index in the result of the parity transformation indicates that the time-like and space-like components transform with a different sign with respect to spatial reflections. This confirms that $V^{\mu a}(x)$ really has the properties of a vector $J^{PC} = 1^{-+}$ object.

Finally, in a similar way as in the previous cases, we find for the axial-vector bilinear form the following properties under parity and charge-conjugation transformations:

$$A^{\mu a}(x) \xrightarrow{P} A^{\mu a\prime}(x') \equiv \sqrt{2}\bar{q}(t,-\boldsymbol{r})\gamma^0\gamma^\mu\gamma_5 T^a\gamma^0 q(t,-\boldsymbol{r}) \equiv -A^a_\mu(t,-\boldsymbol{r}) \ , \tag{2.48}$$

$$\begin{aligned} A^{\mu a}(x) \xrightarrow{C} A^{\mu a\prime}(x') &\equiv \sqrt{2}q^T(x)C\gamma^\mu\gamma_5 T^a C\bar{q}^T(x) \\ &= \sqrt{2}q^T(x)\gamma^{\mu T}\gamma_5^T T^a\bar{q}^T(x) \\ &= \sqrt{2}\bar{q}(x)\gamma^\mu\gamma_5 T^{aT}q(x) \\ &\equiv A^{\mu aT}(x) \ , \end{aligned} \tag{2.49}$$

which reveals that $A^{\mu a}(x)$ really has the properties of an axial-vector $J^{PC} = 1^{++}$ object.

2) eLSM implementation

The bilinear forms of Eqs. (2.36) and (2.37) can be used to represent the different kinds of mesons. Even so, the theoretical implementation of the eLSM uses different mathematical objects which, however, are related to these quark-antiquark quadratic forms. We introduce the matrix

$$\Phi_{ij}(x) \equiv \sqrt{2}\bar{q}_{jR}(x)q_{iL}(x) \ , \tag{2.50}$$

incorporating the (pseudo)scalar degrees of freedom, as well as the left- and right-handed matrices

$$L^\mu_{ij}(x) \equiv \sqrt{2}\bar{q}_{jL}(x)\gamma^\mu q_{iL}(x) \quad \text{and} \quad R^\mu_{ij}(x) \equiv \sqrt{2}\bar{q}_{jR}(x)\gamma^\mu q_{iR}(x) \ , \tag{2.51}$$

which describe the (axial-)vector degrees of freedom. The indices i, j are flavor indices. The spinors q_L and q_R describe the left- and right-handed quark fields, respectively. Using the chiral projection operators (1.129), we show that the objects (2.50) and (2.51) are related to the bilinear forms (2.36) and (2.37). We rewrite the expression (2.50) as

$$\begin{aligned} \Phi_{ij}(x) &\equiv \sqrt{2}\bar{q}_j(x)\mathcal{P}_L\mathcal{P}_L q_i(x) \\ &= \frac{1}{\sqrt{2}}\left[\bar{q}_j(x)q_i(x) + i\bar{q}_j(x)i\gamma_5 q_i(x)\right] \\ &\equiv S_{ij}(x) + iP_{ij}(x) \ , \end{aligned} \tag{2.52}$$

where we introduced the hermitian scalar and pseudoscalar mesonic matrices

$$S_{ij}(x) \equiv \frac{1}{\sqrt{2}}\bar{q}_j(x)q_i(x) \ , \qquad P_{ij}(x) \equiv \frac{1}{\sqrt{2}}\bar{q}_j(x)i\gamma_5 q_i(x) \ . \tag{2.53}$$

On the other hand, we expand the matrix (2.50) in the algebra of $U(3)$, i.e., in terms of the $U(3)$ generators:

$$\Phi(x) = \Phi^a(x)T^a = [S^a(x) + iP^a(x)]T^a \ , \tag{2.54}$$

where the coefficients $S^a(x)$ and $P^a(x)$ are identical to the expressions (2.36). This can be seen by multiplying the expansion with T^b from the right,

$$\begin{aligned}
[S^a(x) + iP^a(x)]T^a_{ij}T^b_{jk} &= [S_{ij}(x) + iP_{ij}(x)]T^b_{jk} \\
&\equiv \Big[\frac{1}{\sqrt{2}}\bar{q}_j(x)q_i(x) + i\frac{1}{\sqrt{2}}\bar{q}_j(x)i\gamma_5 q_i(x)\Big]T^b_{jk} \\
&= \frac{1}{\sqrt{2}}\bar{q}_j(x)T^b_{jk}q_i(x) + i\frac{1}{\sqrt{2}}\bar{q}_j(x)i\gamma_5 T^b_{jk}q_i(x) \ , \qquad (2.55)
\end{aligned}$$

taking the trace, and using the orthogonality of the generators, $\mathrm{Tr}(T^aT^b) = \delta^{ab}/2$,

$$\begin{aligned}
[S^a(x) + iP^a(x)]T^a_{ij}T^b_{ji} &= \frac{1}{2}[S^b(x) + iP^b(x)] \\
&\equiv \frac{1}{\sqrt{2}}\bar{q}_j(x)T^b_{ji}q_i(x) + i\frac{1}{\sqrt{2}}\bar{q}_j(x)i\gamma_5 T^b_{ji}q_i(x) \\
&= \frac{1}{\sqrt{2}}\bar{q}(x)T^b q(x) + i\frac{1}{\sqrt{2}}\bar{q}(x)i\gamma_5 T^b q(x) \ . \qquad (2.56)
\end{aligned}$$

A comparison of the real and imaginary parts proves that the matrix $\Phi_{ij}(x)$ [Eq. (2.50)] indeed contains (pseudo)scalar mesons.

Considering the Eq. (2.54) and remembering the explicit form of the Gell-Mann matrices T^a, we obtain

$$\Phi \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{(\sigma_N + a_0^0) + i(\eta_N + \pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & K_0^{*+} + iK^+ \\ a_0^- + i\pi^- & \frac{(\sigma_N - a_0^0) + i(\eta_N - \pi^0)}{\sqrt{2}} & K_0^{*0} + iK^0 \\ K_0^{*-} + iK^- & \bar{K}_0^{*0} + i\bar{K}^0 & \sigma_S + i\eta_S \end{pmatrix} , \qquad (2.57)$$

where the fields with superscripts 0 and $\pm$ are linear combinations of the isovector components:

$$a_0^{\pm} = \frac{1}{\sqrt{2}}(a_{0,1} \mp ia_{0,2}) \ , \qquad a_0^0 = a_{0,3} \ , \qquad (2.58)$$

$$\pi^{\pm} = \frac{1}{\sqrt{2}}(\pi_1 \mp i\pi_2) \ , \qquad \pi_0^0 = \pi_3 \ , \qquad (2.59)$$

$$K_0^{*\pm} = \frac{1}{\sqrt{2}}(K_{0,1}^{*\pm} \mp iK_{0,2}^{*\pm}) \ , \qquad K_0^{*0} = \frac{1}{\sqrt{2}}(K_{0,1}^{*0} - iK_{0,2}^{*0}) \ , \qquad (2.60)$$

$$\bar{K}_0^{*0} = \frac{1}{\sqrt{2}}(K_{0,1}^{*0} + iK_{0,2}^{*0}) \ , \qquad (2.61)$$

$$K^{\pm} = \frac{1}{\sqrt{2}}(K_1^{\pm} \mp iK_2^{\pm}) \ , \qquad K^0 = \frac{1}{\sqrt{2}}(K_1^0 - iK_2^0)) \ , \qquad (2.62)$$

$$\bar{K}^0 = \frac{1}{\sqrt{2}}(K_1^0 + iK_2^0) \ . \qquad (2.63)$$

	chiral transformation $U(3)_L \times U(3)_R$	parity transformation P	charge conjugation C
$\Phi(x)$	$U_L \Phi(x) U_R^\dagger$	$\Phi^\dagger(t, -\boldsymbol{r})$	$\Phi^T(x)$
$L^\mu(x)$	$U_L L^\mu(x) U_L^\dagger$	$R_\mu(t, -\boldsymbol{r})$	$-(R^\mu)^T(x)$
$R^\mu(x)$	$U_R R^\mu(x) U_R^\dagger$	$L_\mu(t, -\boldsymbol{r})$	$-(L^\mu)^T(x)$

Table 2.2. Transformation properties of Φ, L^μ, and R^μ.

In a similar way as in the (pseudo)scalar sector, we obtain the dependence of the left- and right-handed matrices on the bilinear forms of Eq. (2.37),

$$L_{ij}^\mu(x) = [V^{\mu a}(x) + A^{\mu a}(x)] T_{ij}^a \ , \qquad R_{ij}^\mu(x) = [V^{\mu a}(x) - A^{\mu a}(x)] T_{ij}^a \ , \tag{2.64}$$

showing that the matrices $L_{ij}^\mu(x)$ and $R_{ij}^\mu(x)$ really represent vector and axial-vector objects. The expressions $V^\mu = V^{\mu a} T^a$ and $A^\mu = A^{\mu a} T^a$ in matrix form are given by

$$V^\mu \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} & K^{*\mu +} \\ \rho^{\mu -} & \frac{\omega_N^\mu - \rho^{\mu 0}}{\sqrt{2}} & K^{*\mu 0} \\ K^{*\mu -} & \bar{K}^{*\mu 0} & \omega_S^\mu \end{pmatrix} , \tag{2.65}$$

$$A^\mu \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1N}^\mu + a_1^{\mu 0}}{\sqrt{2}} & a_1^{\mu +} & K_1^{\mu +} \\ a_1^{\mu -} & \frac{f_{1N}^\mu - a_1^{\mu 0}}{\sqrt{2}} & K_1^{\mu 0} \\ K_1^{\mu -} & \bar{K}_1^{\mu 0} & f_{1S}^\mu \end{pmatrix} , \tag{2.66}$$

where similar relations as in the pseudo(scalar) sector define the fields with indices ^{0} and $^\pm$.

Finally, we give the chiral, parity, and charge-conjugation transformations of the objects (2.50) and (2.51). Using the chiral transformations of the left(right)-handed (anti)quark fields [Eqs. (1.136) and (1.137)], as well as Eqs. (2.42) and the following ones for the parity and the charge-conjugation transformations of the (pseudo)scalar and (axial-)vector bilinear forms, we obtain the transformation properties as summarized in Tab. 2.2.

2.2.3. The Lagrangian

The eLSM Lagrangian follows from requiring the symmetry under parity, charge-conjugation, (global) chiral, and dilatation transformations as well as featuring the explicit, spontaneous, and anomalous symmetry breaking in accordance with QCD. For $N_f = 3$, mesons are described by [3]

$$\begin{aligned}\mathcal{L}_{\text{meson}} = &\operatorname{Tr}\left[(D^\mu\Phi)^\dagger D_\mu\Phi\right] - m_0^2 \operatorname{Tr}\left(\Phi^\dagger\Phi\right) - \lambda_1\left[\operatorname{Tr}\left(\Phi^\dagger\Phi\right)\right]^2 - \lambda_2 \operatorname{Tr}\left[\left(\Phi^\dagger\Phi\right)^2\right] \\ &- \frac{1}{4}\operatorname{Tr}\left(L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}\right) + \operatorname{Tr}\left[\left(\frac{m_1^2}{2} + \Delta\right)\left(L_\mu L^\mu + R_\mu R^\mu\right)\right] \\ &+ \operatorname{Tr}\left[H\left(\Phi + \Phi^\dagger\right)\right] + c_1\left(\det\Phi - \det\Phi^\dagger\right)^2 \\ &+ i\frac{g_2}{2}\left[\operatorname{Tr}\left(L_{\mu\nu}\left[L^\mu, L^\nu\right]\right) + \operatorname{Tr}\left(R_{\mu\nu}\left[R^\mu, R^\nu\right]\right)\right] \\ &+ \frac{h_1}{2}\operatorname{Tr}\left(\Phi^\dagger\Phi\right)\operatorname{Tr}\left(L_\mu L^\mu + R_\mu R^\mu\right) + h_2 \operatorname{Tr}\left[(L_\mu\Phi)^\dagger(L^\mu\Phi) + (\Phi R_\mu)^\dagger(\Phi R^\mu)\right] \\ &+ 2h_3 \operatorname{Tr}\left(\Phi R^\mu \Phi^\dagger L^\mu\right) + g_3\left[\operatorname{Tr}\left(L_\mu L_\nu L^\mu L^\nu\right) + \operatorname{Tr}\left(R_\mu R_\nu R^\mu R^\nu\right)\right] \\ &+ g_4\left[\operatorname{Tr}\left(L_\mu L^\mu L_\nu L^\nu\right) + \operatorname{Tr}\left(R_\mu R^\mu R_\nu R^\nu\right)\right] + g_5 \operatorname{Tr}\left(L_\mu L^\mu\right)\operatorname{Tr}\left(R_\nu R^\nu\right) \\ &+ g_6\left[\operatorname{Tr}\left(L_\mu L^\mu\right)\operatorname{Tr}\left(L_\nu L^\nu\right) + \operatorname{Tr}\left(R_\mu R^\mu\right)\operatorname{Tr}\left(R_\nu R^\nu\right)\right] \ . \end{aligned} \tag{2.67}$$

The covariant derivative includes the constant g_1 parametrizing the coupling of scalar and pseudoscalar mesonic fields with vector and axial-vector fields,

$$D^\mu\Phi = \partial^\mu\Phi - ig_1\left(L^\mu\Phi - \Phi R^\mu\right) \ , \tag{2.68}$$

and, in the case of global chiral symmetry, the left- and right-handed field-strength tensors are given by

$$R^{\mu\nu} = \partial^\mu R^\nu - \partial^\nu R^\mu \ , \qquad L^{\mu\nu} = \partial^\mu L^\nu - \partial^\nu L^\mu \ . \tag{2.69}$$

(Note, it is also possible to extend the covariant derivative and the field-strength tensors, in order to include electromagnetic interactions. For details of this approach, compare Ref. [1].) The matrices H and Δ are defined as

$$H = \operatorname{diag}\left(\frac{h_{0N}}{2}, \frac{h_{0N}}{2}, \frac{h_{0S}}{\sqrt{2}}\right) \ , \qquad \Delta = \operatorname{diag}(\delta_N, \delta_N, \delta_S) \ . \tag{2.70}$$

In this form the mesonic part of the eLSM Lagrangian contains the following 18 parameters:

$$m_0^2,\ m_1^2,\ c_1,\ \delta_N,\ \delta_S,\ g_1,\ g_2,\ g_3,\ g_4,\ g_5,\ g_6,\ h_{0N},\ h_{0S},\ h_1,\ h_2,\ h_3,\ \lambda_1,\ \lambda_2\ . \tag{2.71}$$

However, not all of these are important. As we explain in Sec. 2.2.5, only the following eleven parameters have to be considered [3]:

$$\begin{aligned} C_1 = m_0^2 + \lambda_1(\phi_N^2 + \phi_S^2)\ , \quad C_2 = m_1^2 + \frac{h_1}{2}(\phi_N^2 + \phi_S^2)\ , \\ c_1\ ,\quad \delta_S\ ,\quad g_1\ ,\quad g_2\ ,\quad \phi_N\ ,\quad \phi_S\ ,\quad h_2\ ,\quad h_3\ ,\quad \lambda_2\ . \end{aligned} \tag{2.72}$$

In the following paragraphs we briefly discuss the terms of the eLSM Lagrangian (2.67).

Kinetic and mass terms

It is well known that the dynamics of (pseudo)scalar fields should be described by Klein-Gordon Lagrangians. The implementation in terms of the matrix Φ has a similar form, but requires the usage of traces in order to ensure the invariance under chiral transformations. The kinetic and mass terms of scalar and pseudoscalar fields read

$$\mathrm{Tr}[(\partial_\mu\Phi)^\dagger\partial^\mu\Phi] - m_0^2\,\mathrm{Tr}(\Phi^\dagger\Phi)\ . \tag{2.73}$$

The constant m_0^2 has dimension [energy^2] and thus, in this from the mass term is not dilatation invariant. However, after incorporating the dilaton field (1.163) [which is predominantly assigned to $f_0(1710)$ [4]], the dilatation symmetry is manifest again, see Refs. [2, 3].

For one more reason, one has to give special attention to m_0, because by choosing $m_0^2 < 0$ it allows to model SSB. According to Goldstone's theorem [39], this causes the emergence of eight Nambu-Goldstone bosons, due to the eight broken generators of the $SU(3)_A$ symmetry ($U(1)_A$ is anomalously broken). However, since the eLSM features also the explicit breaking of chiral symmetry, we are left with Pseudo-Nambu-Goldstone bosons. They can be identified with the pseudoscalar octet.

Using the field-strength tensors (2.69) the kinetic and mass terms of the vector and axial-vector fields are given by the usual form, however, equipped with a trace:

$$-\frac{1}{4}\,\mathrm{Tr}(L_{\mu\nu}L^{\mu\nu} + R_{\mu\nu}R^{\mu\nu}) + \mathrm{Tr}\left[\frac{m_1^2}{2}(L_\mu L^\mu + R_\mu R^\mu)\right] + \mathrm{Tr}\left[\Delta\left(L_\mu L^\mu + R_\mu R^\mu\right)\right]\ , \tag{2.74}$$

where the constant m_1^2 as well as the parameters $\delta_N \sim m_u^2 = m_d^2$ and $\delta_S \sim m_s^2$ (contained in Δ) have dimension [energy^2], and thus break the dilatation symmmetry. In the case of δ_N and δ_S this is expected since they parametrize bare quark masses which generate an explicit breaking of dilatation symmetry. The invariance of the m_1^2 term, however, is restored when incorporating the dilaton field into the model, see Ref. [2, 3].

Scalar and pseudoscalar self-interactions

The interaction terms involving only (pseudo)scalar fields are straightforward extensions of the mass term $\sim \mathrm{Tr}(\Phi^\dagger\Phi)$. One can either the complete trace or first take the square of $\Phi^\dagger\Phi$ and then take the trace. Note, as seen from the perspective of P, C, and chiral symmetry requirements self-interaction terms of higher order in $\Phi^\dagger\Phi$ are also possible, but the number of possible terms is restricted by assuming dilatation invariance [2, 3]. We obtain

$$-\lambda_1\left[\mathrm{Tr}\left(\Phi^\dagger\Phi\right)\right]^2 - \lambda_2\,\mathrm{Tr}\left[\left(\Phi^\dagger\Phi\right)^2\right]\ . \tag{2.75}$$

Both constants, λ_1 and λ_2, are dimensionless parameters. [However, the first term is expected to be smaller than the second one, because it is large-N_c suppressed, see Ref. [3].]

Vector and axial-vector self-interactions

The Lagrangian contains five interaction terms involving only (axial-)vector degrees of freedom. They are parametrized by the dimensionless constants g_2, g_3, g_4, g_5, and g_6,

$$i\frac{g_2}{2}\left[\text{Tr}\left(L_{\mu\nu}\left[L^\mu, L^\nu\right]\right) + \text{Tr}\left(R_{\mu\nu}\left[R^\mu, R^\nu\right]\right)\right] + g_3\left[\text{Tr}\left(L_\mu L_\nu L^\mu L^\nu\right) + \text{Tr}\left(R_\mu R_\nu R^\mu R^\nu\right)\right] \\ + g_4\left[\text{Tr}\left(L_\mu L^\mu L_\nu L^\nu\right) + \text{Tr}\left(R_\mu R^\mu R_\nu R^\nu\right)\right] + g_5\,\text{Tr}\left(L_\mu L^\mu\right)\text{Tr}\left(R_\nu R^\nu\right) \\ + g_6\left[\text{Tr}\left(L_\mu L^\mu\right)\text{Tr}\left(L_\nu L^\nu\right) + \text{Tr}\left(R_\mu R^\mu\right)\text{Tr}\left(R_\nu R^\nu\right)\right] \; . \tag{2.76}$$

Differently from the (pseudo)scalar sector, here it is possible to define interaction terms which contain three field variables, although Lorentz symmetry requires the inclusion of a field-strength tensor. This entails the emergence of derivatively coupled interaction terms.

Another chirally invariant possibility is the combination of four left- and right-handed fields. The first realization are the terms proportional to g_3 and g_4, which contain either four left- or four right-handed fields and differ only in the order of the Lorentz indices. Since the commutators of left- and right-handed fields is non-zero, these are indeed different structures. On the other hand, instead of using one trace, we can utilize the product of two independent traces. This allows the combination of left- and right-handed fields in the interaction term proportional to g_5, but also allows for a further possibility to couple either four left- or four right-handed fields as realized in the term proportional to g_6. [The terms parametrized by g_5 and g_6 are large-N_c suppressed, see Ref. [3].]

Scalar/pseudoscalar and vector/axial-vector interactions

Finally, the last type of interaction terms includes both scalar and pseudoscalar as well as vector and axial-vector fields:

$$+ g_1^2\,\text{Tr}\left[(L_\mu\Phi - \Phi R_\mu)^\dagger(L^\mu\Phi - \Phi R^\mu)\right] + \frac{h_1}{2}\,\text{Tr}\left(\Phi^\dagger\Phi\right)\text{Tr}\left(L_\mu L^\mu + R_\mu R^\mu\right) \\ + h_2\,\text{Tr}\left[(L_\mu\Phi)^\dagger(L^\mu\Phi) + (\Phi R_\mu)^\dagger(\Phi R^\mu)\right] + 2h_3\,\text{Tr}\left(\Phi R^\mu\Phi^\dagger L^\mu\right) \; . \tag{2.77}$$

The term parametrized by the dimensionless constant g_1 arises from the covariant derivative (2.68). Further interaction terms are parametrized by the dimensionless coupling constants h_1 [which is, however, large-N_c suppressed, see Ref. [3]], as well as h_2 and h_3.

Explicit symmetry breaking and anomaly

The eLSM also includes explicit symmetry breaking terms in order to avoid massless pseudoscalar degrees of freedom which arise after SSB as Goldstone bosons, compare the discussion of the LSM in Sec. 2.1. Indeed, in Nature the chiral symmetry is explicitly broken due to non-vanishing quark masses, see Sec. 1.3.4.

The explicit breaking of chiral symmetry is modeled by the term

$$\text{Tr}\left[H\left(\Phi + \Phi^\dagger\right)\right] \; , \tag{2.78}$$

where the matrix H [Eq. (2.70)] contains the parameters $h_{0N} \sim m_u = m_d$ and $h_{0S} \sim m_s$ which are related to the quark masses and have dimension [energy^3]. Thus, they also explicitly break dilatation symmetry, as expected from parameters describing the bare quark masses.

Furthermore, the eLSM Lagrangian also implements the $U(1)_A$ anomaly, which is crucial to understand the mesons η and η'. It is implemented by the term

$$c_1 \left(\det \Phi - \det \Phi^\dagger\right)^2 \ , \tag{2.79}$$

where the parameter c_1 has dimension [energy^{-2}].[2] The term (2.79) breaks the symmetry under $U(1)_A$ transformations, because the determinant is invariant under $SU(3)_{L(R)}$ transformations but not under $U(1)_A$ transformations:

$$\begin{aligned} c_1(\det \Phi - \det \Phi^\dagger)^2 \ &\xrightarrow{\chi}\ c_1(\det \Phi' - \det \Phi^{\dagger\prime})^2 \\ &= c_1 \left[\det(U_L \Phi U_R^\dagger) - \det(U_R \Phi^\dagger U_L^\dagger)\right]^2 \\ &= c_1 \left[\det(U_L)\det(\Phi)\det(U_R^\dagger) - \det(U_R)\det(\Phi^\dagger)\det(U_L^\dagger)\right]^2 \\ &= c_1 \left\{\det\left[e^{i(\theta_R^0-\theta_L^0)T_0}\Phi\right] - \det\left[e^{-i(\theta_R^0-\theta_L^0)T_0}\Phi^\dagger\right]\right\}^2 \\ &\sim c_1 \left\{\det\left[e^{i2\theta_A^0 T_0}\Phi\right] - \det\left[e^{-i2\theta_A^0 T_0}\Phi^\dagger\right]\right\}^2 \\ &\neq c_1(\det \Phi - \det \Phi^\dagger)^2 \ , \end{aligned} \tag{2.80}$$

where we have used that the determinants of $U_{L(R)} \in SU(3)_{L(R)}$ are equal to one. Furthermore, we applied Eq. (1.144) to identify the remaining complex phases as $U(1)_A$ transformations. This proves that the $U(1)_A$ symmetry is indeed broken. The anomaly term influences the masses of the (pseudo)scalar mesons and is responsible for the large mass of η' (around 1 GeV).

2.2.4. Spontaneous symmetry breaking in the extended linear sigma model

The spontaneous breaking of chiral symmetry is implemented by choosing the parameter $m_0^2 < 0$. Then the potential is of a similar form as in the Lagrangian (1.66) and the two $I(J^{PC}) = 0(0^{++})$ fields σ_N and σ_S develop non-zero VEVs

$$\langle \sigma_N \rangle = \phi_N \neq 0 \ , \tag{2.81}$$
$$\langle \sigma_S \rangle = \phi_S \neq 0 \ , \tag{2.82}$$

which entails the SSB. The subsequent procedure is in principle the same as in Sec. 1.2.4. We expand the σ_N and σ_S fields around the VEVs,

$$\sigma_N \ \to\ \sigma_N + \phi_N \ , \qquad \sigma_S \ \to\ \sigma_S + \phi_S \ , \tag{2.83}$$

[2] Note, dilatation invariance does not apply for the anomaly.

and substitute these shifts into the Lagrangian. It may sound trivially the same as in previously discussed SSB procedures, but in the eLSM the situation is a bit more difficult. The difference arises from the more complicated term structure of the eLSM Lagrangian, which will lead to additional terms inducing a mixing between different types of mesonic fields. The subtle point is the term $\mathrm{Tr}[(D_\mu\Phi)^\dagger D^\mu\Phi]$. It delivers the following mixing terms between pseudoscalar and axial-vector fields and scalar and vector fields [3]:

$$\begin{aligned}
&- g_1\phi_N(f_{1N}^{\mu}\partial_\mu\eta_N + \boldsymbol{a_1}^{\mu}\cdot\partial_\mu\boldsymbol{\pi}) - \sqrt{2}g_1\phi_S f_{1S}^{\mu}\partial_\mu\eta_S \\
&- \frac{g_1}{2}(\phi_N + \sqrt{2}\phi_S)(K_1^{0\mu}\partial_\mu\bar{K}^0 + K_1^{+\mu}\partial_\mu K^- + \bar{K}_1^{0,\mu}\partial_\mu K^0 + K_1^{-\mu}\partial_\mu K^+) \\
&+ i\frac{g_1}{2}(\phi_N - \sqrt{2}\phi_S)(K^{*0\mu}\partial_\mu\bar{K}_0^{*0} + K^{*+\mu}\partial_\mu K_0^{*-} - \bar{K}^{*0\mu}\partial_\mu K_0^{*0} - K^{*-\mu}\partial_\mu K_0^{*+}) \ . \qquad (2.84)
\end{aligned}$$

This causes non-diagonal contributions in the quadratic part of the Lagrangian. It is advantageous to eliminate the mixing terms (2.84) have to be eliminated. To this end, we perform the following shifts of the axial-vector fields,

$$f_{1N/S}^{\mu} \rightarrow f_{1N/S}^{\mu} + w_{f_{1N/S}}\partial^\mu\eta_{N/S} \ , \qquad (2.85)$$

$$a_1^{\mu[\pm,0]} \rightarrow a_1^{\mu[\pm,0]} + w_{a_1}\partial^\mu\pi^{[\pm,0]} \ , \qquad (2.86)$$

$$K_1^{\mu[\pm,0,\bar{0}]} \rightarrow K_1^{\mu[\pm,0,\bar{0}]} + w_{K_1}\partial^\mu K^{[\pm,0,\bar{0}]} \ , \qquad (2.87)$$

and vector fields,

$$K^{*\mu[\pm,0,\bar{0}]} \rightarrow K^{*\mu[\pm,0,\bar{0}]} + w_{K^*}\partial^\mu K_0^{*[\pm,0,\bar{0}]} \ , \qquad (2.88)$$

where we have used a short-hand notation to group together the isotriplet and isodoublet states using the index "$[\pm, 0\ (,\bar{0})]$", where $\bar{0}$ refers to $\bar{K}$ or $\bar{K}_0^*$. The constants w_{a_1}, $w_{f_{1N}}$, $w_{f_{1S}}$, w_{K_1}, and w_{K^*} are adjusted such that the shifts (2.85) to (2.88) generate terms which cancel the terms (2.84):

$$w_{f_{1N}} = w_{a_1} = \frac{g_1\phi_N}{m_{a_1}^2} \ , \qquad (2.89)$$

$$w_{f_{1S}} = \frac{\sqrt{2}g_1\phi_S}{m_{f_{1S}}^2} \ , \qquad (2.90)$$

$$w_{K_1} = \frac{g_1(\phi_N + \sqrt{2}\phi_S)}{2m_{K_1}^2} \ , \qquad (2.91)$$

$$w_{K^*} = \frac{ig_1(\phi_N - \sqrt{2}\phi_S)}{2m_{K^*}^2} \ , \qquad (2.92)$$

where the notation is simplified by using the expressions of the squared tree-level masses of the mesons, which we have anticipated from Eqs. (2.113) to (2.120).
These shifts, however, also lead to additional terms contributing to the kinetic terms of η_N, η_S, and $\pi^{[\pm,0]}$, $K^{[\pm,0,\bar{0}]}$, as well as $K_0^{*[\pm,0,\bar{0}]}$. These additional contributions to the usual factor of $\frac{1}{2}$ in the kinetic terms are problematic, because it becomes impossible to

interpret the operators emerging from the Fourier decomposition of these fields as creation and annihilation operators of normalized one-meson states. In order to solve this problem, i.e., to obtain canonically normalized kinetic terms, we have to introduce the following field renormalizations:

$$\pi^{[\pm,0]} \rightarrow Z_\pi \pi^{[\pm,0]} \ , \tag{2.93}$$
$$K^{[\pm,0,\bar{0}]} \rightarrow Z_K K^{[\pm,0,\bar{0}]} \ , \tag{2.94}$$
$$\eta_{N/S} \rightarrow Z_{\eta_{N/S}} \eta_{N/S} \ , \tag{2.95}$$
$$K_0^{*\mu[\pm,0,\bar{0}]} \rightarrow Z_{K_0^*} K_0^{*\mu[\pm,0,\bar{0}]} \ , \tag{2.96}$$

with

$$Z_\pi = Z_{\eta_N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2\phi_N^2}} = (1 - g_1\phi_N w_{a_1})^{-\frac{1}{2}} \ , \tag{2.97}$$
$$Z_{\eta_S} = \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2\phi_S^2}} = (1 - \sqrt{2}g_1\phi_S w_{f_{1S}})^{-\frac{1}{2}} \ , \tag{2.98}$$
$$Z_K = \frac{m_{K_1}}{\sqrt{4m_{K_1}^2 - g_1^2(\phi_N^2 + \sqrt{2}\phi_S)^2}} = [1 - \frac{g_1}{2}(\phi_N + \sqrt{2}\phi_S)w_{K_1}]^{-\frac{1}{2}} \ , \tag{2.99}$$
$$Z_{K_0^*} = \frac{m_{K^*}}{\sqrt{4m_{K^*}^2 - g_1^2(\phi_N^2 - \sqrt{2}\phi_S)^2}} = [1 + i\frac{g_1}{2}(\phi_N + \sqrt{2}\phi_S)w_{K^*}]^{-\frac{1}{2}} \ , \tag{2.100}$$

where have we used the definitions of w_{a_1}, $w_{f_{1N}}$, $w_{f_{1S}}$, w_{K_1} and w_{K^*} in Eqs. (2.89) to (2.92).

$$m_\pi^2 = Z_\pi^2 \left[m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 + \lambda_1\phi_S^2\right] \equiv \frac{Z_\pi^2 h_{0N}}{\phi_N} \ , \tag{2.101}$$
$$m_K^2 = Z_K^2 \left[m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 - \frac{\lambda_2}{\sqrt{2}}\phi_N\phi_S + (\lambda_1 + \lambda_2)\phi_S^2\right] \ , \tag{2.102}$$
$$m_{\eta_N}^2 = Z_\pi^2 \left[m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 + \lambda_1\phi_S^2 + c_1\phi_N^2\phi_S^2\right] \equiv Z_\pi^2\left(\frac{h_{0N}}{\phi_N} + c_1\phi_N^2\phi_S^2\right) \ , \tag{2.103}$$
$$m_{\eta_S}^2 = Z_{\eta_S}^2 \left[m_0^2 + \lambda_1\phi_N^2 + (\lambda_1 + \lambda_2)\phi_S^2 + \frac{c_1}{4}\phi_N^4\right] \equiv Z_{\eta_S}^2\left(\frac{h_{0S}}{\phi_S} + \frac{c_1}{4}\phi_N^4\right) \ , \tag{2.104}$$
$$m_{\eta_{NS}}^2 = Z_\pi Z_{\pi_S}\frac{c_1}{2}\phi_N^3\phi_S \ , \tag{2.105}$$

while the squared masses of the scalars are

$$m_{a_0}^2 = m_0^2 + \left(\lambda_1 + \frac{3}{2}\lambda_2\right)\phi_N^2 + \lambda_1\phi_S^2 \ , \tag{2.106}$$

$$m_{K_0^*}^2 = Z_{K_0^*}^2\left[m_0^2 + \left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 + \frac{\lambda_2}{\sqrt{2}}\phi_N\phi_S + (\lambda_1+\lambda_2)\phi_S^2\right] \ , \tag{2.107}$$

$$m_{\sigma_N}^2 = m_0^2 + 3\left(\lambda_1 + \frac{\lambda_2}{2}\right)\phi_N^2 + \lambda_1\phi_S^2 \ , \tag{2.108}$$

$$m_{\sigma_S}^2 = m_0^2 + \lambda_1\phi_N^2 + 3(\lambda_1+\lambda_2)\phi_S^2 \ , \tag{2.109}$$

$$m_{\sigma_{NS}}^2 = 2\lambda_1\phi_N\phi_S \ . \tag{2.110}$$

For η_N and η_S, as well as for σ_N and σ_S mixing terms between the strange and non-strange sector appear. The related quantities $m_{\eta_{NS}}^2$ and $m_{\sigma_{NS}}^2$ can be removed by orthogonal transformations, which result in the mass eigenvalues of η and η' as well as f_0^H and f_0^L [3]:

$$m_{\eta/\eta'}^2 = \frac{1}{2}\left[m_{\eta_N}^2 + m_{\eta_S}^2 \pm \sqrt{(m_{\eta_N}^2 - m_{\eta_S}^2)^2 + 4m_{\eta_{NS}}}\right] \ , \tag{2.111}$$

$$m_{f_0^H/f_0^L}^2 = \frac{1}{2}\left[m_{\sigma_N}^2 + m_{\sigma_S}^2 \pm \sqrt{(m_{\sigma_N}^2 - m_{\sigma_S}^2)^2 + 4m_{\sigma_{NS}}^2}\right] \ . \tag{2.112}$$

Finally, the squared masses of the vectors are given by

$$m_{\omega_N}^2 = m_\rho^2 = m_1^2 + \frac{1}{2}(h_1+h_2+h_3)\phi_N^2 + \frac{h_1}{2}\phi_S^2 + 2\delta_N \ , \tag{2.113}$$

$$m_{\omega_S}^2 = m_1^2 + \frac{h_1}{2}\phi_N^2 + \left(\frac{h_1}{2} + h_2 + h_3\right)\phi_S^2 + 2\delta_S \ , \tag{2.114}$$

$$m_{K^*}^2 = m_1^2 + \frac{1}{4}(g_1^2 + 2h_1 + h_2)\phi_N^2 + \frac{1}{\sqrt{2}}\phi_N\phi_S(h_3 - g_1)^2 \tag{2.115}$$

$$+ \frac{1}{2}(g_1^2 + h_1 + h_2)\phi_S^2 + \delta_N + \delta_S \ , \tag{2.116}$$

and the squared masses of the axial-vectors read

$$m_{f_{1N}}^2 = m_{a_1}^2 = m_1^2 + \frac{1}{2}(2g_1^2 + h_1 + h_2 - h_3)\phi_N^2 + \frac{h_1}{2}\phi_S^2 + 2\delta_N \ , \tag{2.117}$$

$$m_{f_{1S}}^2 = m_1^2 + \frac{h_1}{2}\phi_N^2 + (2g_1^2 + \frac{h_1}{2} + h_2 - h_3)\phi_S^2 + 2\delta_S \ , \tag{2.118}$$

$$m_{K_1}^2 = m_1^2 + \frac{1}{4}(g_1^2 + 2h_1 + h_2)\phi_N^2 - \frac{1}{\sqrt{2}}\phi_N\phi_S(h_3 - g_1^2) \tag{2.119}$$

$$+ \frac{1}{2}(g_1^2 + h_1 + h_2)\phi_S^2 + \delta_N + \delta_S \ . \tag{2.120}$$

Finally, we summarize the important points of procedure entailed from SSB in the eLSM:

- Investigating fluctuations of σ_N and σ_S around their non-zero VEVs (2.83) leads to non-diagonal terms in some bilinear terms (2.84).

- These non-physical terms have to be eliminated by performing the shifts (2.85) to (2.88).
- Finally, in order to ensure canonically normalized kinetic terms, the field renormalizations (2.93) to (2.96) have to be performed.

2.2.5. Results

We briefly present results obtained from the eLSM as studied in Ref. [3]. As already mentioned, not all parameters of the Lagrangian in the original form (2.67) are important. The following restrictions can be made:

- The constants g_3, g_4, g_5 and g_6 are not considered, because these parameters do not influence any of the investigated decays.
- The exploration of the vector-meson mass term $\mathrm{Tr}[(m_1^2/2+\Delta)(L_\mu L^\mu + R_\mu R^\mu)]$ shows that one can redefine the parameter m_1 such that only the linear combination $\delta_S - \delta_N$ appears in the mass formulas. That is, we can determine only this difference by a fit of the (axial-)vector masses. Then we can also set $\delta_N = 0$ from the very beginning and fix only δ_S with the fit.
- In the (pseudo)scalar sector of the explicit symmetry breaking, we observe that the parameters h_{0N} and h_{0S} are uniquely determined from the mass terms of the pion (2.101) and the strange part of the eta meson η_S (2.104). Thus, the VEVs ϕ_N and ϕ_S are used instead.
- Omitting isoscalar-scalar mesons, the constants m_0 and λ_1, as well as m_1 and h_1 can be determined only in the combinations $C_1 = m_0^2 + \lambda_1(\phi_N^2 + \phi_S^2)$ and $C_2 = m_1^2 + \frac{h_1}{2}(\phi_N^2 + \phi_S^2)$.

Finally, we are left with the eleven parameters in Eq. (2.72) which are fitted and compared to 21 experimental known quantities [58]: the masses and decay widths of (pseudo)scalar and (axial-)vector mesons, as well as the weak decay constants:

$$f_\pi = (92.2 \pm 4.6)\,\mathrm{MeV}\ , \qquad f_K = (155.6/\sqrt{2} \pm 5.5)\,\mathrm{MeV}\ , \tag{2.121}$$

which are related to the renormalization factors Z_π, Z_K, and the VEVs through the formulas [3, 59]

$$Z_\pi = \frac{\phi_N}{f_\pi}\ , \qquad Z_K = \frac{\phi_N + \sqrt{2}\phi_S}{2 f_K}\ . \tag{2.122}$$

The resulting parameter values are summarized in Tab. 2.3.

Parameter	Value
C_1 [GeV^2]	-0.9183 ± 0.0006
C_2 [GeV^2]	0.4135 ± 0.0147
c_1 [GeV^{-2}]	450.5420 ± 7.0339
δ_S [GeV^2]	0.1511 ± 0.0038
g_1	5.8433 ± 0.0176
g_2	3.0250 ± 0.2329
ϕ_N [GeV]	0.1646 ± 0.0001
ϕ_S [GeV]	0.1262 ± 0.0001
h_2	9.8796 ± 0.6627
h_3	4.8667 ± 0.0864
λ_2	68.2972 ± 0.0435

Table 2.3. Parameter values of the mesonic sector of the eLSM [3].

The numerical values of the renormalization constants are [3]

$$Z_\pi = 1.785 \;, \tag{2.123}$$
$$Z_K = 1.559 \;, \tag{2.124}$$
$$Z_{\eta_S} = 1.466 \;, \tag{2.125}$$

and those of the w-parameters read [3]

$$w_{a_1} = 0.636 \, \frac{1}{\text{GeV}} \;, \tag{2.126}$$
$$w_{K_1} = 0.620 \, \frac{1}{\text{GeV}} \;, \tag{2.127}$$
$$w_{f_{1S}} = 0.513 \, \frac{1}{\text{GeV}} \;. \tag{2.128}$$

In this form, the eLSM has been proven to describe mesonic masses and decays of quark-antiquark mesons up to 1.7 GeV within reasonable accuracy ($\sim 5\%$) [3].

2.3. The nucleon and its chiral partner in the two-flavor extended linear sigma model

Now, we turn to the baryonic sector of the eLSM for the case $N_f = 2$, which was first developed in Ref. [12] and further investigated in Refs. [13, 14, 15]. In contrast to the

LSM, it contains not only the nucleons but also describes the chiral partner of the nucleon ($J^P = 1/2^-$). In the LSM the nucleon mass was generated solely (apart from small corrections arising from the explicit symmetry breaking) by SSB, see Sec. 2.1. The situation is different in the eLSM where the chiral partner of the nucleon is included in the so-called "mirror assignment" [10, 11, 60, 61], which assumes that the chiral partner transforms in a "mirror" way under chiral transformations compared to the nucleon (in contrast to the so-called "naive assignment", where the chiral partners transforms as the nucleon). In this case, it is possible to construct an additional chirally symmetric mass term contributing to the masses of the nucleon and its chiral partner. Physically this parametrizes contributions arising from other condensates, such as e.g., a gluon or a four-quark condensate.

After introducing the mathematical representation of the nucleon and its chiral partner in the naive and the mirror assignment in Sec. 2.3.1, we present the eLSM Lagrangian for both cases in Secs. 2.3.2 and 2.3.2. Then, Sec. 2.3.3 summarizes the results and draws conclusions.

2.3.1. Mathematical representation of the nucleon and its chiral partner

Besides the nucleon N with $J^P = 1/2^+$, the eLSM contains also its chiral partner N^* with $J^P = 1/2^-$. According to Ref. [18], the chiral partner can be assigned to $N(1535)$ or $N(1650)$.

The nucleon and its chiral partner are implemented by the positive-parity spinor Ψ_1 and the negative-parity spinor Ψ_2:

$$\Psi_1(t,\boldsymbol{r}) \stackrel{P}{\to} \gamma^0\Psi_1(t,-\boldsymbol{r}) \; , \qquad \Psi_2(t,\boldsymbol{r}) \stackrel{P}{\to} -\gamma^0\Psi_2(t,-\boldsymbol{r}) \; , \tag{2.129}$$

$$\bar{\Psi}_1(t,\boldsymbol{r}) \stackrel{P}{\to} \bar{\Psi}_1(t,-\boldsymbol{r})\gamma^0 \; , \qquad \bar{\Psi}_2(t,\boldsymbol{r}) \stackrel{P}{\to} -\bar{\Psi}_2(t,-\boldsymbol{r})\gamma^0 \; , \tag{2.130}$$

which mix to produce the physical states N and N^*. Their charge-conjugation transformations are given by

$$\Psi_1 \stackrel{C}{\to} C\bar{\Psi}_1^T \; , \qquad \Psi_2 \stackrel{C}{\to} -C\bar{\Psi}_2^T \; , \tag{2.131}$$

$$\bar{\Psi}_1 \stackrel{C}{\to} \Psi_1^T C \; , \qquad \bar{\Psi}_2 \stackrel{C}{\to} -\Psi_2^T C \; . \tag{2.132}$$

Considering the chiral transformations of Ψ_1 and Ψ_2, two possibilities exist:

- the naive assignment where both fields transform in the same way,

$$\Psi_{1L} \stackrel{\chi}{\to} U_L\Psi_{1L} \; , \qquad \Psi_{1R} \stackrel{\chi}{\to} U_R\Psi_{1R} \; ,$$
$$\Psi_{2L} \stackrel{\chi}{\to} U_L\Psi_{2L} \; , \qquad \Psi_{2R} \stackrel{\chi}{\to} U_R\Psi_{2R} \; , \tag{2.133}$$

- and the mirror assignment where the field Ψ_2 transforms in a "mirror" way compared to Ψ_1. The left-handed component of Ψ_2 transforms with the right-handed chiral transformation matrix U_R, and vice versa:

$$\begin{aligned} \Psi_{1L} &\xrightarrow{\chi} U_L \Psi_{1L} , \qquad & \Psi_{1R} &\xrightarrow{\chi} U_R \Psi_{1R} , \\ \Psi_{2L} &\xrightarrow{\chi} U_R \Psi_{2L} , \qquad & \Psi_{2R} &\xrightarrow{\chi} U_L \Psi_{2R} . \end{aligned} \tag{2.134}$$

The latter assignment was first introduced in Ref. [10] and further investigated in Refs. [11, 60, 61]. Most notably, it allows for a mass term within the Lagrangian. It has been investigated not only in the vacuum [11, 12, 13, 16, 62], but also at non-zero density [60, 63].

For $N_f = 2$ the (pseudo)scalar and (axial-)vector degrees of freedom are included in the following matrices, which are the $N_f = 2$ reduced versions of Eqs. (2.57), (2.65), and (2.66), see Ref. [1]:

$$\Phi \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{(\sigma_N + a_0^0) + i(\eta_N + \pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ \\ a_0^- + i\pi^- & \frac{(\sigma_N - a_0^0) + i(\eta_N - \pi^0)}{\sqrt{2}} \end{pmatrix} , \tag{2.135}$$

$$V^\mu \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} \\ \rho^{\mu -} & \frac{\omega_N^\mu - \rho^{\mu 0}}{\sqrt{2}} \end{pmatrix} , \quad A^\mu \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1N}^\mu + a_1^{\mu 0}}{\sqrt{2}} & a_1^{\mu +} \\ a_1^{\mu -} & \frac{f_{1N}^\mu - a_1^{\mu 0}}{\sqrt{2}} \end{pmatrix} . \tag{2.136}$$

They are commonly written in the basis of the four $U(2)$ generators, $T^0 = \mathbb{1}/2$ and $T^a = \tau^a/2 \; \forall \; a \in \{1, 2, 3\}$. The two-flavor (pseudo)scalar mesons are included in

$$\Phi = (\sigma_N + i\eta_N) T^0 + (\boldsymbol{a}_0 + i\boldsymbol{\pi}) \cdot \boldsymbol{T} , \tag{2.137}$$

while the two-flavor left- and right-handed matrices, including vector and axial-vector mesonic degrees of freedom, are given by

$$L^\mu = (\omega^\mu + f_1^\mu) T^0 + (\boldsymbol{\rho}^\mu + \boldsymbol{a}_1^\mu) \cdot \boldsymbol{T} , \tag{2.138}$$
$$R^\mu = (\omega^\mu - f_1^\mu) T^0 + (\boldsymbol{\rho}^\mu - \boldsymbol{a}_1^\mu) \cdot \boldsymbol{T} . \tag{2.139}$$

2.3.2. The Lagrangian

Within both assignments, the naive (2.133) as well as the mirror one (2.134), it is possible to construct chirally symmetric as well as parity and charge-conjugation invariant Lagrangians describing the nucleon and its chiral partner. Although the eLSM prefers the mirror assignment [13], it is useful to investigate both assignments.

Case 1: Lagrangian in the mirror assignment

Regarding the chiral transformations in Eq. (2.134), we consider the following chirally invariant Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{bar, m}}^{N_f=2} =&\bar{\Psi}_{1L}i\gamma_\mu D_{1L}^\mu\Psi_{1L} + \bar{\Psi}_{1R}i\gamma_\mu D_{1R}^\mu\Psi_{1R} + \bar{\Psi}_{2L}i\gamma_\mu D_{2R}^\mu\Psi_{2L} + \bar{\Psi}_{2R}i\gamma_\mu D_{2L}^\mu\Psi_{2R} \\ &- \hat{g}_1(\bar{\Psi}_{1L}\Phi\Psi_{1R} + \bar{\Psi}_{1R}\Phi^\dagger\Psi_{1L}) - \hat{g}_2(\bar{\Psi}_{2L}\Phi^\dagger\Psi_{2R} + \bar{\Psi}_{2R}\Phi\Psi_{2L}) \\ &- \hat{m}_0(\bar{\Psi}_{1L}\Psi_{2R} - \bar{\Psi}_{1R}\Psi_{2L} - \bar{\Psi}_{2L}\Psi_{1R} + \bar{\Psi}_{2R}\Psi_{1L}) \;, \end{aligned} \tag{2.140}$$

where the covariant derivatives are given by

$$D_{kL}^\mu = \partial^\mu - i\hat{c}_k L^\mu \quad \text{and} \quad D_{kR}^\mu = \partial^\mu - i\hat{c}_k R^\mu \qquad \forall\, k \in \{1,2\} \;. \tag{2.141}$$

They include the dimensionless coupling constants $\hat{c}_1$ and $\hat{c}_2$, which determine the strength of the interaction of the nucleon and its chiral partner with (axial-)vector degrees of freedom. The interaction with scalar and pseudoscalar fields is parametrized by the dimensionless constants $\hat{g}_1$ and $\hat{g}_2$.

The term proportional to the mass parameter $\hat{m}_0$ is unique for the mirror assignment. It generates a mixing between Ψ_1 and Ψ_2. Note, it is the only term in Eq. (2.140) which is not dilatation invariant. Nevertheless, it can be rendered so by including the dilaton field G [which is predominantly assigned to $f_0(1710)$ [4]] as well as a four-quark field χ [which is predominantly assigned to $f_0(500)$ [64, 65]]:

$$\hat{m}_0(\bar{\Psi}_2\gamma_5\Psi_1 - \bar{\Psi}_1\gamma_5\Psi_2) \quad \rightarrow \quad (a\chi + bG)(\bar{\Psi}_2\gamma_5\Psi_1 - \bar{\Psi}_1\gamma_5\Psi_2) \;, \tag{2.142}$$

where we have introduced the dimensionless parameters a and b and used the chiral projection operators (1.129) to obtain the baryon field combination

$$\begin{aligned}&\bar{\Psi}_{1L}\Psi_{2R} - \bar{\Psi}_{1R}\Psi_{2L} - \bar{\Psi}_{2L}\Psi_{1R} + \bar{\Psi}_{2R}\Psi_{1L} \\ &= \bar{\Psi}_1\mathcal{P}_R\Psi_2 - \bar{\Psi}_1\mathcal{P}_L\Psi_2 - \bar{\Psi}_2\mathcal{P}_R\Psi_1 + \bar{\Psi}_2\mathcal{P}_L\Psi_1 \\ &= \bar{\Psi}_2\gamma_5\Psi_1 - \bar{\Psi}_1\gamma_5\Psi_2 \;. \end{aligned} \tag{2.143}$$

Then, under the assumption that a symmetry-breaking mechanism induces a non-vanishing VEV for the gluon and/or the four-quark field, it generates the mass parameter $\hat{m}_0$. Shifting the glueball and/or the four-quark states by their VEVs, $G \to G + G_0$ and $\chi \to \chi + \chi_0$, respectively, we obtain

$$\hat{m}_0 = a\chi_0 + bG_0 \;. \tag{2.144}$$

For a more detailed description utilizing the inclusoin of a tetraquark into the eLSM, see Refs. [14, 66].

Besides $\hat{m}_0$, further mass terms are generated by SSB. Substituting the shift $\sigma_N \to \sigma_N + \phi_N$, we finally collect the following mass terms:

$$\begin{aligned}\mathcal{L}_{\text{mass,m}} &= -\frac{\hat{g}_1\phi_N}{2}\bar{\Psi}_1\Psi_1 - \frac{\hat{g}_2\phi_N}{2}\bar{\Psi}_2\Psi_2 - \hat{m}_0(\bar{\Psi}_2\gamma_5\Psi_1 - \bar{\Psi}_1\gamma_5\Psi_2) \\ &= -\left(\bar{\Psi}_1, -\bar{\Psi}_2\gamma_5\right)\begin{pmatrix} \frac{\hat{g}_1\phi_N}{2} & \hat{m}_0 \\ \hat{m}_0 & -\frac{\hat{g}_2\phi_N}{2}\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \gamma_5\Psi_2\end{pmatrix} \\ &\equiv -\bar{\mathbf{\Psi}}M\mathbf{\Psi} \ ,\end{aligned} \tag{2.145}$$

where we have defined the mass matrix

$$M = \begin{pmatrix} \frac{\hat{g}_1\phi_N}{2} & \hat{m}_0 \\ \hat{m}_0 & -\frac{\hat{g}_2\phi_N}{2}\end{pmatrix} \tag{2.146}$$

and introduced the vector

$$\mathbf{\Psi} = (\Psi_1, \gamma_5\Psi_2)^T \ , \tag{2.147}$$

as well as its Dirac-adjoint

$$\bar{\mathbf{\Psi}} = (\bar{\Psi}_1, -\bar{\Psi}_2\gamma_5) \ . \tag{2.148}$$

The γ_5 matrix in $\mathbf{\Psi}$ (and $\bar{\mathbf{\Psi}}$) ensures that both vector components have the same parity, such that the mass matrix (2.146) contains only numbers. The negative sign in the second component of the adjoint vector $\bar{\mathbf{\Psi}}$ results from the anticommutation of γ_5 and γ_0, namely

$$\overline{\gamma_5\Psi_2} = (\gamma_5\Psi_2)^\dagger\gamma_0 = (\Psi_2)^\dagger\gamma_5\gamma_0 = -(\Psi_2)^\dagger\gamma_0\gamma_5 = -\bar{\Psi}_2\gamma_5 \ .$$

The non-diagonal form of the mass matrix (2.146) implies that Ψ_1 and Ψ_2 cannot be assigned to physical particles. Instead, they mix to form the physical states N and N^* (describing the nucleon and its chiral partner).

We diagonalize the mass matrix (2.146) by solving the eigenvalue problem

$$0 \overset{!}{=} \det(M - m\mathbb{1}) = m^2 - m\left[\frac{\phi_N}{2}(\hat{g}_1 - \hat{g}_2)\right] - \hat{m}_0^2 + \frac{1}{4}\phi_N^2\hat{g}_1\hat{g}_2 \ . \tag{2.149}$$

The two solutions

$$\begin{aligned} m_\pm &= \frac{1}{2}\left[\frac{1}{2}\phi_N(\hat{g}_1 - \hat{g}_2) \pm \sqrt{\frac{1}{4}\phi_N^2(\hat{g}_1 - \hat{g}_2)^2 + 4\hat{m}_0^2 + \phi_N^2\hat{g}_1\hat{g}_2}\right] \\ &= \frac{1}{4}\phi_N(\hat{g}_1 - \hat{g}_2) \pm \sqrt{\hat{m}_0^2 + \left[\frac{1}{4}\phi_N(\hat{g}_1 + \hat{g}_2)\right]^2} \end{aligned} \tag{2.150}$$

are the eigenvalues of the two eigenstates N and $\gamma_5 N^*$ (note the γ_5) which are related to the masses of the physical states N and N^* as follows:

$$m_N \equiv m_+ = \sqrt{\hat{m}_0^2 + \left[\frac{1}{4}\phi_N(\hat{g}_1 + \hat{g}_2)\right]^2} + \frac{1}{4}\phi_N(\hat{g}_1 - \hat{g}_2) \ , \tag{2.151}$$

$$m_{N^*} \equiv -m_- = \sqrt{\hat{m}_0^2 + \left[\frac{1}{4}\phi_N(\hat{g}_1 + \hat{g}_2)\right]^2} - \frac{1}{4}\phi_N(\hat{g}_1 - \hat{g}_2) \ , \tag{2.152}$$

where an additional minus sign appears for the mass of the chiral partner, because of

$$\bar{N}^* N^* = -(\overline{N^*\gamma_5})(\gamma_5 N^*) \tag{2.153}$$

due to the negative sign in the second component of the adjoint spinor vector [compare the discussion following Eq. (2.148)].

The results (2.151) and (2.152) show that the masses of the nucleon and its chiral partner contain a part arising from SSB $\sim \phi_N$, but they also depend on the mass parameter $\hat{m}_0$. This indicates that the masses are not solely generated by SSB, but also other sources contribute to the masses, such as G_0 and χ_0 (as described above). However, we should emphasize that the mass equation is not linear, i.e., it is not simply a linear sum of the various contributions.

Furthermore, from Eqs. (2.151) and (2.152) it is visible that the masses of the nucleon and its chiral partner are equal in the limit of chiral symmetry restoration

$$\phi_N \to 0 \ : \qquad m_N = m_{N^*} = \hat{m}_0 \ . \tag{2.154}$$

The physical states N and N^* are mixtures of Ψ_1 and Ψ_2, which we express as follows:

$$\begin{pmatrix} N \\ N^* \end{pmatrix} = U \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \equiv \frac{1}{\sqrt{2\cosh\delta}} \begin{pmatrix} e^{\delta/2} & \gamma_5 e^{-\delta/2} \\ \gamma_5 e^{-\delta/2} & -e^{\delta/2} \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \ , \tag{2.155}$$

where we introduced the rotation matrix U. The transformation of the adjoint vector of spinor fields is then given by

$$\begin{aligned} \left(\bar{N}, \bar{N}^*\right) &= \frac{1}{\sqrt{2\cosh\delta}} \left(\bar{\Psi}_1, \bar{\Psi}_2\right) \begin{pmatrix} e^{\delta/2} & -\gamma_5 e^{-\delta/2} \\ -\gamma_5 e^{-\delta/2} & -e^{\delta/2} \end{pmatrix} \\ &\equiv \left(\bar{\Psi}_1, \bar{\Psi}_2\right) \bar{U}^T \ , \end{aligned} \tag{2.156}$$

where due to $\{\gamma_5, \gamma^\mu\} = 0$ an additional minus signs appear in the off-diagonal elements of the transformation matrix. To simplify the notation, we label the resulting orthogonal matrix $\bar{U}$.

The rotation depends on the mixing parameter δ which we choose such that the Lagrangian is diagonal in the physical fields N and N^*. We rewrite Eq. (2.145) as follows:

$$\begin{aligned}\mathcal{L}_{\text{mass,m}} &= -\left(\bar{\Psi}_1, \bar{\Psi}_2\right) M \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} \\ &= -\left(\bar{\Psi}_1, \bar{\Psi}_2\right) \bar{U}^T \bar{U} M U^T U \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} \\ &= -\left(\bar{N}, \bar{N}^*\right) \bar{U} M U^T \begin{pmatrix} N \\ N^*\end{pmatrix} \ . \end{aligned} \tag{2.157}$$

We demand that the transformed mass matrix $\bar{U}MU^T$ is a diagonal matrix with the eigenvalues (2.151) and (2.152) on the main diagonal:

$$\begin{aligned}&\bar{U}MU^T = \\ &= \frac{1}{2\cosh\delta}\begin{pmatrix} \frac{\phi_N}{2}(\hat{g}_1 e^{\delta} - \hat{g}_2 e^{-\delta}) + 2\hat{m}_0 & \frac{\phi_N}{2}\gamma_5(\hat{g}_1 + \hat{g}_2) + \gamma_5 \hat{m}_0 (e^{\delta} - e^{-\delta}) \\ -\frac{\phi_N}{2}\gamma_5(\hat{g}_1 + \hat{g}_2) + \gamma_5 \hat{m}_0 (e^{\delta} - e^{-\delta}) & -\frac{\phi_N}{2}(\hat{g}_1 e^{\delta} - \hat{g}_2 e^{-\delta}) + 2\hat{m}_0 \end{pmatrix} \\ &\overset{!}{=} \text{diag}(m_N, m_{N^*}) \ . \end{aligned} \tag{2.158}$$

Using the definition of the hyperbolic functions $\cosh\delta = (e^{\delta} + e^{-\delta})/2$ we find that the mixing parameter has to fulfill

$$\cosh\delta = \frac{m_N + m_{N^*}}{2\hat{m}_0} \ . \tag{2.159}$$

Finally, we express the Lagrangian (2.140) as a function of physical fields. We have to regard SSB in the mesonic sector ($\sigma_N \to \phi_N + \sigma_N$), which entails the following shifts and renormalizations:

$$\begin{aligned} f_{1N}^{\mu} &\to f_{1N}^{\mu} + Zw\partial^{\mu}\eta_N \ , \\ a^{[0,\pm]\mu} &\to a^{[0,\pm]\mu} + Zw\partial^{\mu}\pi^{[0,\pm]} \ , \\ \pi^{[0,\pm]} &\to Z\pi^{[0,\pm]} \ , \\ \eta_N &\to Z\eta_N \ , \end{aligned} \tag{2.160}$$

where we have used that

$$w_{f_{1N}} = w_{a_1} \equiv w \qquad \text{and} \qquad Z_{\pi} = Z_{\eta_N} \equiv Z \ , \tag{2.161}$$

see Eqs. (2.85) to (2.97) for $N_f = 2$. Thus, we obtain

$$
\begin{aligned}
\mathcal{L}_{\text{bar, m}}^{N_f=2} =& \bar{\Psi}_1 i\gamma_\mu \partial^\mu \Psi_1 + \bar{\Psi}_2 i\gamma_\mu \partial^\mu \Psi_2 \\
&+ \frac{\hat{c}_1}{2}\ \bar{\Psi}_1 \gamma_\mu \big(\omega_N^\mu + \boldsymbol{\rho}^\mu \cdot \boldsymbol{\tau} - \gamma_5 \{ f_{1N}^\mu + \boldsymbol{a}_1^\mu \cdot \boldsymbol{\tau} + Zw[\partial^\mu \eta_N + (\partial^\mu \boldsymbol{\pi}) \cdot \boldsymbol{\tau}]\}\big)\Psi_1 \\
&+ \frac{\hat{c}_2}{2}\ \bar{\Psi}_2 \gamma_\mu \big(\omega_N^\mu + \boldsymbol{\rho}^\mu \cdot \boldsymbol{\tau} + \gamma_5 \{ f_{1N}^\mu + \boldsymbol{a}_1^\mu \cdot \boldsymbol{\tau} + Zw[\partial^\mu \eta_N + (\partial^\mu \boldsymbol{\pi}) \cdot \boldsymbol{\tau}]\}\big)\Psi_2 \\
&- \frac{\hat{g}_1}{2}\ \bar{\Psi}_1 [\sigma_N + \boldsymbol{a}_0 \cdot \boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi} \cdot \boldsymbol{\tau})]\Psi_1 - \frac{\hat{g}_1}{2}\ \phi_N \bar{\Psi}_1 \Psi_1 \\
&- \frac{\hat{g}_2}{2}\ \bar{\Psi}_2 [\sigma_N + \boldsymbol{a}_0 \cdot \boldsymbol{\tau} - i\gamma_5 Z(\eta_N + \boldsymbol{\pi} \cdot \boldsymbol{\tau})]\Psi_2 - \frac{\hat{g}_2}{2}\ \phi_N \bar{\Psi}_2 \Psi_2 \\
&- \hat{m}_0 (\bar{\Psi}_1 \gamma_5 \Psi_2 - \bar{\Psi}_2 \gamma_5 \Psi_1)\ .
\end{aligned}
\tag{2.162}
$$

Inserting Eqs. (2.155) and (2.156), which can be written as

$$\Psi_1 = (2\cosh\delta)^{-\frac{1}{2}} \big(N e^{\frac{\delta}{2}} + \gamma_5 N^* e^{-\frac{\delta}{2}}\big)\ , \tag{2.163}$$

$$\Psi_2 = (2\cosh\delta)^{-\frac{1}{2}} \big(\gamma_5 N e^{-\frac{\delta}{2}} - N^* e^{\frac{\delta}{2}}\big)\ , \tag{2.164}$$

$$\bar{\Psi}_1 = (2\cosh\delta)^{-\frac{1}{2}} \big(\bar{N} e^{\frac{\delta}{2}} - \bar{N}^* \gamma_5 e^{-\frac{\delta}{2}}\big)\ , \tag{2.165}$$

$$\bar{\Psi}_2 = (2\cosh\delta)^{-\frac{1}{2}} \big(-\bar{N} \gamma_5 e^{-\frac{\delta}{2}} - \bar{N}^* e^{\frac{\delta}{2}}\big)\ , \tag{2.166}$$

the Lagrangian as a function of physical fields reads

$$
\begin{aligned}
\mathcal{L}^{N_f=2}_{\text{bar, m}} = {} & \bar{N} i\gamma_\mu \partial^\mu N + \bar{N}^* i\gamma_\mu \partial^\mu N^* \\
& + \frac{\hat{c}_1}{4\cosh\delta}\Big[\bar{N}\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)Ne^{\delta} \\
& \qquad + \bar{N}^*\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)N^*e^{-\delta} \\
& \qquad + \bar{N}\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N^* \\
& \qquad + \bar{N}^*\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N\Big] \\
& + \frac{\hat{c}_2}{4\cosh\delta}\Big[\bar{N}\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} + \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)Ne^{-\delta} \\
& \qquad + \bar{N}^*\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} + \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)N^*e^{\delta} \\
& \qquad + \bar{N}\gamma_\mu\big\{-\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N^* \\
& \qquad + \bar{N}^*\gamma_\mu\big\{-\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N\Big] \\
& - \frac{\hat{g}_1}{4\cosh\delta}\Big\{\bar{N}[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]Ne^{\delta} \\
& \qquad - \bar{N}^*[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^*e^{-\delta} \\
& \qquad + \bar{N}[\gamma_5(\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau}) + iZ(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^* \\
& \qquad - \bar{N}^*[\gamma_5(\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau}) + iZ(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N\Big\} \\
& - \frac{\hat{g}_2}{4\cosh\delta}\Big\{-\bar{N}[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} - i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]Ne^{-\delta} \\
& \qquad + \bar{N}^*[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} - i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^*e^{\delta} \\
& \qquad + \bar{N}[\gamma_5(\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau}) - iZ(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^* \\
& \qquad - \bar{N}^*[\gamma_5(\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau}) - iZ(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N\Big\} \\
& - \frac{(\hat{g}_1 e^{\delta} - \hat{g}_2 e^{-\delta})\phi_N}{4\cosh\delta}\bar{N}N + \frac{(\hat{g}_1 e^{-\delta} - \hat{g}_2 e^{\delta})\phi_N}{4\cosh\delta}\bar{N}^*N^* - \frac{\hat{m}_0}{\cosh\delta}(\bar{N}N + \bar{N}^*N^*)\ ,
\end{aligned}
\tag{2.167}
$$

where we used the condition (2.159) to see that mixing terms between N and N^* vanish.

Note, the coupling constants $\hat{g}_1$ and $\hat{g}_2$ are uniquely determined by $\hat{m}_0$, m_N, and m_{N^*}. Namely, using the results (2.151) and (2.152), it is possible to express $\hat{g}_1$ and $\hat{g}_2$ as a function of m_N and m_{N^*}:

$$
\begin{aligned}
m_N + m_{N^*} &= 2\sqrt{\hat{m}_0^2 + [\frac{1}{4}\phi_N(\hat{g}_1 + \hat{g}_2)]^2} \\
\Leftrightarrow \qquad \hat{g}_1 + \hat{g}_2 &= \frac{2}{\phi_N}\sqrt{(m_N + m_{N^*})^2 - 4m_0^2}\ .
\end{aligned}
\tag{2.168}
$$

On the other hand, the following expression holds:

$$\begin{aligned} m_N - m_{N^*} &= \frac{1}{2}\phi_N(\hat{g}_1 - \hat{g}_2) \\ \Leftrightarrow \qquad \hat{g}_1 - \hat{g}_2 &= \frac{2}{\phi_N}(m_N - m_{N^*}) \; . \end{aligned} \tag{2.169}$$

Combining Eqs. (2.168) and (2.169), we obtain:

$$\hat{g}_{1/2} = \pm\frac{1}{\phi_N}(m_N - m_{N^*}) + \frac{1}{\phi_N}\sqrt{(m_N + m_{N^*})^2 - 4\hat{m}_0^2} \; . \tag{2.170}$$

Case 2: Lagrangian in the naive assignment

The baryonic eLSM Lagrangian in the naive assignment (2.133) reads

$$\begin{aligned} \mathcal{L}_{\text{bar, n}}^{N_f=2} =& \bar{\Psi}_{1L} i\gamma_\mu D_{1L}^\mu \Psi_{1L} + \bar{\Psi}_{1R} i\gamma_\mu D_{1R}^\mu \Psi_{1R} + \bar{\Psi}_{2L} i\gamma_\mu D_{2L}^\mu \Psi_{2L} + \bar{\Psi}_{2R} i\gamma_\mu D_{2R}^\mu \Psi_{2R} \\ & - \hat{c}_{12}(\bar{\Psi}_{1L}\gamma_\mu L^\mu \Psi_{2L} - \bar{\Psi}_{1R}\gamma_\mu R^\mu \Psi_{2R} + \bar{\Psi}_{2L}\gamma_\mu L^\mu \Psi_{1L} - \bar{\Psi}_{2R}\gamma_\mu R^\mu \Psi_{1R}) \\ & - \hat{g}_1(\bar{\Psi}_{1L}\Phi\Psi_{1R} + \bar{\Psi}_{1R}\Phi^\dagger\Psi_{1L}) - \hat{g}_2(\bar{\Psi}_{2L}\Phi\Psi_{2R} + \bar{\Psi}_{2R}\Phi^\dagger\Psi_{2L}) \\ & - \hat{g}_{12}(\bar{\Psi}_{1L}\Phi\Psi_{2R} - \bar{\Psi}_{1R}\Phi^\dagger\Psi_{2L} - \bar{\Psi}_{2L}\Phi\Psi_{1R} + \bar{\Psi}_{2R}\Phi^\dagger\Psi_{1L}) \; , \end{aligned} \tag{2.171}$$

with the covariant derivatives in Eq. (2.141). In contrast to the Lagrangian in the mirror assignment (2.140), chirally invariant Yukawa interaction terms mixing Ψ_1 and Ψ_2 with (axial-)vector and (pseudo)scalar fields ($\propto \hat{c}_{12}$ and $\hat{g}_{12}$) can be constructed. A mass term, however, is forbidden.
After SSB in the mesonic sector, the following mass terms occur:

$$\begin{aligned} \mathcal{L}_{\text{mass,n}} &= -\frac{\hat{g}_1\phi_N}{2}\bar{\Psi}_1\Psi_1 - \frac{\hat{g}_2\phi_N}{2}\bar{\Psi}_2\Psi_2 - \hat{g}_{12}\frac{\phi_N}{2}(\bar{\Psi}_1\gamma_5\Psi_2 - \bar{\Psi}_2\gamma_5\Psi_1) \\ &= -\left(\bar{\Psi}_1, -\bar{\Psi}_2\gamma_5\right)\begin{pmatrix} \frac{\hat{g}_1\phi_N}{2} & \frac{\hat{g}_{12}\phi_N}{2} \\ \frac{\hat{g}_{12}\phi_N}{2} & -\frac{\hat{g}_2\phi_N}{2} \end{pmatrix}\begin{pmatrix} \Psi_1 \\ \gamma_5\Psi_2 \end{pmatrix} \equiv -\bar{\mathbf{\Psi}} M \mathbf{\Psi} \; . \end{aligned} \tag{2.172}$$

Diagonalization yields the following masses of the nucleon and its chiral partner:

$$m_N = \phi_N\left[\frac{1}{2}\sqrt{\hat{g}_{12}^2 + \frac{1}{4}(\hat{g}_1 + \hat{g}_2)^2} + \frac{1}{4}(\hat{g}_1 - \hat{g}_2)\right] \propto \phi_N \tag{2.173}$$

$$m_{N^*} = \phi_N\left[\frac{1}{2}\sqrt{\hat{g}_{12}^2 + \frac{1}{4}(\hat{g}_1 + \hat{g}_2)^2} - \frac{1}{4}(\hat{g}_1 - \hat{g}_2)\right] \propto \phi_N \; , \tag{2.174}$$

which are generated solely by SSB.

The spinors Ψ_1 and Ψ_2 mix to form the physical fields as given in Eq. (2.155) but with the mixing angle being determined by the condition

$$\cosh\delta = \frac{m_N + m_{N^*}}{\hat{g}_{12}\phi_N} \ . \tag{2.175}$$

Thus, it is proportional to ϕ_N and can be completely eliminated.

The Lagrangian as a function of physical fields reads

$$\begin{aligned}
\mathcal{L}^{N_f=2}_{\text{bar, n}} = \; & \bar{N} i\gamma_\mu \partial^\mu N + \bar{N}^* i\gamma_\mu \partial^\mu N^* \\
& + \frac{\hat{c}_1}{4\cosh\delta}\Big[\bar{N}\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)Ne^{\delta} \\
& \qquad + \bar{N}^*\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)N^*e^{-\delta} \\
& \qquad + \bar{N}\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N^* \\
& \qquad + \bar{N}^*\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N\Big] \\
& + \frac{\hat{c}_2}{4\cosh\delta}\Big[\bar{N}\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)Ne^{-\delta} \\
& \qquad + \bar{N}^*\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)N^*e^{\delta} \\
& \qquad - \bar{N}\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N^* \\
& \qquad - \bar{N}^*\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N\Big] \\
& + \frac{\hat{c}_{12}}{2\cosh\delta}\Big[\bar{N}\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)N \\
& \qquad - \bar{N}^*\gamma_\mu\big(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau} - \gamma_5\{f_{1N}^\mu + \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} + Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\}\big)N^*\Big] \\
& - \frac{\hat{c}_{12}}{2}\tanh\delta\Big(\bar{N}\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N^* \\
& \qquad + \bar{N}^*\gamma_\mu\big\{\gamma_5(\omega_N^\mu + \boldsymbol{\rho}^\mu\cdot\boldsymbol{\tau}) - f_{1N}^\mu - \boldsymbol{a}_1^\mu\cdot\boldsymbol{\tau} - Zw[\partial^\mu\eta_N + (\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\big\}N\Big) \\
& - \frac{\hat{g}_1}{4\cosh\delta}\Big\{\bar{N}[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]Ne^{\delta} \\
& \qquad - \bar{N}^*[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^*e^{-\delta}\Big\} \\
& - \frac{\hat{g}_2}{4\cosh\delta}\Big\{-\bar{N}[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]Ne^{-\delta} \\
& \qquad + \bar{N}^*[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^*e^{\delta}\Big\}
\end{aligned}$$

$$- \frac{\hat{g}_{12}}{2\cosh\delta}\Big\{\bar{N}[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N + \bar{N}^*[\sigma_N + \boldsymbol{a}_0\cdot\boldsymbol{\tau} + i\gamma_5 Z(\eta_N + \boldsymbol{\pi}\cdot\boldsymbol{\tau})]N^*\Big\}$$

$$- \frac{(\hat{g}_1 e^{\delta} - \hat{g}_2 e^{-\delta})\phi_N}{4\cosh\delta}\bar{N}N + \frac{(\hat{g}_1 e^{-\delta} - \hat{g}_2 e^{\delta})\phi_N}{4\cosh\delta}\bar{N}^*N^* - \frac{\hat{g}_{12}\phi_N}{2\cosh\delta}(\bar{N}N + \bar{N}^*N^*)\ , \quad (2.176)$$

Note, using Eq. (2.175) the mixing terms between N and N^* as well as interaction terms between N, N^* and (pseudo)scalar mesonic fields (without derivative) vanish. Differences of this Lagrangian compared to the Lagrangian in the mirror assignment (2.167) are encircled. The Lagrangian in the naive assignment contains only derivative couplings of N and N^* with the pseudoscalars η_N and π resulting from SSB.

2.3.3. Results

We briefly summarize the results[3] and conclusions as presented in Refs. [12, 13]. The aim of these works was to study the origin of the nucleon mass and the identification of its chiral partner. Decay properties and pion-nucleon scattering lengths were evaluated. To this end, first a mirror assignment of the nucleon and its chiral partner was investigated in Ref [12]. Four years later, in Ref. [13], also the naive assignment was elaborated and compared to the results of the model with mirror assignment of the nucleon and its chiral partner.

Case 1: mirror assignment

The baryonic eLSM Lagrangian in the mirror assignment (2.140) contains four parameters: $\hat{m}_0$, c_1, c_2, Z. Their values are obtained from a fitting procedure which involves the experimentally measured [18] decay widths of $N^* \to N\pi$ and $a_1 \to \pi\gamma$, as well as the axial coupling constants g_A^N and $g_A^{N^*}$ [where the latter is evaluated on the lattice [68]]. With the identification of the resonance $N(1535)$ as chiral partner of the nucleon, a standard χ^2 procedure yields the following values, see Ref. [13]:

$$\hat{m}_0 = (459 \pm 117)\ \text{MeV}\ , \quad (2.177)$$

$$c_1 = -2.65 \pm 0.18\ , \quad (2.178)$$

$$c_2 = 10.2 \pm 2.6\ , \quad (2.179)$$

$$Z = 1.81 \pm 0.07\ . \quad (2.180)$$

The sizable value of $\hat{m}_0$ shows that the mass of the nucleon is not solely generated by SSB ($\sigma_N \to \phi_N + \sigma_N$), but that the contributions from other sources are important (as

[3]Note, updated results can be found in Ref. [67], where the model with mirror assignment (just recently) has been revistited to investigate four-quark and glueball states in pion-pion and pion-nucleon scattering.

for instance, a gluon or a four-quark condensate). However, the relation between the nucleon mass m_N and $\hat{m}_0$ is non-linear, see Eq. (2.151). Thus, when setting $\hat{m}_0$ to zero, the nucleon mass will not be simply reduced by the amount of $\hat{m}_0 \simeq 459$ MeV, but will be only slightly smaller than the physical mass of 939 MeV, namely $m_N\big|_{\hat{m}_0} = \hat{g}_1\phi_N/2 \simeq 850$ MeV.
The coupling constants $\hat{g}_1$ and $\hat{g}_2$ are determined by Eq. (2.170). With the values (2.177) to (2.180), we obtain

$$\hat{g}_1 = 10.2 \pm 0.7 \ , \tag{2.181}$$

$$\hat{g}_2 = 17.3 \pm 0.8 \ . \tag{2.182}$$

The predicted isospin-odd and isospin-even πN scattering lengths can be compared to the experimentally known values [69]:

$$a_{0,\text{exp}}^{(-)} = (6.4 \pm 0.1) \cdot 10^{-4} \text{ MeV}^{-1} \tag{2.183}$$

$$a_{0,\text{exp}}^{(+)} = (-8.8 \pm 7.2) \cdot 10^{-6} \text{ MeV}^{-1} \ . \tag{2.184}$$

The isospin-odd scattering length

$$a_0^{(-)} = (6.41 \pm 0.17) \cdot 10^{-4} \text{ MeV}^{-1} \tag{2.185}$$

is found to be in good agreement with experiment. The isospin-even scattering length $a_0^{(+)}$ depends strongly on the mass of the σ meson, for which the assignment to a specific scalar listed in Ref. [18] is not clear, compare Refs. [1, 3, 55]. Therefore, the isospin-even scattering cannot be easily evaluated. In Refs. [12, 13] a plot of $a_0^{(+)}$ as function of m_σ is presented. It shows that the model produces values of $a_0^{(+)}$ which are in agreement with the experiment, if $m_\sigma < 500$ MeV. This indicates that we need a low-energy scalar field, which can be assigned to $f_0(500)$ and interpreted as a four-quark or a molecular state [55]. The coupling of such a state to the model is investigated in Refs. [12, 14].

Furthermore, the model allows to calculate the decay width of $N^* \to N\eta$. This reveals a clear problem of the model. Namely, we obtain

$$\Gamma_{N^* \to N\eta} = (4.9 \pm 0.8) \text{ Mev} \ , \tag{2.186}$$

which is by far too small when compared to the experimental value [18]

$$\Gamma_{N^* \to N\eta}^{\text{exp}} = (63 \pm 10.5) \text{ MeV} \ . \tag{2.187}$$

From this mismatch, the question arises: maybe the resonance $N(1650)$ instead of $N(1535)$ is assigned as the chiral partner of the nucleon? In this scenario [12], the results for the scattering lengths are qualitatively the same, but considering the decay width $N^* \to N\eta$ the assignment $N^* \equiv N(1650)$ is slightly favored.

Case 2: naive assignment (only very briefly)

The investigation of the baryonic eLSM in the naive assignment is different in two main points. First, the mass is created only by SSB (since a mass term is no longer possible). This is a clear disadvantage of the naive assignment compared to the mirror assignment, where this mass term was interpreted as a result of couplings between the nucleon and a glueball and/or four-quark fields. Second, the interaction of N and N^* with (pseudo)scalar mesons reduces to couplings with derivatives of η_N and $\boldsymbol{\pi}$. These terms arise from the shift of (axial-)vector fields after SSB in the mesonic sector. Thus, the decay of $N^* \to N\pi$ is only possible when (axial-)vector degrees of freedom are included. In this respect, the naive assignment is much more restrictive than the mirror assignment.
Considering the isospin-odd scattering length $a_0^{(-)}$, the model with naive assignment yields too small results compared to experimental values. Investigating the decay widths $N^* \to N\eta$, reveals (again) that $N(1650)$ is preferred as chiral partner of the nucleon, compared to the assignment with $N(1535)$.

Conclusions and outlook

The eLSM in the mirror assignment produces values for the isospin-even scattering length which are in good agreement with the experiment. This is not the case for the eLSM in the naive assignment. Considering the decay width of $N^* \to N\eta$, both assignments lead to too small values compared to the experiment, when $N(1535)$ is assigned as chiral partner of the nucleon. If $N(1650)$ is assigned to N^*, the results for $\Gamma_{N^* \to N\eta}$ improve in both models.
In conclusion, the studies of Refs. [12, 13] clearly prefer the mirror assignment and slightly favor the resonance $N(1650)$ as chiral partner of the nucleon.

However, in the scenario with $N(1650)$ as the chiral partner, it is not clear how to fit $N(1535)$ into the baryonic resonance spectrum. A possible improvement of the theoretical description is already presented in the outlook of Ref. [12]: one can extend the baryonic sector by including further baryon states, such that both resonances $N(1535)$ and $N(1650)$ are described simultaneously. In fact, this idea is the motivation for the present work.

Chapter 3.

Mathematical representation of baryons in the three-flavor case

In this chapter, we construct three-flavor baryon fields as three-quark states, see Sec. 3.1. Then in Sec. 3.2, we restrict the discussion to the octet baryons exploiting the quark-diquark picture [see Refs. [70, 71]]. Finally, in Sec. 3.3, we develop a chiral representation in order to incorporate the octet baryons into the chiral effective eLSM approach.

3.1. Baryons as three-quark states (flavor irreducible decomposition)

A very general expression of local baryon fields results from the construction as three-quark objects requiring an appropriate flavor content and a color wave function producing a singlet in color space:

$$B \equiv \epsilon_{abc}\; q_{a,i}^T C\Gamma_1 q_{b,j}\; \Gamma_2 q_{c,k}\ , \tag{3.1}$$

where i, j, k denote flavor indices and a, b, c run over fundamental color indices. The contraction of color indices with the totally anti-symmetric tensor ϵ_{abc} ensures that the baryon is a singlet in color space, in agreement with color confinement [31]. The C operator is the charge-conjugation matrix and Γ_1 and Γ_2 represent matrices according to the desired Lorentz-structure:

$$\Gamma_1, \Gamma_2\ \in\ \left\{1,\ \gamma_5,\ \gamma^\mu,\ \gamma^\mu\gamma_5,\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]\right\}\ . \tag{3.2}$$

We consider $N_f = 3$. A group-theoretical analysis of the $SU(3)_V$ shows that three-flavor baryons can be classified into a singlet with a totally antisymmetric flavor-wave function, into two octets with mixed symmetry, and a decuplet with a symmetric flavor-wave function under the exchange of any two quarks, see Eq. (1.14). Following the approach of Ref. [72], we can take advantage of this knowledge to specify the general form (3.1) into the these $SU(3)_V$ multiplets considering their characteristic permutation symmetries. To this end, we build permutation-symmetry representations of the three-quark flavor combination $q_i q_j q_k$ (we suppress color indices) using the permutation operator

$$P_{ab} \quad \forall\ a, b \in \{1, 2, 3\}\ , \tag{3.3}$$

P	1	2	3	4	5	6	7	8	9	10
q_i, q_j, q_k	uuu	uud	udd	ddd	uus	uds	dds	uss	dss	sss
baryon field	Δ^{++}	Δ^{+}	Δ^{0}	Δ^{-}	Σ^{*+}	Σ^{*0}	Σ^{*-}	Ξ^{*0}	Ξ^{*-}	Ω^{-}

Table 3.1. Non-zero components of S^P_{ijk} and decuplet-baryon assignment.

which exchanges the positions of the a^{th} and b^{th} quark fields. (The properties $P_{ab} = P_{ba}$ and $P^2_{ab} = 1$ are immediately clear.)
We start with the construction of the decuplet and singlet states due to their simple structure and later on we describe the octet states which are more complicated in this context.
Note, this subsection reports the investigations of Ref. [72] in a detailed form.

Totally symmetric decuplet states

We construct a three-quark combination which is symmetric under the exchange of any two quarks. To this end, we add up all possible permutations of $q_i q_j q_k$:

$$\begin{aligned}\Psi_A &= \mathcal{N}(1 + P_{12} + P_{13} + P_{23} + P_{12}P_{13} + P_{12}P_{23}) q_i q_j q_k \\ &= \mathcal{N}(q_i q_j q_k + q_j q_i q_k + q_k q_j q_i + q_i q_k q_j + P_{12} q_k q_j q_i + P_{12} q_i q_k q_j) \\ &= \mathcal{N}(q_i q_j q_k + q_j q_i q_k + q_k q_j q_i + q_i q_k q_j + q_j q_k q_i + q_k q_i q_j) \ ,\end{aligned} \tag{3.4}$$

where we considered only permutations that lead to different three-quark combinations. We omit for instance $P_{13}P_{12} = P_{12}P_{23}$, as well as $P_{12}P_{13}P_{23} = P_{13}$, and so on. The normalization constant $\mathcal{N}$ depends on the number of different terms which is related to the considered set of flavors. Following Ref. [72], we introduce a totally symmetric tensor S^P_{ijk} $\forall\ P \in \{1, 2, \ldots, 10\}$, whose non-zero elements are given in Tab. 3.1. Then we find that the baryon fields of Eq. (3.1) which are classified into the flavor decuplet $[10]_S$ are described by

$$\Delta^P \equiv S^P_{ijk} \epsilon_{abc} \left[q^T_{a,i} C \Gamma_1 q_{b,j} \right] \Gamma_2 q_{c,k} \ . \tag{3.5}$$

The assignment to physical baryon fields is given in Tab. 3.1.

Totally antisymmetric singlet state

The totally antisymmetric three-quark combination is constructed similarly, but for permutations generated by an odd number of transpositions a minus sign appears:

$$\begin{aligned}\Psi_A &= \mathcal{N}(1 - P_{12} - P_{13} - P_{23} + P_{12}P_{13} + P_{12}P_{23}) q_i q_j q_k \\ &= \mathcal{N}(q_i q_j q_k + q_j q_k q_i + q_k q_i q_j - q_j q_i q_k - q_i q_k q_j - q_k q_j q_i) \ ,\end{aligned} \tag{3.6}$$

with the normalization constant $\mathcal{N} = 1/\sqrt{6}$. As expected for the antisymmetric state, the only non-zero flavor combination is the one where the flavors of all three quarks are different, $q_i, q_j, q_k = u, d, s$. Using the totally antisymmetric tensor ϵ_{ijk} to represent the antisymmetric flavor structure, we obtain the baryon field classified into the $SU(3)_V$ singlet $[1]_A$:

$$\Lambda \equiv \epsilon_{ijk}\epsilon_{abc}\left[q^T_{a,i}C\Gamma_1 q_{j,b}\right]\Gamma_2 q_{k,c} \; . \tag{3.7}$$

In the quark model the corresponding field can be assigned to $\Lambda(1405)$. Note, due to Pauli's principle [27], ground-state baryons whose spatial wave function is symmetric cannot be represented by flavor singlets, see Sec. 1.1.2.

The octet states with mixed symmetry

The construction of three-quark combinations which obey a mixed symmetry, M_S or M_A, is more elaborate, because the corresponding S_3 basis functions are not uniquely determined. Using the symmetrizing operator

$$S_{ab} = 1 + P_{ab} \tag{3.8}$$

and the anti-symmetrizing operator

$$A_{ab} = 1 - P_{ab} \; , \tag{3.9}$$

we construct the following mixed-symmetric three-quark combinations:

$$\Psi_{\{2[1\}3]} = A_{13}S_{12}\; q_iq_jq_k = q_iq_jq_k + q_jq_iq_k - q_kq_jq_i - q_kq_iq_j \; , \tag{3.10}$$
$$\Psi_{\{1[2\}3]} = A_{23}S_{12}\; q_iq_jq_k = q_iq_jq_k + q_jq_iq_k - q_iq_kq_j - q_jq_kq_i \; , \tag{3.11}$$
$$\Psi_{\{1[3\}2]} = A_{12}S_{13}\; q_iq_jq_k = q_iq_jq_k + q_kq_jq_i - q_jq_iq_k - q_jq_kq_i \; , \tag{3.12}$$
$$\Psi_{\{3[1\}2]} = A_{23}S_{13}\; q_iq_jq_k = q_iq_jq_k + q_kq_jq_i - q_iq_kq_j - q_kq_iq_j \; , \tag{3.13}$$

where the index notation should point out that first the quarks in the positions $\{\cdot\,\cdot\}$ are symmetrized and then the quarks in the positions $[\cdot\,\cdot]$ are anti-symmetrized. A further possibility is to first symmetrize the second and third quark and then anti-symmetrize the remaining expression, but this delivers just linear combinations of the already introduced fields:

$$\Psi_{\{3[2\}1]} = A_{12}S_{23}\; q_iq_jq_k = \Psi_{\{2[1\}3]} - \Psi_{\{1[2\}3]} + \Psi_{\{1[3\}2]} \; , \tag{3.14}$$
$$\Psi_{\{2[3\}1]} = A_{13}S_{23}\; q_iq_jq_k = \Psi_{\{2[1\}3]} + \Psi_{\{1[3\}2]} - \Psi_{\{3[1\}2]} \; . \tag{3.15}$$

Thus, it is sufficient to consider only the four different three-quark combinations in Eqs. (3.10) to (3.13). However, since we know that there are only two octets, the appearance of four mixed symmetric three-quark objects still seems to be redundant. Indeed, we realize that the permutation operator transforms the expressions (3.10) and (3.11) into each other,

$$\Psi_{\{2[1\}3]} = P_{12}\Psi_{\{1[2\}3]} \; . \tag{3.16}$$

In a similar way, the three-quark combinations (3.12) and (3.13) are transformed into each other by

$$\Psi_{\{1[3\}2]} = P_{13}\Psi_{\{3[1\}2]} \; . \tag{3.17}$$

Thus, the pairs $(\Psi_{\{2[1\}3]}, \Psi_{\{1[2\}3]})$ and $(\Psi_{\{1[3\}2]}, \Psi_{\{3[1\}2]})$ form doublets of the permutation group S_3. These two invariant mixed-symmetric subspaces are spanned by the linear combinations

$$\lambda\,\Psi_{\{2[1\}3]} + \mu\,\Psi_{\{1[2\}3]} \qquad \forall\,\mu\,,\lambda\,\in\mathbb{R}\;, \tag{3.18}$$

$$\lambda'\Psi_{\{1[3\}2]} + \mu'\Psi_{\{3[1\}2]} \qquad \forall\,\mu',\lambda'\in\mathbb{R}\;. \tag{3.19}$$

However, the basis functions (3.10) to (3.13) are not yet orthogonal. Using Gram-Schmidt's process [73] we construct the following orthogonal states:

$$\Psi_{[\{12\}3],1} = \Psi_{\{2[1\}3]} = q_iq_jq_k + q_jq_iq_k - q_kq_jq_i - q_kq_iq_j \;, \tag{3.20}$$

$$\begin{aligned}\Psi_{[\{12\}3],2} &= \Psi_{\{1[2\}3]} - \frac{\langle\Psi_{[\{12\}3],1}|\Psi_{\{1[2\}3]}\rangle}{\langle\Psi_{[\{12\}3],1}|\Psi_{[\{12\}3],1}\rangle}\Psi_{[\{12\}3],1} \\ &= \Psi_{\{1[2\}3]} - \frac{1}{2}\Psi_{[\{12\}3],1} \\ &= \frac{1}{2}(q_iq_jq_k + q_jq_iq_k + q_kq_jq_i + q_kq_iq_j) - q_iq_kq_j - q_jq_kq_i \;.\end{aligned} \tag{3.21}$$

One can check that applying the six permutations of S_3 to the three-quark combinations (3.20) and (3.21) always leads to a linear combination of Eqs. (3.20) and (3.21). Thus these functions form an orthogonal basis of a two-dimensional irreducible representation of S_3.

Using the Gram-Schmidt's process further, with the expressions (3.12) and (3.13) we construct

$$\begin{aligned}\Psi_{[\{31\}2],1} &= \Psi_{\{1[3\}2]} + \frac{1}{4}\Psi_{[\{12\}3],1} - \frac{1}{2}\Psi_{[\{12\}3],2} \\ &= q_iq_jq_k - q_jq_iq_k + \frac{1}{2}(q_kq_jq_i + q_iq_kq_j - q_kq_iq_j - q_jq_kq_i) \;,\end{aligned} \tag{3.22}$$

$$\begin{aligned}\Psi_{[\{31\}2],2} &= \Psi_{\{3[1\}2]} - \frac{1}{4}(\Psi_{[\{12\}3],1} + 2\Psi_{[\{12\}3],2} + 2\Psi_{[\{31\}2],1}) \\ &= \frac{3}{4}(q_jq_kq_i + q_kq_jq_i - q_iq_kq_j - q_kq_iq_j) \;,\end{aligned} \tag{3.23}$$

which form the orthogonal basis of the second two-dimensional irreducible representation of S_3. [Again all S_3 permutations of the three-quark combinations (3.22) and (3.23) can be identified as linear combinations of these functions.]

The basis functions (3.20) and (3.21) span the mixed-symmetric subspace M_S, and the subspace spanned by Eqs. (3.22) and (3.23) includes all the M_A mixed-symmetric states, which get's immediately visible if consider the elements

$$
\begin{aligned}
\Psi_{M_S} &= \frac{3}{2}\Psi_{[\{12\}3],1} + \Psi_{[\{12\}3],2} \\
&\sim 2(q_iq_j + q_jq_i)q_k - (q_jq_k + q_kq_j)q_i - (q_kq_i + q_iq_k)q_j \ , \qquad (3.24) \\
\Psi_{M_A} &= 2\Psi_{[\{31\}2],1} \\
&\sim 2(q_iq_j - q_jq_i)q_k - (q_jq_k - q_kq_j)q_i - (q_kq_i - q_iq_k)q_j \ , \qquad (3.25)
\end{aligned}
$$

see Fig. 1.3.

Baryon fields with the flavor structures as in Eqs. (3.24) and (3.25) respectively are given by

$$
\begin{aligned}
B^a_{M_S} &\equiv \epsilon_{jkl}T^a_{li} \ \left(q_i^T C\Gamma_1 q_j\right)\Gamma_2 q_k \ , \qquad (3.26) \\
B^a_{M_A} &\equiv \epsilon_{ijl}T^a_{lk} \ \left(q_i^T C\Gamma_1 q_j\right)\Gamma_2 q_k \ , \qquad (3.27)
\end{aligned}
$$

where we expanded the fields in the algebra of $SU(3)$, i.e., in terms of the generators $T^a \ \forall \ a \in \{1, 2, \ldots, 8\}$ of $SU(3)$.

We check that these expressions indeed have the correct flavor structure by considering specific cases. As an example, we compute

$$
\begin{aligned}
B^8_{M_A} &= \epsilon_{ijl}\frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}_{DC} \left(q_i^T C\Gamma_1 q_j\right) \ \Gamma_2 q_k \\
&= \frac{1}{2\sqrt{3}}\left[\epsilon_{ij1}\left(q_i^T C\Gamma_1 q_j\right)\Gamma_2 u + \epsilon_{ij2}\left(q_i^T C\Gamma_1 q_j\right)\Gamma_2 d - 2\epsilon_{ij3}\left(q_i^T C\Gamma_1 q_j\right)\Gamma_2 s\right] \\
&= -\frac{1}{2\sqrt{3}}\Big[2\left(u^T C\Gamma_1 d - d^T C\Gamma_1 u\right)\Gamma_2 s \\
&\qquad - \left(d^T C\Gamma_1 s - s^T C\Gamma_1 d\right)\Gamma_2 u \\
&\qquad - \left(s^T C\Gamma_1 u - u^T C\Gamma_1 s\right)\Gamma_2 d\Big] \ , \qquad (3.28)
\end{aligned}
$$

which indeed corresponds to the M_A flavor structure of Eq. (3.25) for $q_i, q_j, q_k = u, d, s$.

The physical baryon fields are mixtures of the M_S and M_A symmetric three-quark combinations (3.26) and (3.27), see Eq. (1.22). However, without loss of generality we can restrict the following investigations to the M_A symmetric field (3.27), because using Fierz transformations to interchange the second and third quark fields, the result contains the M_S symmetric fields, see Ref. [72].
Then, suppressing the symmetry subscript, we determine the following assignment to

octet-baryon states:

$$B^1 \pm iB^2 \equiv \Sigma^\mp \ , \tag{3.29}$$
$$B^3 \equiv \Sigma^0 \ , \tag{3.30}$$
$$B^4 \pm iB^5 \equiv (\Xi^-, p) \ , \tag{3.31}$$
$$B^6 \pm iB^7 \equiv (\Xi^0, n) \ , \tag{3.32}$$
$$B^8 \equiv \Lambda \ . \tag{3.33}$$

3.2. Octet-baryon fields in the quark-diquark picture

We specify the octet-baryon representations further by taking their Lorentz structure into account. Namely, we restrict our investigations to octet baryons which are described by Dirac spinors which (at first) have positive parity. (Note, in this work we consider only the copies of the ground-state baryons and therefore, we do not study the singlet.) Considering the baryon field in Eq. (3.27), we find the following five $J^P = 1/2^+$ fields :

$$B_1^a \equiv \epsilon_{ijl} T_{lk}^a \left(q_i^T C q_j\right) \gamma_5 q_k \ , \tag{3.34}$$
$$B_2^a \equiv \epsilon_{ijl} T_{lk}^a \left(q_i^T C \gamma_5 q_j\right) q_k \ , \tag{3.35}$$

$$B_3^a \equiv \epsilon_{ijl} T_{lk}^a \left(q_i^T C \gamma_\mu \gamma_5 q_j\right) \gamma^\mu q_k = B_3^a(B_1^a, B_2^a) \ , \tag{3.36}$$
$$B_4^a \equiv \epsilon_{ijl} T_{lk}^a \left(q_i^T C \gamma_\mu q_j\right) \gamma^\mu \gamma_5 q_k = 0 \ , \tag{3.37}$$
$$B_5^a \equiv \epsilon_{ijl} T_{lk}^a \left(q_i^T C \sigma_{\mu\nu} q_j\right) \sigma^{\mu\nu} \gamma_5 q_k = 0 \ . \tag{3.38}$$

However, as shown in Ref. [72], the fields (3.37) and (3.38) are zero due to Pauli's principle and the field (3.36) depends on the fields (3.34) and (3.35). In conclusion, positive-parity[1] octet baryons are described by the two independent fields (3.34) and (3.35).

Note, linear combinations of these fields can be assigned to representations of the chiral group as follows, see Ref. [72]:

$$B_1^a - B_2^a \ : \quad ([\bar{3}],[3]) \oplus ([3],[\bar{3}]) \ , \tag{3.40}$$
$$B_1^a + B_2^a \ : \quad ([8],[1]) \oplus ([1],[8]) \ , \tag{3.41}$$

[1] The transformation behavior of these fields can be checked using the Lorentz (1.105) and parity transformations (1.111) of the constituent quarks. For instance, the parity transformation of the octet field (3.34) is calculated as

$$\begin{aligned} B_1^a(t,\boldsymbol{r}) &\to \epsilon_{ijl} T_{lk}^a \left[q_i^T(t,-\boldsymbol{r})\gamma_0^T C \gamma_0 q_j(t,-\boldsymbol{r})\right] \gamma_5 \gamma_0 q_k(t,-\boldsymbol{r}) \\ &= \epsilon_{ijl} T_{lk}^a \left[q_i^T(t,-\boldsymbol{r})\gamma_0 C \gamma_0^T C^{-1} C q_j(t,-\boldsymbol{r})\right] (-\gamma_0)\gamma_5 q_k(t,-\boldsymbol{r}) \\ &= \epsilon_{ijl} T_{lk}^a \left[q_i^T(t,-\boldsymbol{r}) C q_j(t,-\boldsymbol{r})\right] \gamma_0 \gamma_5 q_k(t,-\boldsymbol{r}) \\ &= \gamma_0 B_1^N(t,-\boldsymbol{r}) \ , \end{aligned} \tag{3.39}$$

where in the last step we have used that the two-quark state in front (a so-called diquark, see the following discussion) is a scalar in Dirac space, which allows to interchange it with the Dirac matrix γ_0.

where we have used the notation $([\cdot]_{SU(3)_L}, [\cdot]_{SU(3)_R})$.

In matrix form, the three-quark combinations in Eqs. (3.34) and (3.35) are given by

$$B_{1,ij} \equiv \epsilon_{klj} \left(q_k^T C q_l\right) \gamma_5 q_i \ , \tag{3.42}$$
$$B_{2,ij} \equiv \epsilon_{klj} \left(q_k^T C \gamma_5 q_l\right) q_i \ . \tag{3.43}$$

These representations can be understood in the a so-called quark-diquark picture, see Ref. [70] and in particular Ref. [71]. Here, baryons are considered to be combinations of a quark and a diquark, where the latter is a (colored) state that consists of two quarks and lives in the color-antitriplet representation (i.e., diquarks have an antisymmetric color-wave function). The scalar and pseudoscalar (in the nomenclature of Jaffe [74] "good") diquarks [71] are given by

$$J^P = 0^+ : \qquad D_i \equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T C\gamma^5 q_k \ , \tag{3.44}$$
$$J^P = 0^- : \qquad \tilde{D}_i \equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T C q_k \ . \tag{3.45}$$

(The color structure of these objects is formally identical to the flavor structure.) We compute their behavior under parity and charge-conjugation transformations considering the transformation properties of the constituent quark fields.
Using the quark-field parity transformation (1.111) we obtain

$$\begin{aligned} D_i(x) \equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T(x) C\gamma^5 q_k(x) \ \stackrel{P}{\to}\ D_i'(x') &\equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T(t,-\boldsymbol{x})\gamma^0 C\gamma^5\gamma^0 q_k(t,-\boldsymbol{x}) \\ &\equiv D_i(t,-\boldsymbol{x}) \ , \end{aligned} \tag{3.46}$$
$$\begin{aligned} \tilde{D}_i(x) \equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T(x) C q_k(x) \ \stackrel{P}{\to}\ \tilde{D}_i'(x') &\equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T(t,-\boldsymbol{x})\gamma^0 C\gamma^0 q_k(t,-\boldsymbol{x}) \\ &\equiv -\tilde{D}_i(t,-\boldsymbol{x}) \ , \end{aligned} \tag{3.47}$$

and using the charge-conjugation transformation (1.113) we find:

$$\begin{aligned} D_i \equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T C\gamma^5 q_k \ \stackrel{C}{\to}\ D_i' &\equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^\dagger \gamma^0 C^{-1} C\gamma^5 C\bar{q}_k^T \\ &= \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^\dagger \gamma^0\gamma^5\gamma^0 C^\dagger q_k^{T\dagger} \\ &= -\frac{1}{\sqrt{2}}\epsilon_{ijk} (q_k^T C\gamma^5 q_j)^\dagger \\ &\equiv -D_i^\dagger \ , \end{aligned} \tag{3.48}$$

	$D_i(x)$	$\tilde{D}_i(x)$
Parity P :	$D_i(t, -\boldsymbol{x})$	$-\tilde{D}_i(t, -\boldsymbol{x})$
Charge conjugation C :	$-D_i^\dagger(x)$	$\tilde{D}_i^\dagger(x)$

Table 3.2. Parity and charge-conjugation transformations of the (pseudo)scalar diquark fields.

$$\begin{aligned}\tilde{D}_i \equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^T C q_k \;\stackrel{C}{\to}\; \tilde{D}_i' &\equiv \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^\dagger \gamma^0 C^{-1} C C \bar{q}_k^T \\ &= \frac{1}{\sqrt{2}}\epsilon_{ijk} q_j^\dagger C^\dagger q_k^{T\dagger} \\ &= \frac{1}{\sqrt{2}}\epsilon_{ijk} (q_k^T C q_j)^\dagger \\ &\equiv \tilde{D}_i^\dagger \; . \end{aligned} \tag{3.49}$$

where we have used that $q^T \stackrel{C}{\to} \bar{q} C^T = q^\dagger \gamma^0 C^T$ and $C^T = C^{-1}$. Table 3.2 summarizes these transformation properties.

Using the diquark expressions (3.44) and (3.45), we rewrite the two positive-parity octet-baryonic states in Eqs. (3.42) and (3.43) as

$$J^P = 1/2^+ : \qquad B_{1,ij} \equiv \sqrt{2}\; \tilde{D}_j \gamma^5 q_i \; , \tag{3.50}$$

$$J^P = 1/2^+ : \qquad B_{2,ij} \equiv \sqrt{2}\; D_j q_i \; . \tag{3.51}$$

Baryon fields with negative parity are constructed by including an additional γ_5 matrix between the diquark and quark fields in Eqs. (3.50) and (3.51):

$$J^P = 1/2^- : \qquad B_{1,ij}^* \equiv \sqrt{2} \tilde{D}_j q_i \; , \tag{3.52}$$

$$J^P = 1/2^- : \qquad B_{2,ij}^* \equiv \sqrt{2} D_j \gamma^5 q_i \; . \tag{3.53}$$

The baryon fields in Eqs. (3.50) to (3.53) are 3×3 matrices in flavor space. Regarding the flavor structure of the diquark fields,

$$[d, s] \; , \; -[u, s] \; , \; [u, d] \; , \tag{3.54}$$

we obtain the following flavor content of the baryon matrices:

$$\begin{pmatrix} u[d,s] & u[s,u] & u[u,d] \\ d[d,s] & d[s,u] & d[u,d] \\ s[d,s] & s[s,u] & s[u,d] \end{pmatrix} \leftrightarrow \begin{pmatrix} \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} . \tag{3.55}$$

where we have already indicated the assignment to physical baryons.

3.3. Chiral representation of octet baryons in the mirror assignment

In order to embed baryons into the chirally symmetric approach of the eLSM, it is necessary to construct the baryonic fields from quarks and diquarks with definite behavior under chiral transformations, i.e., in the quark-diquark picture from left- and right-handed quark as well as left- and right-handed diquark fields.

From the scalar- and pseudoscalar-diquark expressions (3.44) and (3.45) we can construct left- and right-handed diquark fields as

$$D_i^L = \frac{1}{\sqrt{2}}\left(\tilde{D}_i - D_i\right) \equiv \epsilon_{ijk} q_j^T C \mathcal{P}_L q_k = \epsilon_{ijk} q_{j,L}^T C q_{k,L} \ , \tag{3.56}$$

$$D_i^R = \frac{1}{\sqrt{2}}\left(\tilde{D}_i + D_i\right) \equiv \epsilon_{ijk} q_j^T C \mathcal{P}_R q_k = \epsilon_{ijk} q_{j,R}^T C q_{k,R} \ , \tag{3.57}$$

where we have used the chiral projection operators (1.129). The chiral transformations of the constituent quarks (1.136) lead to the following behavior of the left- and right-handed diquark fields:

$$D_i^L \to D_j^L U_{L,ji}^\dagger \ , \qquad D_i^R \to D_j^R U_{R,ji}^\dagger \ , \tag{3.58}$$

where we have used that[2]

$$\epsilon^{ijk} U_{jj'} U_{kk'} = \epsilon_{i'j'k'} U_{i'i}^\dagger \ , \qquad \forall\ U \in SU(3) \ . \tag{3.64}$$

The parity and charge-conjugation transformations of the left- and right-handed diquark fields are summarized in Tab. 3.3. They can be obtained considering the transformation properties of the constituent diquark fields as given in Tab. 3.2, or considering the transformations of the constituent quark fields as given in Eqs. (1.111) and (1.113).

The transformation properties in Eq. (3.58) show that a left-handed diquark D_i^L transforms as a left-handed antiquark and the right-handed diquark D_i^R as a right-handed

[2] Proof of $\epsilon^{ijk} U_{jj'} U_{kk'} = \epsilon_{i'j'k'} U_{ii'}^\dagger\ \forall\ U \in SU(3)$:

$$\epsilon^{a'b'c'} \epsilon^{abc} U_{aa'} U_{bb'} U_{cc'} = \epsilon^{a'b'c'} \left(U_{1a'}U_{2b'}U_{3c'} + U_{3a'}U_{1b'}U_{2c'} + U_{2a'}U_{3b'}U_{1c'} + \right. \tag{3.59}$$

$$\left. - \ U_{1a'}U_{3b'}U_{2c'} - U_{2a'}U_{1b'}U_{3c'} - U_{3a'}U_{2b'}U_{1c'}\right) \tag{3.60}$$

$$= 3 \det U - (-3 \det U) = 6 \det U = 6 = \epsilon^{a'b'c'} \epsilon_{a'b'c'} \tag{3.61}$$

$$\leftrightarrow \qquad \epsilon^{abc} U_{aa'} U_{bb'} U_{cc'} = \epsilon_{a'b'c'} \tag{3.62}$$

$$\leftrightarrow \qquad \epsilon^{abc} U_{bb'} U_{cc'} = \epsilon_{a'b'c'} U_{a'a}^\dagger \ , \tag{3.63}$$

where we have used $\det U = \epsilon^{abc} U_{1a} U_{2b} U_{3c} \equiv 1$ and $\epsilon^{a'b'c'} \epsilon_{a'b'c'} = \sum_{b'c'} (\delta_{b'b'}\delta_{c'c'} - \delta_{b'c'}\delta_{c'b'}) = \sum_{b'c'} \delta_{b'b'}\delta_{c'c'} - \sum_{c'} \delta_{c'c'} = 6$. In the last step, we have multiplied the equation from the right-hand side with $U_{a'a}^\dagger$.

	$D_i^L(x)$	$D_i^R(x)$
Parity P :	$-D_i^R(t,-\boldsymbol{x})$	$-D_i^L(t,-\boldsymbol{x})$
Charge conjugation C :	$D_i^{R\dagger}(x)$	$D_i^{L\dagger}(x)$
Chiral χ :	$D_i^L(x)U_L^\dagger$	$D_i^R(x)U_R^\dagger$

Table 3.3. Parity, charge-conjugation and chiral transformations of the left- and right-handed diquark fields.

antiquark. Thus, constructing baryon fields as combinations of a left- or right-handed diquark and a left- or right-handed quark,

$$N_{1L,ij} \equiv D_j^R q_{i,L} \ , \tag{3.65}$$
$$N_{1R,ij} \equiv D_j^R q_{i,R} \ , \tag{3.66}$$
$$N_{2L,ij} \equiv D_j^L q_{i,L} \ , \tag{3.67}$$
$$N_{2R,ij} \equiv D_j^L q_{i,R} \tag{3.68}$$

is in close analogy to the construction of meson fields as combinations of a left-/right-handed quark and a right-/left-handed antiquark, see Eq. (2.50). Note, the fields (3.65) to (3.68) can also be introduced as linear combinations of the previously discussed octet fields in Eqs.(3.50)–(3.53):

$$N_{1L} \equiv \frac{1}{2\sqrt{2}}\left(B_1^* - B_1 + B_2 - B_2^*\right) \ , \tag{3.69}$$
$$N_{1R} \equiv \frac{1}{2\sqrt{2}}\left(B_1^* + B_1 + B_2 + B_2^*\right) \ , \tag{3.70}$$
$$N_{2L} \equiv \frac{1}{2\sqrt{2}}\left(B_1^* - B_1 - B_2 + B_2^*\right) \ , \tag{3.71}$$
$$N_{2R} \equiv \frac{1}{2\sqrt{2}}\left(B_1^* + B_1 - B_2 - B_2^*\right) \ . \tag{3.72}$$

Using the chiral transformation behavior of the constituent quark (1.136) and diquark (3.58), we compute the chiral transformation of the baryonic fields:

$$N_{1L} \to \ U_L N_{1L} U_R^\dagger \ , \qquad N_{1R} \to \ U_R N_{1R} U_R^\dagger \ , \tag{3.73}$$
$$N_{2L} \to \ U_L N_{2L} U_L^\dagger \ , \qquad N_{2R} \to \ U_R N_{2R} U_L^\dagger \ , \tag{3.74}$$

where the chiral transformation from the left follows the naive assignment (it is determined by the transformation behavior of the quark field), while the one from the right results from the transformation of the diquark field ($1 \leftrightarrow R$, $2 \leftrightarrow L$). The chiral transformation of their Dirac-adjoint versions is given by:

$$\bar{N}_{1L} \to \ U_R \bar{N}_{1L} U_L^\dagger \ , \qquad \bar{N}_{1R} \to \ U_R \bar{N}_{1R} U_R^\dagger \ , \tag{3.75}$$
$$\bar{N}_{2L} \to \ U_L \bar{N}_{2L} U_L^\dagger \ , \qquad \bar{N}_{2R} \to \ U_L \bar{N}_{2R} U_R^\dagger \ . \tag{3.76}$$

	N_{1L}	N_{1R}	N_{2L}	N_{2R}
Parity P :	$-\gamma^0 N_{2R}$	$-\gamma^0 N_{2L}$	$-\gamma^0 N_{1R}$	$-\gamma^0 N_{1L}$
Charge conjugation C :	$C\bar{N}_{2R}^T$	$C\bar{N}_{2L}^T$	$C\bar{N}_{1R}^T$	$C\bar{N}_{1L}^T$
Chiral χ :	$U_L N_{1L} U_R^\dagger$	$U_R N_{1R} U_R^\dagger$	$U_L N_{2L} U_L^\dagger$	$U_R N_{2R} U_L^\dagger$

	M_{1L}	M_{1R}	M_{2L}	M_{2R}
Parity P :	$-\gamma^0 M_{2R}$	$-\gamma^0 M_{2L}$	$-\gamma^0 M_{1R}$	$-\gamma^0 M_{1L}$
Charge conjugation C :	$C\bar{M}_{2R}^T$	$C\bar{M}_{2L}^T$	$C\bar{M}_{1R}^T$	$C\bar{M}_{1L}^T$
Chiral χ :	$U_R M_{1L} U_R^\dagger$	$U_L M_{1R} U_R^\dagger$	$U_R M_{2L} U_L^\dagger$	$U_L M_{2R} U_L^\dagger$

Table 3.4. Parity, charge-conjugation), chiral transformations of $N_{1/2}$ and $M_{1/2}$.

The parity and charge-conjugation transformations of the baryon fields are given in Tab. 3.4. They follow from the respective transformations of the constituent quark and diquark as given in Eqs. (1.111) and (1.113) and Tab. 3.3. As an example, we calculate the parity and charge-conjugation transformations of N_{1L}:

$$N_{1L,ij} \equiv D_j^R q_{iL} \xrightarrow{P} N'_{1L,ij} \equiv -D_j^L \gamma^0 q_{iL} \equiv -\gamma^0 N_{2R,ij} \; , \tag{3.77}$$

$$\begin{aligned} N_{1L,ij} \equiv D_j^R q_{iL} \xrightarrow{C} N'_{1L,ij} &\equiv D_j^{L\dagger} \mathcal{P}_L C \bar{q}_i^T \\ &= C\gamma^0 (-D_j^L)^{T\dagger} q_{iR}^{\dagger T} \\ &= C\gamma^0 (D_j^L q_{iR})^{\dagger T} \\ &= C\bar{N}_{2R,ij}^T \; , \end{aligned} \tag{3.78}$$

where Dirac matrices commute with the diquark, because the latter is a scalar in Dirac space. Furthermore, we have used that $D_j^{L/R} = -(D_j^{L/R})^T$, since two quarks, i.e., fermions are exchanged.

In order to construct a baryonic eLSM Lagrangian which features a chirally invariant mass term, baryon fields have to be included in the mirror assignment, see Refs. [10, 11, 12] and Sec. 2.3.
To this end, we introduce further baryon fields whose chiral transformation from the left is "mirror-like" when compared to Eqs. (3.73) and (3.74). Such transformation properties are achieved by including an additional Dirac matrix between the diquark and quark fields in Eqs. (3.65) to (3.68). This generates a switched transformation behavior from the left (because the left- and right-handed projection operators acting on the quark fields are converted into right- and left-handed ones, respectively, due to $[\gamma^5, \gamma^\mu] = 0$). The occuring

free Lorentz index is contracted with a partial derivative[3], such that the baryonic fields are still Dirac spinors. Finally, we obtain the following mathematical structure of the "mirror-like" fields:

$$M_{1L,ij} \equiv D_i^R i\gamma^\mu \partial_\mu q_{j,L} \ , \tag{3.79}$$
$$M_{1R,ij} \equiv D_i^R i\gamma^\mu \partial_\mu q_{j,R} \ , \tag{3.80}$$
$$M_{2L,ij} \equiv D_i^L i\gamma^\mu \partial_\mu q_{j,L} \ , \tag{3.81}$$
$$M_{2R,ij} \equiv D_i^L i\gamma^\mu \partial_\mu q_{j,R} \ . \tag{3.82}$$

Their chiral transformations read

$$M_{1R} \rightarrow \ U_L M_{1R} U_R^\dagger \ , \qquad M_{1L} \rightarrow \ U_R M_{1L} U_R^\dagger \ , \tag{3.83}$$
$$M_{2R} \rightarrow \ U_L M_{2R} U_L^\dagger \ , \qquad M_{2L} \rightarrow \ U_R M_{2L} U_L^\dagger \ , \tag{3.84}$$

and the chiral transformations of their Dirac adjoint versions are given by

$$\bar{M}_{1L} \rightarrow \ U_R \bar{M}_{1L} U_R^\dagger \ , \qquad \bar{M}_{1R} \rightarrow \ U_R \bar{M}_{1R} U_L^\dagger \ , \tag{3.85}$$
$$\bar{M}_{2L} \rightarrow \ U_L \bar{M}_{2L} U_R^\dagger \ , \qquad \bar{M}_{2R} \rightarrow \ U_L \bar{M}_{2R} U_L^\dagger \ . \tag{3.86}$$

Indeed, the transformations from the left in Eqs. (3.83) and (3.84) are now mirror-like. The transformations from the right, again, result from the transformation of the diquark field ($1 \leftrightarrow R$, $2 \leftrightarrow L$).

Furthermore, we have included an additional phase factor i into the fields (3.79) to (3.82), in order to ensure that the M_1 and M_2 fields behave in the same way as the N_1 and N_2 fields under parity and charge-conjugation transformations. As an example, we calculate the discrete transformations of M_{1L}:

$$M_{1L,ij} \equiv D_j^R i\gamma^\mu \partial_\mu q_{iL} \overset{P}{\rightarrow} \ M'_{1L,ij} \equiv -D_j^L i\gamma^\mu \partial^\mu \gamma^0 q_{iL} \equiv -\gamma^0 M_{2R,ij} \ , \tag{3.87}$$

$$\begin{aligned} M_{1L,ij} \equiv D_j^R i\gamma^\mu \partial_\mu q_{iL} \overset{C}{\rightarrow} \ M'_{1L,ij} &\equiv D_j^{L\dagger} i\gamma^\mu \partial_\mu \mathcal{P}_L C \bar{q}_i^T \\ &= -(D_j^{L\dagger})^T iCC^{-1}\gamma^\mu C \partial_\mu \mathcal{P}_L \gamma^0 q_i^{\dagger T} \\ &= -CD_j^{L\dagger T} i(-\gamma^\mu)^T \gamma^0 \partial_\mu \mathcal{P}_R q_i^{\dagger T} \\ &= C\gamma^0 D_j^{L\dagger T} i\gamma^{\mu T\dagger} \partial_\mu q_{iR}^{\dagger T} \\ &= C\gamma^0 (D_j^L i\gamma^\mu \partial_\mu q_{iR})^{\dagger T} \\ &\equiv C\gamma^0 \bar{M}_{2R}^T \ , \end{aligned} \tag{3.88}$$

where we have used that $\gamma^\mu \gamma^0 = \gamma^0 \gamma^{\mu\dagger}$.

[3]Note, in order to obtain a current which is invariant under color transformations, one can use a covariant $D^\mu = \partial^\mu - g_{\text{QCD}} A^\mu$, compare Ref. [75].

Table 3.4 summarizes the parity and charge-conjugation transformations of all baryonic fields.

Note, parity eigenstates can be constructed as linear combinations of N_1 and N_2, as well as M_1 and M_2, see Sec. 4.4.

Finally, one should note that while for mesons the flavor increase from $N_f = 2$ to $N_f = 3$ is a straightforward process of enlarging the 2×2 matrices [Eqs. (2.135) to (2.139)] to 3×3 matrices [Eqs. (2.57), (2.65), and (2.66)], in the baryonic sector the generalization is more difficult, because the transition from a spinor isodoublet (2.1) to the 3×3 matrix in Eq. (3.55) hast to be performed:

$$\begin{pmatrix} p \\ n \end{pmatrix} \qquad \rightarrow \qquad \begin{pmatrix} \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} . \tag{3.89}$$

Furthermore, the inclusion of a chiral partner $J^P = \frac{1}{2}^-$ is also not as straightforward as in the $N_f = 2$ case.

Chapter 4.

Baryonic eLSM Lagrangian for $N_f = 3$

This section performs the extension of the baryonic eLSM from $N_f = 2$ to $N_f = 3$. While this extension has already been done in the mesonic sector, in the baryonic sector, up to now only the two-flavor studies of Refs. [12, 13] exist, see Sec. 2.3. We believe that these investigations might be useful for future projects in hadron physics as well as astrophysics, addressing for instance scattering processes involving strange hadrons [76, 77, 78, 79] or the hyperon puzzle for compact stars [64, 80].

The baryonic chiral three-flavor Lagrangian is presented in Sec. 4.1. Then, Sec. 4.2 is inserted to discuss the large-N_c scaling properties of the parameters. Subsequently, in Sec. 4.3, we show how to prove the chiral and CP invariance of the Lagrangian. Finally, we perform the transition to the Lagrangian as a function of parity eigenstates which can be assigned to physical fields, see Sec. 4.4.

4.1. Chiral Lagrangian for baryons

In order to describe three-flavor baryons and their interactions with mesons, we investigate the following Lagrangian, which is invariant under chiral, as well as parity and charge-conjugation transformations:

$$\begin{aligned}
\mathcal{L}_{\text{bar}} = &\operatorname{Tr}(\bar{N}_{1L} i\gamma_\mu D^\mu_{2L} N_{1L} + \bar{N}_{1R} i\gamma_\mu D^\mu_{1R} N_{1R} + \bar{N}_{2L} i\gamma_\mu D^\mu_{1L} N_{2L} + \bar{N}_{2R} i\gamma_\mu D^\mu_{2R} N_{2R} \\
&+ \bar{M}_{1L} i\gamma_\mu D^\mu_{4R} M_{1L} + \bar{M}_{1R} i\gamma_\mu D^\mu_{3L} M_{1R} + \bar{M}_{2L} i\gamma_\mu D^\mu_{3R} M_{2L} + \bar{M}_{2R} i\gamma_\mu D^\mu_{4L} M_{2R}) \\
&- g_N \operatorname{Tr}(\bar{N}_{1L} \Phi N_{1R} + \bar{N}_{1R} \Phi^\dagger N_{1L} + \bar{N}_{2L} \Phi N_{2R} + \bar{N}_{2R} \Phi^\dagger N_{2L}) \\
&- g_M \operatorname{Tr}(\bar{M}_{1L} \Phi^\dagger M_{1R} + \bar{M}_{1R} \Phi M_{1L} + \bar{M}_{2L} \Phi^\dagger M_{2R} + \bar{M}_{2R} \Phi M_{2L}) \\
&- m_{0,1} \operatorname{Tr}(\bar{N}_{1L} M_{1R} + \bar{M}_{1R} N_{1L} + \bar{N}_{2R} M_{2L} + \bar{M}_{2L} N_{2R}) \\
&- m_{0,2} \operatorname{Tr}(\bar{N}_{1R} M_{1L} + \bar{M}_{1L} N_{1R} + \bar{N}_{2L} M_{2R} + \bar{M}_{2R} N_{2L}) \\
&- \kappa_1 \operatorname{Tr}(\bar{N}_{1R} \Phi^\dagger N_{2L} \Phi + \bar{N}_{2L} \Phi N_{1R} \Phi^\dagger) - \kappa_1' \operatorname{Tr}(\bar{N}_{1L} \Phi N_{2R} \Phi + \bar{N}_{2R} \Phi^\dagger N_{1L} \Phi^\dagger) \\
&- \kappa_2 \operatorname{Tr}(\bar{M}_{1R} \Phi M_{2L} \Phi + \bar{M}_{2L} \Phi^\dagger M_{1R} \Phi^\dagger) - \kappa_2' \operatorname{Tr}(\bar{M}_{1L} \Phi^\dagger M_{2R} \Phi + \bar{M}_{2R} \Phi M_{1L} \Phi^\dagger)\ .
\end{aligned} \tag{4.1}$$

The dynamics of the baryon fields are, as usually, described by Dirac Lagrangians (equipped by traces to ensure the invariance under chiral transformations). Moreover, the following covariant derivatives have been introduced:

$$D^\mu_{kR} = \partial^\mu - ic_k R^\mu \ , \qquad D^\mu_{kL} = \partial^\mu - ic_k L^\mu \ , \quad \forall\ k \in \{1,2,3,4\} \ . \tag{4.2}$$

They generate baryon-(axial-)vector interactions proportional to the dimensionless coupling constants c_1, c_2, c_3, and c_4. The third and fourth lines in Eq. (4.1) contain couplings of baryon fields to (pseudo)scalar mesons with the dimensionless coupling constants g_N and g_M. Explicit mass terms parametrized by $m_{0,1}$ and $m_{0,2}$ are possible, since a mirror assignment is realized, see Eqs. (3.73) and (3.83). Finally, in the last two lines four-point interactions proportional to κ_1, κ_2, κ_1', κ_2' appear.

In total, the Lagrangian contains twelve parameters:

$$c_1 \ , \ c_2 \ , \ c_3 \ , \ c_4 \ , \ g_N \ , \ g_M \ , \ m_{0,1} \ , \ m_{0,2} \ , \ \kappa_1 \ , \ \kappa_1' \ , \ \kappa_2 \ , \ \kappa_2' \ . \tag{4.3}$$

The constants c_1 to c_4, g_N, and g_M are dimensionless. The corresponding terms are of scaling dimension four[1]. Thus, for $m_{0,1} = 0 = m_{0,2} = \kappa_1 = \kappa_1' = \kappa_2 = \kappa_2'$ the baryonic part of the model is dilatation invariant.

In contrast, the mass terms are of naive scaling dimension three, implying that

$$[m_{0,1}] = [m_{0,2}] = [\text{energy}] \ . \tag{4.5}$$

However, we can render these terms dilatation-invariant (as already discussed in Sec. 2.3.2) by assuming them to arise from (dilatation-invariant) interactions of a glueball and/or a four-quark state with baryons, see Eq. (2.144) and the previous discussion.

The terms in the last two lines in Eq. (4.1) also break dilatation invariance, because they have a naive scaling dimension five, i.e., the corresponding coupling constants are of dimension

$$[\kappa_1] = [\kappa_1'] = [\kappa_2] = [\kappa_2'] = [\text{energy}^{-1}] \ . \tag{4.6}$$

Thus, to render these terms dilatation invariant κ_1 to κ_2' need to be proportional to inverse powers of a gluon and/or a four-quark condensate, which means taht the corresponding terms can only arise from non-analytic baryons-glueballs/four-quark interactions. Although such terms should be avoided in a Lagrangian description, we have to include

[1] For instance, the dimension analysis of the g_N and g_M terms reads:

$$\begin{aligned} \text{terms} \ \propto g_{N/M} : \quad & [(\text{fermionic field})(\text{scalar field})(\text{fermionic field})] \\ & = [\text{energy}^{\frac{3}{2}+1+\frac{3}{2}}] = [\text{energy}^4] \ . \end{aligned} \tag{4.4}$$

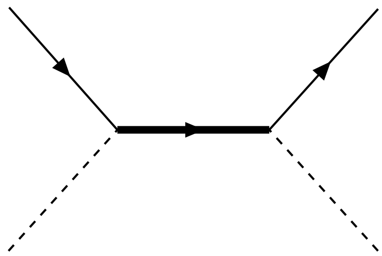


Figure 4.1. Effective four-point interaction arising from two three-point interaction vertices between a meson (dashed line) and baryon (solid line) and a heavier baryon (thick line).

them, because otherwise the baryonic fields become (at odds with experiment) pairwise degenerate in mass, see Sec. 5.2. We may consider these terms to be effective four-point interactions arising from two (dilatation-invariant) three-point meson-baryon-baryon interaction vertices, where one of the baryons is much heavier then the typical energy scale where the Lagrangian (4.1) is valid. If these vertices are connected by the propagator of the heavier baryon, see Fig. 4.1, the connecting propagator can be considered as static and homogeneous, which leads to the terms parametrized by κ_1 to κ_2'.

Note, the Lagrangian (4.1) contains only the large-N_c dominant terms (except for the terms parametrized by κ_1, κ_2, κ_1', κ_2', scaling as N_c^{-1}, see Sec. 4.2). Further chirally, parity, and charge-conjugation invariant terms, which in the large-N_c expansion are suppressed, are given by

$$\begin{aligned}
&- n_1 \operatorname{Tr}(\bar{N}_{1L} M_{2R} \Phi + \bar{M}_{2R} N_{1L} \Phi^\dagger + \bar{M}_{1L} N_{2R} \Phi + \bar{N}_{2R} M_{1L} \Phi^\dagger) \\
&- n_2 \operatorname{Tr}(\bar{N}_{1R} M_{2L} \Phi + \bar{M}_{2L} N_{1R} \Phi^\dagger + \bar{M}_{1R} N_{2L} \Phi + \bar{N}_{2L} M_{1R} \Phi^\dagger) \\
&- \epsilon_1 [\operatorname{Tr}(\bar{N}_{1L} \Phi) \operatorname{Tr}(N_{2R} \Phi) + \operatorname{Tr}(\bar{N}_{2R} \Phi^\dagger) \operatorname{Tr}(N_{1L} \Phi^\dagger)] \\
&- \epsilon_2 [\operatorname{Tr}(\bar{M}_{1R} \Phi) \operatorname{Tr}(M_{2L} \Phi) + \operatorname{Tr}(\bar{M}_{2L} \Phi^\dagger) \operatorname{Tr}(M_{1R} \Phi^\dagger)] \\
&- \epsilon_3 \operatorname{Tr}(\Phi^\dagger \Phi) \operatorname{Tr}(\bar{N}_{1L} M_{1R} + \bar{M}_{1R} N_{1L} + \bar{N}_{2R} M_{2L} + \bar{M}_{2L} N_{2R}) \\
&- \epsilon_4 \operatorname{Tr}(\Phi^\dagger \Phi) \operatorname{Tr}(\bar{N}_{1R} M_{1L} + \bar{M}_{1L} N_{1R} + \bar{N}_{2L} M_{2R} + \bar{M}_{2R} N_{2L}) \; . \qquad (4.7)
\end{aligned}$$

The constants n_1 and n_2 scale as $N_c^{-1/2}$ and the constants ϵ_1 to ϵ_4 scale as N_c^0. The related terms are suppressed compared to terms parametrized by g_N and g_M which scale as $N_c^{1/2}$. We investigate the large-N_c scaling properties in more detail in the following section.

4.2. Large-N_c scaling properties of the parameters

In a generalized $SU(N_c)$ color gauge theory, baryons are completely antisymmetric N_c-quark states in color space:

$$B \equiv \epsilon_{a_1 a_2 \ldots a_{N_c}} q_{a_1} q_{a_2} \ldots q_{a_{N_c}} \; , \qquad (4.8)$$

where $a_i \in \{1, 2, \ldots, N_c\}$ are color indices. Flavor indices are suppressed. (Note, the following considerations will be are carried out for the physical case $N_f = N_c = 3$. They can be straightforwardly extended to arbitrary N_c where, however, $N_f = N_c$. Extensions where $N_f \neq N_c$ are also possible, but this would go beyond the scope of the present section, which only aims to present the large-N_c arguments needed to distinguish between dominant and subdominant terms.)

As a consequence of Eq. (4.8), the mass of a baryon grows with N_c, see Refs. [40, 41]:

$$m_B \propto N_c \ . \tag{4.9}$$

Heuristically, we can understand this utilizing the quark-model picture. Here, we assume that the baryon mass is a sum of the quark masses, the quark kinetic energy, and the quark-quark potential energy. Then, in Ref. [41] the large-N_c dependence is explained as follows: "Since there are N_c quarks in the baryon, the quark mass contributes an amount [proportional] to N_c. For the quark kinetic energy, we may guess that kinetic energy of N_c quarks is N_c times the the kinetic energy of one quark. For the potential energy, the interaction between one pair of quarks is of order $1/N_c$, but the total potential energy is a sum of all of the pair interactions, and there are $\frac{1}{2}N_c^2$ pairs. [These] factors N_c^2 and $1/N_c$ combine to give a contribution of order N_c. [Thus, all contributions to the mass are] of the same order of magnitude [and] the entire baryon mass is of order N_c."

Considering baryons as quark-diquark objects like N_1, N_2, M_1, and M_2 [Eqs (3.65) to (3.68) and (3.79) to (3.82)], we assume that they are represented by the combination of a $(N_c - 1)$-quark state D_{a_1} and a quark in the large-N_c limit, i.e., (suppressing different chiralities):

$$N_1, N_2, M_1, M_2 \sim D_{a_1} q^{a_1} \ , \tag{4.10}$$

where D_{a_1} is the generalization of the 'good' diquark ($N_c = 3$) in Eqs. (3.44) and (3.45) to arbitrary N_c:

$$D_{a_1} \equiv \epsilon_{a_1 a_2 \ldots a_{N_c}} q^{a_2} \ldots q^{a_{N_c}} \ . \tag{4.11}$$

Following the same arguments as for the baryon mass, this entails that the mass of a generalized diquark grows as $N_c - 1$. Thus, for N_c being large we can assume that it scales as

$$m_D \propto N_c \ . \tag{4.12}$$

In order to investigate the large-N_c scaling properties of the coupling constants in Eq. (4.1), it is useful to study the related Feynman diagrams in the large-N_c limit. The basic diagram is displayed in Fig. 4.2 (where for simplicity $N_c = 3$ quark lines are drawn). It corresponds to the mass energy of the baryon. Since the mass of the baryon scales as N_c,

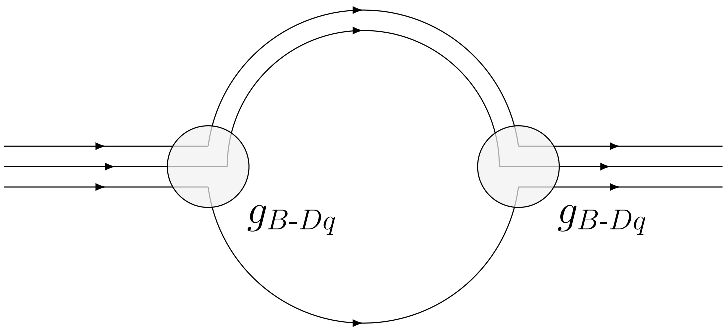


Figure 4.2. Mass energy of the baryon - $m_{0,1}, m_{0,2}$-terms.

the combinatoric factor of the diagram in Fig. 4.2 should also scale as N_c:

$$\left(g_{B\text{-}Dq} \, \frac{1}{m_D} \, g_{B\text{-}Dq} \right) N_c \overset{!}{\propto} N_c \tag{4.13}$$

where $g_{B\text{-}Dq}$ parametrizes the coupling of a baryon to a generalized diquark and a quark, and the generalized diquark (being heavy) contributes as $1/m_D \propto 1/N_c$. The overall factor N_c arises from the circulating color $a_1 = 1, 2, \ldots, N_c$. As a consequence, the baryon-diquark-quark coupling has to grow as

$$g_{B\text{-}Dq} \propto \sqrt{N_c} \; . \tag{4.14}$$

The diagram in Fig. 4.2 corresponds to the terms parametrized by $m_{0,1}$ and $m_{0,2}$. Thus the large-N_c dependence of these parameters is given by

$$m_{0,1} \propto N_c \; , \qquad m_{0,2} \propto N_c \; . \tag{4.15}$$

In order to find the scaling behavior of the remaining model parameters, we have to include mesons into the considerations. In contrast to baryons, mesons are quark-antiquark objects (regardless of N_c). Their masses are independent of N_c, see Ref.: [41]:

$$m_M \propto N_c^0 \; , \tag{4.16}$$

and the coupling of a standard meson to $\bar{q}q$ scales as, see Refs. [40, 41],

$$g_{M\text{-}\bar{q}q} \propto 1/\sqrt{N_c} \; . \tag{4.17}$$

The dominant interaction of a baryon with a meson is represented by the diagram in Fig. 4.3. It scales as

$$\left(g_{B\text{-}Dq} \, \frac{1}{m_D} \, g_{M\text{-}\bar{q}q} \, g_{B\text{-}Dq} \right) \; N_c \propto \sqrt{N_c} \; , \tag{4.18}$$

where the overall factor N_c emerges for the same reasons as in Fig. 4.2. The diagram in Fig. 4.3 corresponds to the interaction terms parametrized by c_1 to c_4, g_N, and g_M. We

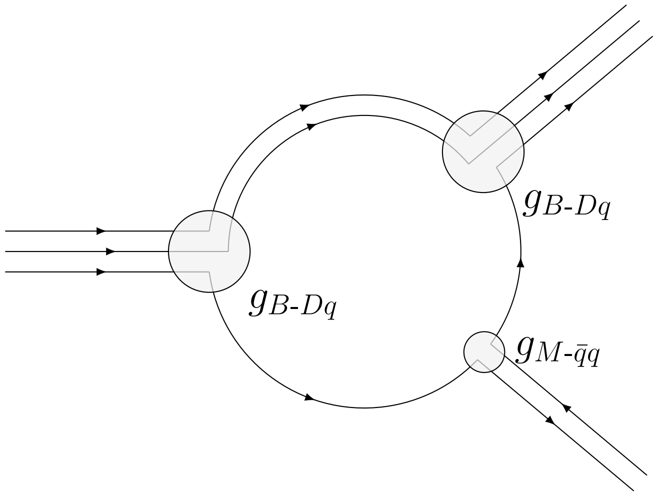


Figure 4.3. Dominant baryon-meson interaction term - c_k, g_N, g_M-terms.

exemplarily prove this for $g_N \,\mathrm{Tr}(\bar{N}_{1L}\Phi N_{1R})$ by investigating the flavor indices in the flavor trace. For the sake of clarity, we omit the indices $1, L, R$ of the baryon fields and display only the flavor indices:

$$\begin{aligned}
\mathrm{Tr}(\bar{N}\Phi N) &= \bar{N}_{ij}\Phi_{jk}N_{ki} \\
&\sim (\bar{q}_j\bar{D}_i)(\bar{q}_k q_j)(D_i q_k) \\
&\sim [\bar{q}_j(\epsilon_{ilm}\bar{q}_l\bar{q}_m)]\,(\bar{q}_k q_j)\,[(\epsilon_{ino}q_n q_o)q_k] \\
&= (\delta_{ln}\delta_{mo} - \delta_{lo}\delta_{mn})\,[\bar{q}_j(\bar{q}_l\bar{q}_m)]\,(\bar{q}_k q_j)\,[(q_n q_o)q_k] \\
&= [\bar{q}_j(\bar{q}_l\bar{q}_m)]\,(\bar{q}_k q_j)\,[(q_l q_m)q_k] - [\bar{q}_j(\bar{q}_l\bar{q}_m)]\,(\bar{q}_k q_j)\,[(q_m q_l)q_k] \ .
\end{aligned} \tag{4.19}$$

In Eq. (4.19) the quarks of the diquark fields are linked (same indices l, m), while the (separate) quark lines of the in- and outgoing baryons are connected to the produced meson, just as illustrated in Fig. 4.3, q.e.d.. Thus, according to Eq. (4.18) for $N_c \to \infty$ the baryon-baryon-meson coupling constants scale as

$$c_k \propto \sqrt{N_c}\ , \qquad g_N \propto \sqrt{N_c}\ , \qquad g_M \propto \sqrt{N_c} \qquad \forall\ k \in \{1,2,3,4\}\ . \tag{4.20}$$

The terms in Eq. (4.7) which are parametrized by n_1 and n_2 have the following flavor structure:

$$\begin{aligned}
\mathrm{Tr}(\bar{N}M\Phi) &= \bar{N}_{ij}M_{jk}\Phi_{ki} \\
&\sim (\bar{q}_j\bar{D}_i)(D_k q_j)(\bar{q}_i q_k) \\
&\sim [\bar{q}_j(\epsilon_{ilm}\bar{q}_l\bar{q}_m)]\,[(\epsilon_{kno}q_n q_o)q_j]\,(\bar{q}_i q_k) \\
&= \left[\delta_{ik}\,(\delta_{ln}\delta_{mo} - \delta_{lo}\delta_{mn}) - \delta_{in}\,(\delta_{lk}\delta_{mo} - \delta_{lo}\delta_{mk}) + \delta_{io}\,(\delta_{lk}\delta_{mn} - \delta_{ln}\delta_{mk})\right] \\
&\quad\ \times [\bar{q}_j(\bar{q}_l\bar{q}_m)]\,[(q_n q_o)q_j]\,(\bar{q}_i q_k)
\end{aligned} \tag{4.21}$$

where we have exemplarily picked $n_1\,\mathrm{Tr}(\bar{N}_{1L}M_{2R}\Phi)$ (omitting the $1, 2, L, R$ indices of the baryon fields). Already in the second line, we see that the quark-antiquark pair of the meson is connected to the diquark, while the single quark line of the baryon fields (flavor index j) evolves undisturbed. In the third line, recalling the diquark structure $\epsilon_{ijk}q_j q_k$, we find that various flavor structures appear. This also includes flavor-disconnected diagrams, such as for instance the following coupling term ($i = 3 = k$)

$$\bar{D}_3 D_3(\bar{q}_3 q_3) \ , \tag{4.22}$$

which corresponds to the transition

$$[u, d] \to [u, d]\bar{s}s \ . \tag{4.23}$$

Thus, an $\bar{s}s$ pair emerges from a non-strange structure, which is a large-N_c suppressed process.
Nevertheless also some flavor-connected diagrams seem to be present. For instance, for $i = 2$ and $k = 3$ we get the coupling term $\bar{D}_2 D_3(\bar{q}_2 q_3)$, which implies $[u, s] \to [u, d]\bar{d}s$.

However, considering the color flow restricts the number of possible transitions. The color structure of the term (4.21) reads

$$\begin{aligned}\mathrm{Tr}(\bar{N} M \Phi) &\sim (\bar{q}_a \bar{D}_a)(D_a q_a)(\bar{q}_a q_a) \\ &\sim [\bar{q}_a(\epsilon_{ade}\bar{q}_d\bar{q}_e)]\,[(\epsilon_{afg}q_f q_g)q_a]\,(\bar{q}_a q_a) \\ &= (\delta_{df}\delta_{eg} - \delta_{dg}\delta_{ef})\,[\bar{q}_a(\bar{q}_d\bar{q}_e)]\,[(q_f q_g)q_a]\,(\bar{q}_a q_a) \ ,\end{aligned} \tag{4.24}$$

where we keep only one color index for simplicity and the summation over color indices within each hadron is implicit. In the most general case, the color structure is given by $(\bar{q}_a\bar{D}_a)(D_b q_b)(\bar{q}_c q_c)$ with implicit sum over the indices a, b, c. This is because all hadrons have to be singlets in color space, and there cannot be any a priori relation between the color of two quarks or diquarks sitting in two different hadrons. Without loss of generality, we restrict the discussion to $(\bar{q}_a\bar{D}_a)(D_a q_a)(\bar{q}_a q_a)$. Color can be changed by intermediate gluons, but these gluons can be formally absorbed into the hadron-quark vertices (if they are of the same order in large-N_c) or they are further suppressed, such that we are left with $(\bar{q}_a\bar{D}_a)(D_a q_a)(\bar{q}_a q_a)$.

Considering Eq. (4.24) we see that the color-anticolor has to different to the color of the quark, i.e. the interaction term $\mathrm{Tr}(\bar{N} M \Phi)$ allows only for transitions of the following type ($a = 3$):

$$[R, G] \to [R, G]\bar{B}B \ . \tag{4.25}$$

Hence, in the previously mentioned case $i = 2$ and $k = 3$ terms of the form $\mathrm{Tr}(\bar{N} M \Phi)$ correspond to a transition of the type $s_r u_g \to u_r d_g(\bar{d}_b s_b)$. The related Feynman diagram cannot be drawn by simply exchanging quark lines, but color switches generated by additional gluons are inevitable.

However, instead of searching for such appropriate gluon configurations, we exploit a hadronic approach which allows for an alternative way. Namely, we may introduce an

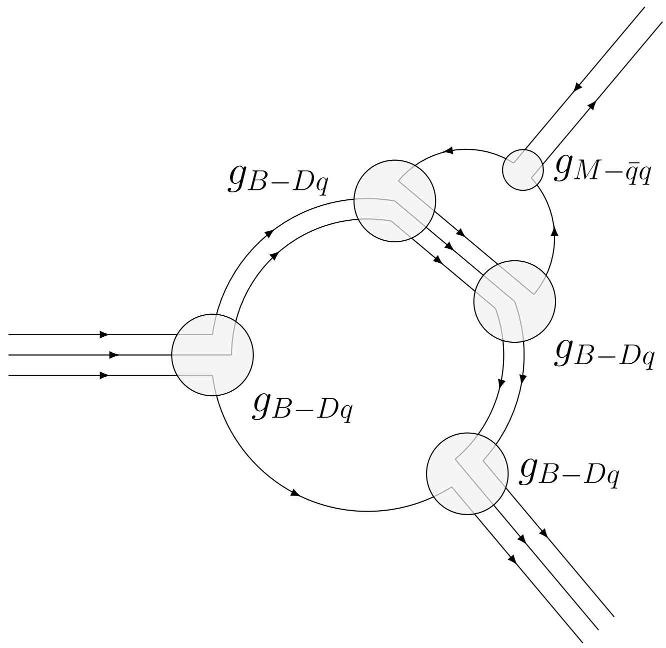


Figure 4.4. Baryon-meson interaction with an intermediate virtual baryon - n_1, n_2 terms.

additional white intermediate virtual baryon, as shown in Fig. 4.4, such that the coupling of the diquark to a quark with the "missing" color is fulfilled due to color confinement. The large-N_c scaling of this diagram is calculated as

$$\left(g_{B\text{-}Dq} \, \frac{1}{m_D} \, g_{B\text{-}Dq} \, g_{M\text{-}\bar{q}q} \, \frac{1}{m_B} \, g_{B\text{-}Dq} \, \frac{1}{m_D} g_{B\text{-}Dq} \right) N_c \propto \frac{1}{\sqrt{N_c}} \; . \tag{4.26}$$

Thus, terms of the type $\mathrm{Tr}(\bar{N} M \Phi)$ are suppressed with N_c and the constants n_1 and n_2 scale as

$$n_1 \; , \; n_2 \; \propto N_c^{-\frac{1}{2}} \; . \tag{4.27}$$

The κ-terms in the Lagrangian (4.1) have the following flavor structure:

$$\begin{aligned} \mathrm{Tr}(\bar{N}\Phi N \Phi) &\sim (\bar{q}_j \bar{D}_i)(q_k q_j)(D_l q_k)(q_i q_l) \\ &\sim [\bar{q}_j(\epsilon_{imn} q_m q_n)](q_k q_j)[(\epsilon_{lpq} q_p q_q) q_k](q_i q_l) \; . \end{aligned} \tag{4.28}$$

The evaluation of these terms is similar to the one of the term (4.21). Taking also the color structure into account, we find that the κ-terms are represented by the diagram in Fig. 4.5 scaling as

$$\left(g^4_{B\text{-}Dq} \, g^2_{M\text{-}\bar{q}q} \, \frac{1}{m_D^2} \frac{1}{m_B} \right) N_c \propto \frac{1}{N_c} \; . \tag{4.29}$$

Thus, the related constant behave as

$$\kappa_i, \kappa_i' \propto N_c^{-1} \quad \forall \; i \in \{1, 2\} \tag{4.30}$$

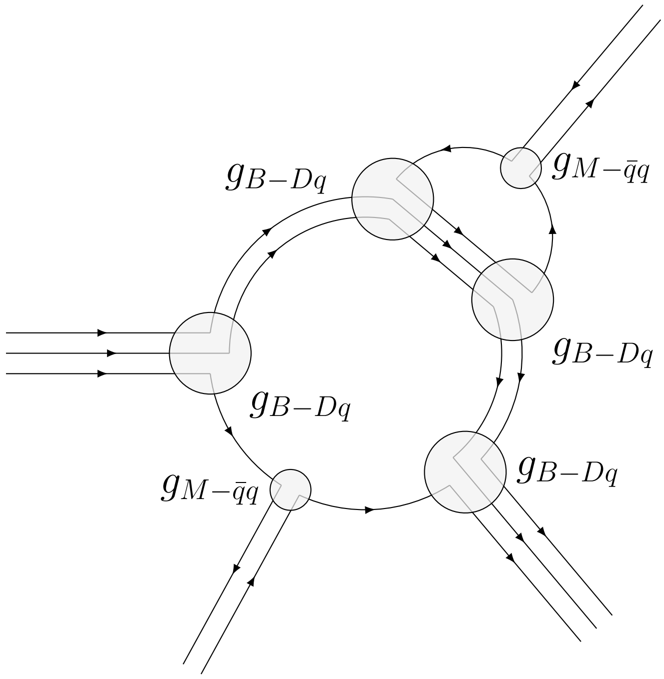


Figure 4.5. Baryon-baryon-meson-meson interaction - κ-terms.

Parameter	$m_{0,1}, m_{0,2}$	c_1, c_2, c_3, c_4	g_N, g_M	$\kappa_1, \kappa'_1, \kappa_2, \kappa'_2$	n_1, n_2	$\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$
Scaling	N_c	$N_c^{1/2}$	$N_c^{1/2}$	N_c^{-1}	$N_c^{-1/2}$	N_c^0

Table 4.1. Large-N_c scaling properties of the parameters.

in the limit of large N_c.

Finally the terms in the Lagrangian (4.7) which are parametrized by $\epsilon_1, \epsilon_2, \epsilon_3$, and ϵ_4 are proportional to a product of two traces. Such terms are usually large-N_c suppressed [with one important exception: the term arising from the axial $U(1)_A$ anomaly, see Sec. 6]. The flavor structure of the ϵ-terms is given by:

$$\mathrm{Tr}(\bar{N}\Phi)\,\mathrm{Tr}(N\Phi) \sim (\bar{q}_j\bar{D}_i)(q_iq_j)(D_lq_k)(q_kq_l) \tag{4.31}$$

$$\mathrm{Tr}(\Phi\Phi)\,\mathrm{Tr}(\bar{N}N) \sim (q_iq_j)(q_jq_i)(\bar{q}_l\bar{D}_k)(D_kq_l)\ . \tag{4.32}$$

A similar procedure as in the previous cases delivers the following large-N_c scaling of the parameters:

$$\epsilon_i \propto N_c^0 \quad \forall\, i \in \{1,2,3,4\}\ . \tag{4.33}$$

The large-N_c scaling properties of all parameters are again summarized in Tab. 4.1.

4.3. Proof of chiral, parity, and charge-conjugation invariance

Using the transformation properties of the baryon field as summarized in Tab. 3.4 and the transformations of the mesonic fields as given in Tab. 2.2, the Lagrangian (4.1) proves to be invariant under chiral, parity and charge-conjugation transformations. In the following, we show the calculations for some representative terms.

Chiral transformations:

The chiral invariance of the kinetic terms is checked as we illustratively show for N_{1L}:

$$\mathrm{Tr}(\bar{N}_{1L} i\gamma_\mu \partial^\mu N_{1L}) \xrightarrow{\chi} \mathrm{Tr}(U_R \bar{N}_{1L} U_L^\dagger i\gamma_\mu \partial^\mu U_L N_{1L} U_R^\dagger) = \mathrm{Tr}(\bar{N}_{1L} i\gamma_\mu \partial^\mu N_{1L}) \ , \tag{4.34}$$

where we used that the trace is invariant under cyclic permutations. The transformations of the baryon-baryon–(axial-)vector mesons interaction terms look similar, for instance

$$\begin{aligned} c_2\,\mathrm{Tr}(\bar{N}_{1L}\gamma_\mu L^\mu N_{1L}) &\xrightarrow{\chi} c_2\,\mathrm{Tr}(U_R\bar{N}_{1L}U_L^\dagger\gamma_\mu U_L L^\mu U_L^\dagger U_L N_{1L} U_R^\dagger) \\ &= c_2\,\mathrm{Tr}(\bar{N}_{1L}\gamma_\mu L^\mu N_{1L}) \ , \\ c_1\,\mathrm{Tr}(\bar{N}_{1R}\gamma_\mu R^\mu N_{1R}) &\xrightarrow{\chi} c_1\,\mathrm{Tr}(U_R\bar{N}_{1R}U_R^\dagger\gamma_\mu U_R R^\mu U_R^\dagger U_R N_{1R} U_R^\dagger) \\ &= c_1\,\mathrm{Tr}(\bar{N}_{1R}\gamma_\mu R^\mu N_{1R}) \ , \end{aligned} \tag{4.35}$$

The mass terms transform as

$$m_{0,1}\,\mathrm{Tr}(\bar{N}_{1L}M_{1R}) \xrightarrow{\chi} m_{0,1}\,\mathrm{Tr}(U_R\bar{N}_{1L}U_L^\dagger U_L M_{1R} U_R^\dagger) = m_{0,1}\,\mathrm{Tr}(\bar{N}_{1L}M_{1R}) \ , \tag{4.36}$$
$$m_{0,2}\,\mathrm{Tr}(\bar{N}_{1R}M_{1L}) \xrightarrow{\chi} m_{0,2}\,\mathrm{Tr}(U_R\bar{N}_{1R}U_R^\dagger U_R M_{1L} U_R^\dagger) = m_{0,2}\,\mathrm{Tr}(\bar{N}_{1R}M_{1L}) \ . \tag{4.37}$$

The invariance of baryon-baryon-(pseudo)scalar couplings is computed as

$$\begin{aligned} g_N\,\mathrm{Tr}(\bar{N}_{1L}\Phi N_{1R}) &\xrightarrow{\chi} g_N\,\mathrm{Tr}(U_R\bar{N}_{1L}U_L^\dagger U_L\Phi U_R^\dagger U_R N_{1R} U_R^\dagger) \\ &= g_N\,\mathrm{Tr}(\bar{N}_{1L}\Phi N_{1R}) \ , \end{aligned} \tag{4.38}$$
$$\begin{aligned} g_M\,\mathrm{Tr}(\bar{M}_{1L}\Phi^\dagger M_{1R}) &\xrightarrow{\chi} g_M\,\mathrm{Tr}(U_R\bar{M}_{1L}U_R^\dagger U_R\Phi^\dagger U_L^\dagger U_L M_{1R} U_R^\dagger) \\ &= g_M\,\mathrm{Tr}(\bar{M}_{1L}\Phi^\dagger M_{1R}) \ . \end{aligned} \tag{4.39}$$

For the interaction terms of two baryon fields with two (pseudo)scalar meson matrices we obtain

$$\begin{aligned} \kappa_1\,\mathrm{Tr}(\bar{N}_{1R}\Phi^\dagger N_{2L}\Phi) &\xrightarrow{\chi} \kappa_1\,\mathrm{Tr}(U_R\bar{N}_{1R}U_R^\dagger U_R\Phi^\dagger U_L^\dagger U_L N_{2L} U_L^\dagger \Phi U_R^\dagger) \\ &= \kappa_1\,\mathrm{Tr}(\bar{N}_{1R}\Phi^\dagger N_{2L}\Phi) \ . \end{aligned} \tag{4.40}$$

The chiral invariance of the remaining terms can be shown similarly.

Parity:

To show the invariance under parity transformations, we have to consider pairs of terms. In the case of the kinetic terms the calculation is of the following form:

$$\begin{aligned}\mathrm{Tr}(\bar{N}_{1L} i\gamma_\mu \partial^\mu N_{1L}) &\overset{P}{\rightarrow} \mathrm{Tr}[(-\gamma_0 N_{2R})^\dagger \gamma_0 i\gamma_\mu \partial_\mu (-\gamma_0 N_{2R})] \\ &= \mathrm{Tr}(\bar{N}_{2R} i\gamma_\mu \partial^\mu N_{2R}) \ , \end{aligned} \tag{4.41}$$

$$\mathrm{Tr}(\bar{N}_{2R} i\gamma_\mu \partial^\mu N_{2R}) \overset{P}{\rightarrow} \mathrm{Tr}(\bar{N}_{1L} i\gamma_\mu \partial^\mu N_{1L}) \ , \tag{4.42}$$

where we have used that $\partial^\mu \rightarrow \partial_\mu$ under parity transformations.

Turning to the transformations of the baryon-baryon-(axial-)vector mesons interaction terms, we give the following transformation as an example:

$$\begin{aligned}c_2 \,\mathrm{Tr}(\bar{N}_{1L} \gamma_\mu L^\mu N_{1L}) &\overset{P}{\rightarrow} c_2 \,\mathrm{Tr}[(-\gamma^0 N_{2R})^\dagger \gamma^0 \gamma_\mu R_\mu (-\gamma^0 N_{2R})] \\ &= c_2 \,\mathrm{Tr}(\bar{N}_{2R} \gamma_\mu R^\mu N_{2R}) \ , \end{aligned} \tag{4.43}$$

$$c_2 \,\mathrm{Tr}(\bar{N}_{2R} \gamma_\mu R^\mu N_{2R}) \overset{P}{\rightarrow} c_2 \,\mathrm{Tr}(\bar{N}_{1L} \gamma_\mu L^\mu N_{1L}) \ . \tag{4.44}$$

The mass terms behave under parity transformations as follows, e.g.:

$$\begin{aligned}m_{0,1} \,\mathrm{Tr}(\bar{N}_{1L} M_{1R}) &\overset{P}{\rightarrow} m_{0,1} \,\mathrm{Tr}[(-\gamma^0 N_{2R})^\dagger \gamma^0 (-\gamma^0 M_{2L})] \\ &= m_{0,1} \,\mathrm{Tr}(\bar{N}_{2R} M_{2L}) \ , \end{aligned} \tag{4.45}$$

$$m_{0,1} \,\mathrm{Tr}(\bar{N}_{2R} M_{2L}) \overset{P}{\rightarrow} m_{0,1} \,\mathrm{Tr}(\bar{N}_{1L} M_{1R}) \ . \tag{4.46}$$

The baryon-baryon-(pseudo)scalar meson coupling terms transform in the following way, e.g.:

$$\begin{aligned}g_N \,\mathrm{Tr}(\bar{N}_{1L} \Phi N_{1R}) &\overset{P}{\rightarrow} g_N \,\mathrm{Tr}[(-\gamma^0 N_{2R})^\dagger \gamma^0 \Phi^\dagger (-\gamma^0 N_{2L})] \\ &= g_N \,\mathrm{Tr}(\bar{N}_{2R} \Phi^\dagger N_{2L}) \ , \end{aligned} \tag{4.47}$$

$$g_N \,\mathrm{Tr}(\bar{N}_{2R} \Phi^\dagger N_{2L}) \overset{P}{\rightarrow} g_N \,\mathrm{Tr}(\bar{N}_{1L} \Phi N_{1R}) \ . \tag{4.48}$$

The coupling terms of two baryonic fields and two (pseudo)scalar mesons transformations read as follows, e.g.:

$$\begin{aligned}\kappa_1 \,\mathrm{Tr}(\bar{N}_{1R} \Phi^\dagger N_{2L} \Phi) &\overset{P}{\rightarrow} \kappa_1 \,\mathrm{Tr}[(-\gamma^0 N_{2L})^\dagger \gamma^0 \Phi (-\gamma^0 N_{1R}) \Phi^\dagger] \\ &= \kappa_1 \,\mathrm{Tr}(\bar{N}_{2L} \Phi N_{1R} \Phi^\dagger) \ , \end{aligned} \tag{4.49}$$

$$\kappa_1 \,\mathrm{Tr}(\bar{N}_{2L} \Phi N_{1R} \Phi^\dagger) \overset{P}{\rightarrow} \kappa_1 \,\mathrm{Tr}(\bar{N}_{1R} \Phi^\dagger N_{2L} \Phi) \ . \tag{4.50}$$

The parity-invariance of the remaining terms can be shown similarly.

Charge conjugation:

The charge-conjugation transformation of the kinetic term describing the dynamics of e.g. N_{1L} reads

$$\begin{aligned}\mathrm{Tr}(\bar{N}_{1L} i\gamma_\mu \partial^\mu N_{1L}) \stackrel{C}{\to} \; & \mathrm{Tr}\left(N_{2R}^T C i\gamma_\mu \partial^\mu C \bar{N}_{2R}^T\right) \\ &= \mathrm{Tr}\left(N_{2R}^T i\gamma_\mu^T \partial^\mu \bar{N}_{2R}^T\right) \\ &= \mathrm{Tr}\left[-(\partial^\mu \bar{N}_{2R}) i\gamma_\mu N_{2R}\right] \\ &= \mathrm{Tr}\left(\bar{N}_{2R} i\gamma_\mu \partial^\mu N_{2R}\right) \;, \end{aligned} \tag{4.51}$$

where we have used

$$\bar{N}_{1L} \stackrel{C}{\to} (C\bar{N}_{2R}^T)^\dagger \gamma^0 = \bar{N}_{2R}^* C^\dagger \gamma^0 = \bar{N}_{2R}^* \gamma^0 C = N_{2R}^T C \tag{4.52}$$

and the properties of the charge-conjugation matrix as given in Eq. (1.115). In the third line, a minus sign appears due to the exchange of two fermions, see discussion following Eq. (1.120). In the last line, an integration by parts took place. The resulting expression in Eq. (4.51) equals the last term of the first line in Eq. (4.1), which in turn transforms as

$$\mathrm{Tr}\left(\bar{N}_{2R} i\gamma_\mu \partial^\mu N_{2R}\right) \stackrel{C}{\to} \mathrm{Tr}\left(\bar{N}_{1L} i\gamma_\mu \partial^\mu N_{1L}\right) \;, \tag{4.53}$$

revealing the charge-conjugation invariance of the kinetic terms.

For the baryon-baryon-(axial-)vector mesons interaction terms the charge-conjugation transformation read as follows:

$$\begin{aligned} c_2\,\mathrm{Tr}\left(\bar{N}_{1L}\gamma_\mu L^\mu N_{1L}\right) \stackrel{C}{\to} \; & c_2\,\mathrm{Tr}\left[N_{2R}^T C\gamma_\mu (-R^\mu)^T C \bar{N}_{2R}^T\right] \\ &= c_2\,\mathrm{Tr}\left(-N_{2R}^T \gamma_\mu^T R^{\mu T} \bar{N}_{2R}^T\right) \\ &= c_2\,\mathrm{Tr}\left(\bar{N}_{2R}\gamma_\mu R^\mu N_{2R}\right) \;. \end{aligned} \tag{4.54}$$

The mass terms behave as follows:

$$\mathrm{Tr}\left(\bar{N}_{1L} M_{1R}\right) \stackrel{C}{\to} \mathrm{Tr}\left(N_{2R}^T C C \bar{M}_{2L}^T\right) = \mathrm{Tr}\left(-N_{2R}^T \bar{M}_{2L}^T\right) = \mathrm{Tr}\left(\bar{M}_{2L} N_{2R}\right) \;. \tag{4.55}$$

The baryon-baryon-(pseudo)scalar meson coupling terms transform under charge conjugation as illustratively shown by

$$\begin{aligned} g_N\,\mathrm{Tr}\left(\bar{N}_{1L}\Phi N_{1R}\right) \stackrel{C}{\to} \; & g_N\,\mathrm{Tr}\left(N_{2R}^T C \Phi^T C \bar{N}_{2L}^T\right) \\ &= g_N\,\mathrm{Tr}\left(-N_{2R}^T \Phi^T \bar{N}_{2L}^T\right) \\ &= g_N\,\mathrm{Tr}\left(\bar{N}_{2L}\Phi N_{2R}\right) \;, \\ g_N\,\mathrm{Tr}\left(\bar{N}_{2L}\Phi N_{2R}\right) \stackrel{C}{\to} \; & g_N\,\mathrm{Tr}\left(\bar{N}_{1L}\Phi N_{1R}\right) \;. \end{aligned} \tag{4.56}$$

The coupling terms of two baryon fields and two (pseudo)scalar mesons behave as

$$\begin{aligned}\kappa_1 \operatorname{Tr}(\bar{N}_{1R}\Phi^\dagger N_{2L}\Phi) \stackrel{C}{\to} \;& \kappa_1 \operatorname{Tr}\left[N_{2L}^T C(\Phi^T)^\dagger C \bar{N}_{1R}^T \Phi^T\right] \\ &= \kappa_1 \operatorname{Tr}\left[-N_{2L}^T (\Phi^\dagger)^T \bar{N}_{1R}^T \Phi^T\right] \\ &= \kappa_1 \operatorname{Tr}\left(\Phi \bar{N}_{1R}\Phi^\dagger N_{2L}\right) \\ &= \kappa_1 \operatorname{Tr}\left(\bar{N}_{1R}\Phi^\dagger N_{2L}\Phi\right) \ . \end{aligned} \tag{4.57}$$

The invariance of the remaining terms can be shown similarly.

4.4. Lagrangian as a function of parity eigenstates

The Lagrangian (4.1) is a function of N_1, N_2, M_1, and M_2. As these fields have a definite behavior under chiral transformations [Tab. 3.4] it is easy to prove that the Lagrangian (4.1) is chirally symmetric. In this form, however, the Lagrangian cannot describe physical quantities, because N_1, N_2, M_1, and M_2 do not have a definite behavior under parity transformations [Tab. 3.4] and thus cannot be directly associated with physical particles or resonances (even in the limit of vanishing mixing).

We introduce baryonic fields with definite behavior under parity transformations as

$$B_N(x) = \frac{1}{\sqrt{2}}\left[N_1(x) - N_2(x)\right] \ , \tag{4.58}$$

$$B_{N_*}(x) = \frac{1}{\sqrt{2}}\left[N_1(x) + N_2(x)\right] \ , \tag{4.59}$$

$$B_M(x) = \frac{1}{\sqrt{2}}\left[M_1(x) - M_2(x)\right] \ , \tag{4.60}$$

$$B_{M_*}(x) = \frac{1}{\sqrt{2}}\left[M_1(x) + M_2(x)\right] \ , \tag{4.61}$$

where now B_N and B_M have positive parity, and B_{N_*} and B_{M_*} have negative parity, as it follows from the transformations of the N_1 to M_2 fields as given in Tab. 3.4. Similarly, one can compute the charge-conjugation and flavor transformations. The results are given in Tab. 4.2.

In the limit of zero mixing, these fields can be assigned to octet-baryon states as follows:

$$B_N \equiv \{N(939), \Lambda(1116), \Sigma(1193), \Xi(1338)\} \ , \tag{4.62}$$

$$B_M \equiv \{N(1440), \Lambda(1600), \Sigma(1660), \Xi(1690)\} \ , \tag{4.63}$$

$$B_{N(M)_*} \equiv \{N(1650), \Lambda(1800), \Sigma(1750), \Xi(?)\} \ , \tag{4.64}$$

$$B_{M(N)_*} \equiv \{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\} \ , \tag{4.65}$$

where the assignment of chiral partners, Eqs. (4.64) and (4.65), is not (yet) clear. (Note, a detailed study of the mixing will be performed in Sec. 5 for the case $N_f = 2$.)

	Parity P	Charge conjugation C	Flavor $SU(3)_V$
B_N	$\gamma^0 B_N$	$-C\bar{B}_N^T$	$U_V B_N U_V^\dagger$
B_{N_*}	$-\gamma^0 B_{N_*}$	$C\bar{B}_{N_*}^T$	$U_V B_{N_*} U_V^\dagger$
B_M	$\gamma^0 B_M$	$-C\bar{B}_M^T$	$U_V B_M U_V^\dagger$
B_{M_*}	$-\gamma^0 B_{M_*}$	$C\bar{B}_{M_*}^T$	$U_V B_{M_*} U_V^\dagger$

Table 4.2. Parity and charge-conjugation transformations of the parity eigenstates.

The fields (4.58) to (4.61) are 3×3 matrices in flavor space, which can be assigned to physical baryon fields as follows, see Eq. (3.55):

$$B_i \equiv \begin{pmatrix} \frac{\Lambda_i}{\sqrt{6}} + \frac{\Sigma_i^0}{\sqrt{2}} & \Sigma_i^+ & p_i \\ \Sigma_i^- & \frac{\Lambda_i}{\sqrt{6}} - \frac{\Sigma_i^0}{\sqrt{2}} & n_i \\ \Xi_i^- & \Xi^{0i} & -\frac{2\Lambda_i}{\sqrt{6}} \end{pmatrix} , \qquad \forall\, i \in \{N, N_*, M, M_*\} , \tag{4.66}$$

where the isovector $\boldsymbol{N}_i = (p_i, n_i)^T$ describes the non-strange isodoublet, the $\mathcal{S} = -2$ isodoublet is $\boldsymbol{\Xi}_i = (\Xi_i^-, \Xi_i^0)^T$, the isosinglet with $\mathcal{S} = -1$ is Λ_i, and the isotriplets with $\mathcal{S} = -1$ is $\boldsymbol{\Sigma_i} = (\Sigma_{i,1}, \Sigma_{i,2}, \Sigma_{i,3})^T$, where combinations of $\Sigma_{i,k}$ for $k = 1, 2, 3$ describe the different charged Σ states: $\Sigma_{i,1} = (\Sigma_i^+ + \Sigma_i^-)/\sqrt{2}$, and $\Sigma_{i,2} = i(\Sigma_i^+ - \Sigma_i^-)/\sqrt{2}$, as well as $\Sigma_{i,3} = \Sigma_i^0$.

Note, the chiral transformations of the parity eigenstates are given by

$$B_{NL} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_L N_{1L} U_R^\dagger - U_L N_{2L} U_L^\dagger \right) , \tag{4.67}$$

$$B_{NR} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_R N_{1R} U_R^\dagger - U_R N_{2R} U_L^\dagger \right) , \tag{4.68}$$

$$B_{N_*L} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_L N_{1L} U_R^\dagger + U_L N_{2L} U_L^\dagger \right) , \tag{4.69}$$

$$B_{N_*R} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_R N_{1R} U_R^\dagger + U_R N_{2R} U_L^\dagger \right) , \tag{4.70}$$

$$B_{ML} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_R M_{1L} U_R^\dagger - U_R M_{2L} U_L^\dagger \right) , \tag{4.71}$$

$$B_{MR} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_L M_{1R} U_R^\dagger - U_L M_{2R} U_L^\dagger \right) , \tag{4.72}$$

$$B_{M_*L} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_R M_{1L} U_R^\dagger - U_R M_{2L} U_L^\dagger \right) , \tag{4.73}$$

$$B_{M_*R} \xrightarrow{\chi} \frac{1}{\sqrt{2}} \left(U_L M_{1R} U_R^\dagger - U_L M_{2R} U_L^\dagger \right) . \tag{4.74}$$

Substituting the relations (4.58) to (4.61) into Eq. (4.1), we obtain the following Lagrangian as a function of the parity eigenstates:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{bar}} = \mathrm{Tr}(&\bar{B}_{NL} i\gamma_\mu D^\mu_{nL} B_{NL} + \bar{B}_{NR} i\gamma_\mu D^\mu_{nR} B_{NR} \\
&+\bar{B}_{N_*L} i\gamma_\mu D^\mu_{nL} B_{N_*L} + \bar{B}_{N_*R} i\gamma_\mu D^\mu_{nR} B_{N_*R} \\
&+\bar{B}_{ML} i\gamma_\mu D^\mu_{mR} B_{ML} + \bar{B}_{MR} i\gamma_\mu D^\mu_{mL} B_{MR} \\
&+\bar{B}_{M_*L} i\gamma_\mu D^\mu_{mR} B_{M_*L} + \bar{B}_{M_*R} i\gamma_\mu D^\mu_{mL} B_{M_*R}) \\
&-c_{AN}\,\mathrm{Tr}(\bar{B}_{NL}\gamma_\mu L^\mu B_{N_*L} + \bar{B}_{N_*L}\gamma_\mu L^\mu B_{NL} - \bar{B}_{NR}\gamma_\mu R^\mu B_{N_*R} - \bar{B}_{N_*R}\gamma_\mu R^\mu B_{NR}) \\
&-c_{AM}\,\mathrm{Tr}(\bar{B}_{ML}\gamma_\mu R^\mu B_{M_*L} + \bar{B}_{M_*L}\gamma_\mu R^\mu B_{ML} - \bar{B}_{MR}\gamma_\mu L^\mu B_{M_*R} - \bar{B}_{M_*R}\gamma_\mu L^\mu B_{MR}) \\
&-g_N\,\mathrm{Tr}(\bar{B}_{NL}\Phi B_{NR} + \bar{B}_{NR}\Phi^\dagger B_{NL} + \bar{B}_{N_*L}\Phi B_{N_*R} + \bar{B}_{N_*R}\Phi^\dagger B_{N_*L}) \\
&-g_M\,\mathrm{Tr}(\bar{B}_{ML}\Phi^\dagger B_{MR} + \bar{B}_{MR}\Phi B_{ML} + \bar{B}_{M_*L}\Phi^\dagger B_{M_*R} + \bar{B}_{M_*R}\Phi B_{M_*L}) \\
&-\frac{m_{0,1}}{2}\,\mathrm{Tr}(\bar{B}_{NL}B_{MR} + \bar{B}_{NR}B_{ML} + \bar{B}_{N_*L}B_{M_*R} + \bar{B}_{N_*R}B_{M_*L} \\
&+\bar{B}_{ML}B_{NR} + \bar{B}_{MR}B_{NL} + \bar{B}_{M_*L}B_{N_*R} + \bar{B}_{M_*R}B_{N_*L} \\
&+\bar{B}_{NL}B_{M_*R} - \bar{B}_{NR}B_{M_*L} - \bar{B}_{M_*L}B_{NR} + \bar{B}_{M_*R}B_{NL} \\
&-\bar{B}_{ML}B_{N_*R} + \bar{B}_{MR}B_{N_*L} + \bar{B}_{N_*L}B_{MR} - \bar{B}_{N_*R}B_{ML}) \\
&-\frac{m_{0,2}}{2}\,\mathrm{Tr}(\bar{B}_{NL}B_{MR} + \bar{B}_{NR}B_{ML} + \bar{B}_{N_*L}B_{M_*R} + \bar{B}_{N_*R}B_{M_*L} \\
&+\bar{B}_{ML}B_{NR} + \bar{B}_{MR}B_{NL} + \bar{B}_{M_*L}B_{N_*R} + \bar{B}_{M_*R}B_{N_*L} \\
&-\bar{B}_{NL}B_{M_*R} + \bar{B}_{NR}B_{M_*L} + \bar{B}_{M_*L}B_{NR} - \bar{B}_{M_*R}B_{NL} \\
&+\bar{B}_{ML}B_{N_*R} - \bar{B}_{MR}B_{N_*L} - \bar{B}_{N_*L}B_{MR} + \bar{B}_{N_*R}B_{ML}) \\
&-\frac{\kappa_1'}{2}\,\mathrm{Tr}(-\bar{B}_{NL}\Phi B_{NR}\Phi - \bar{B}_{NR}\Phi^\dagger B_{NL}\Phi^\dagger + \bar{B}_{N_*L}\Phi B_{N_*R}\Phi + \bar{B}_{N_*R}\Phi^\dagger B_{N_*L}\Phi^\dagger \\
&+\bar{B}_{NL}\Phi B_{N_*R}\Phi - \bar{B}_{NR}\Phi^\dagger B_{N_*L}\Phi^\dagger - \bar{B}_{N_*L}\Phi B_{NR}\Phi + \bar{B}_{N_*R}\Phi^\dagger B_{NL}\Phi^\dagger) \\
&-\frac{\kappa_1}{2}\,\mathrm{Tr}(-\bar{B}_{NL}\Phi B_{NR}\Phi^\dagger - \bar{B}_{NR}\Phi^\dagger B_{NL}\Phi + \bar{B}_{N_*L}\Phi B_{N_*R}\Phi^\dagger + \bar{B}_{N_*R}\Phi^\dagger B_{N_*L}\Phi \\
&-\bar{B}_{NL}\Phi B_{N_*R}\Phi + \bar{B}_{NR}\Phi^\dagger B_{N_*L}\Phi^\dagger + \bar{B}_{N_*L}\Phi B_{NR}\Phi - \bar{B}_{N_*R}\Phi^\dagger B_L\Phi^\dagger) \\
&-\frac{\kappa_2'}{2}\,\mathrm{Tr}(-\bar{B}_{ML}\Phi^\dagger B_{MR}\Phi - \bar{B}_{MR}\Phi B_{ML}\Phi^\dagger + \bar{B}_{M_*L}\Phi^\dagger B_{M_*R}\Phi + \bar{B}_{M_*R}\Phi B_{M_*L}\Phi^\dagger \\
&+\bar{B}_{ML}\Phi^\dagger B_{M_*R}\Phi - \bar{B}_{MR}\Phi B_{M_*L}\Phi^\dagger - \bar{B}_{M_*L}\Phi^\dagger B_{MR}\Phi + \bar{B}_{M_*R}\Phi B_{ML}\Phi^\dagger) \\
&-\frac{\kappa_2}{2}\,\mathrm{Tr}(-\bar{B}_{ML}\Phi^\dagger B_{MR}\Phi^\dagger - \bar{B}_{MR}\Phi B_{ML}\Phi + \bar{B}_{M_*L}\Phi^\dagger B_{M_*R}\Phi^\dagger + \bar{B}_{M_*R}\Phi B_{M_*L}\Phi \\
&-\bar{B}_{ML}\Phi^\dagger B_{M_*R}\Phi^\dagger + \bar{B}_{MR}\Phi B_{M_*L}\Phi + \bar{B}_{M_*L}\Phi^\dagger B_{MR}\Phi^\dagger - \bar{B}_{M_*R}\Phi B_{ML}\Phi)\;, \qquad (4.75)
\end{aligned}
$$

where the "covariant" derivatives are given by

$$
D^\mu_{kR} = \partial^\mu - ic_k R^\mu\;, \qquad D^\mu_{kL} = \partial^\mu - ic_k L^\mu \quad \forall k \in \{N, M\}\;. \qquad (4.76)
$$

The redefined dimensionless coupling constants

$$
c_N = \frac{c_1 + c_2}{2}\;, \qquad c_M = \frac{c_3 + c_4}{2} \qquad (4.77)
$$

parametrize the coupling of two equal-parity baryons to (axial-)vector mesons. The interactions of two different-parity baryons to (axial-)vector mesons is described by the redefined constants

$$c_{A_N} = \frac{c_1 - c_2}{2} , \qquad c_{A_M} = \frac{c_3 - c_4}{2} , \tag{4.78}$$

which are also dimensionless. The remaining constants are already known from the Lagrangian (4.1).

The Lagrangian (4.75) incorporates 16 baryons (four octet-baryon fields incorporating four baryons each), see Eq. (4.66):

$$\begin{aligned} &\boldsymbol{N}_N, \boldsymbol{N}_{N_*}, \boldsymbol{N}_M, \boldsymbol{N}_{M_*} \ , \\ &\Lambda_N, \Lambda_{N_*}, \Lambda_M, \Lambda_{M_*} \ , \\ &\boldsymbol{\Sigma}_N, \boldsymbol{\Sigma}_{N_*}, \boldsymbol{\Sigma}_M, \boldsymbol{\Sigma}_{M_*} \ , \\ &\boldsymbol{\Xi}_N, \boldsymbol{\Xi}_{N_*}, \boldsymbol{\Xi}_M, \boldsymbol{\Xi}_{M_*} \ . \end{aligned} \tag{4.79}$$

After the evaluation of the matrix multiplications and the traces, the Lagrangian becomes very large. Nevertheless, it is clear that, in order to extract any information, we need the Lagrangian at the level of the baryon fields in Eq. (4.79). We introduce the following four vectors (and their adjoint):

$$\boldsymbol{X} = (X_N, \gamma_5 X_{N_*}, X_M, \gamma_5 X_{M_*})^T \ , \tag{4.80}$$

$$\bar{\boldsymbol{X}} = (\bar{X}_N, -\bar{X}_{N_*}\gamma_5, \bar{X}_M, -\bar{X}_{M_*}\gamma_5) \qquad \forall\ X \in \{\boldsymbol{N}, \Lambda, \boldsymbol{\Sigma}, \boldsymbol{\Xi}\} \ . \tag{4.81}$$

Rewriting the Lagrangian as a function of these vectors, we are able to present the Lagrangian in a compact matrix form. The γ_5 matrices in the definitions (4.80) and (4.81) are introduced in order to avoid such matrices in the mass matrices, as we show in the following Sec. 4.4.1, where the matrix form of the Lagrangian (4.75) is investigated term by term.

4.4.1. Mass terms

The eLSM Lagrangian (4.75) contains explicit mass terms for the baryons. Using the chiral projection operators (1.129) we write them as

$$\begin{aligned} &-\frac{m_{0,1} + m_{0,2}}{2} \ \mathrm{Tr}(\bar{B}_N B_M + \bar{B}_{N_*} B_{M_*} + \bar{B}_M B_N + \bar{B}_{M_*} B_{N_*}) \\ &-\frac{m_{0,1} - m_{0,2}}{2} \ \mathrm{Tr}(\bar{B}_N \gamma_5 B_{M_*} - \bar{B}_{M_*} \gamma_5 B_N - \bar{B}_M \gamma_5 B_{N_*} + \bar{B}_{N_*} \gamma_5 B_M) \ . \end{aligned} \tag{4.82}$$

Furthermore, after SSB ($\sigma_N \to \sigma_N + \phi_N$ and $\sigma_S \to \sigma_S + \phi_S$) the following terms arise:

$$\begin{aligned}
&-g_N \operatorname{Tr}(\bar{B}_N \Phi_0 B_N + \bar{B}_{N_*} \Phi_0 B_{N_*}) - g_M \operatorname{Tr}(\bar{B}_M \Phi_0 B_M + \bar{B}_{M_*} \Phi_0 B_{M_*}) \\
&-\frac{\kappa_1' + \kappa_1}{2} \operatorname{Tr}(-\bar{B}_N \Phi_0 B_N \Phi_0 + \bar{B}_{N_*} \Phi_0 B_{N_*} \Phi_0) \\
&-\frac{\kappa_1' - \kappa_1}{2} \operatorname{Tr}(\bar{B}_N \Phi_0 \gamma_5 B_{N_*} \Phi_0 - \bar{B}_{N_*} \Phi_0 \gamma_5 B_N \Phi_0) \\
&-\frac{\kappa_2' + \kappa_2}{2} \operatorname{Tr}(-\bar{B}_M \Phi_0 B_M \Phi_0 + \bar{B} M_* \Phi_0 B_{M_*} \Phi_0) \\
&-\frac{\kappa_2' - \kappa_2}{2} \operatorname{Tr}(\bar{B}_M \Phi_0 \gamma_5 B_{M_*} \Phi_0 - \bar{B}_{M_*} \Phi_0 \gamma_5 B_M \Phi_0) \ ,
\end{aligned} \tag{4.83}$$

where

$$\Phi_0 = \frac{1}{2} \operatorname{diag}(\phi_N,\ \phi_N,\ \sqrt{2}\phi_S) \tag{4.84}$$

includes the VEVs. Evaluating the traces in Eqs. (4.82) and (4.83), we find that baryons with different strangeness do not mix (as expected). For instance, the mass terms involving the nucleonic fields are given by

$$\begin{aligned}
\mathcal{L}_{\text{mass},\boldsymbol{N}} = &- \left(\frac{g_N \phi_N}{2} - \frac{\kappa_1' + \kappa_1}{4\sqrt{2}} \phi_N \phi_S \right) \bar{\boldsymbol{N}}_N \boldsymbol{N}_N - \left(\frac{g_N \phi_N}{2} + \frac{\kappa_1' + \kappa_1}{4\sqrt{2}} \phi_N \phi_S \right) \bar{\boldsymbol{N}}_{N_*} \boldsymbol{N}_{N_*} \\
&- \left(\frac{g_M \phi_N}{2} - \frac{\kappa_2' + \kappa_2}{4\sqrt{2}} \phi_N \phi_S \right) \bar{\boldsymbol{N}}_M \boldsymbol{N}_M - \left(\frac{g_M \phi_N}{2} + \frac{\kappa_2' + \kappa_2}{4\sqrt{2}} \phi_N \phi_S \right) \bar{\boldsymbol{N}}_{M_*} \boldsymbol{N}_{M_*} \\
&- \frac{\kappa_1' - \kappa_1}{4\sqrt{2}} \phi_N \phi_S \left(\bar{\boldsymbol{N}}_N \gamma^5 \boldsymbol{N}_{N_*} - \bar{\boldsymbol{N}}_{N_*} \gamma^5 \boldsymbol{N}_N \right) \\
&- \frac{\kappa_2' - \kappa_2}{4\sqrt{2}} \phi_N \phi_S \left(\bar{\boldsymbol{N}}_M \gamma^5 \boldsymbol{N}_{M_*} - \bar{\boldsymbol{N}}_{M_*} \gamma^5 \boldsymbol{N}_M \right) \\
&- \frac{m_{0,1} + m_{0,2}}{2} \left(\bar{\boldsymbol{N}}_N \boldsymbol{N}_M + \bar{\boldsymbol{N}}_{N_*} \boldsymbol{N}_{M_*} + \bar{\boldsymbol{N}}_M \boldsymbol{N}_N + \bar{\boldsymbol{N}}_{M_*} \boldsymbol{N}_{N_*} \right) \\
&- \frac{m_{0,2} - m_{0,1}}{2} \left(\bar{\boldsymbol{N}}_N \gamma^5 \boldsymbol{N}_{M_*} + \bar{\boldsymbol{N}}_{N_*} \gamma^5 \boldsymbol{N}_M - \bar{\boldsymbol{N}}_M \gamma^5 \boldsymbol{N}_{N_*} - \bar{\boldsymbol{N}}_{M_*} \gamma^5 \boldsymbol{N}_N \right) \ .
\end{aligned} \tag{4.85}$$

Finally, using Eq. (4.80) we write all the mass terms in Eqs. (4.82) and (4.83) in a compact way as

$$\mathcal{L}_{\text{mass}} = -\bar{\boldsymbol{N}} M_N \boldsymbol{N} - \bar{\boldsymbol{\Lambda}} M_\Lambda \boldsymbol{\Lambda} - \bar{\boldsymbol{\Sigma}} M_\Sigma \boldsymbol{\Sigma} - \bar{\boldsymbol{\Xi}} M_\Xi \boldsymbol{\Xi} \ . \tag{4.86}$$

where the mass matrices M_N, M_Λ, M_Σ, M_Ξ are given by

$$M_N = \begin{pmatrix} \frac{\phi_N}{2} g_N - \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_1' + \kappa_1) & \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_1' - \kappa_1) & \frac{m_{0,1}+m_{0,2}}{2} & \frac{m_{0,1}-m_{0,2}}{2} \\ \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_1' - \kappa_1) & -\frac{\phi_N}{2} g_N - \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_1' + \kappa_1) & -\frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} \\ \frac{m_{0,1}+m_{0,2}}{2} & -\frac{m_{0,1}-m_{0,2}}{2} & \frac{\phi_N}{2} g_M - \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_2' + \kappa_2) & \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_2' - \kappa_2) \\ \frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} & \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_2' - \kappa_2) & -\frac{\phi_N}{2} g_M - \frac{\phi_N \phi_S}{4\sqrt{2}}(\kappa_2' + \kappa_2) \end{pmatrix}, \tag{4.87}$$

$$M_\Sigma = \begin{pmatrix} \frac{\phi_N}{2} g_N + \delta m_s - \frac{\phi_N^2}{8}(\kappa_1' + \kappa_1) & \frac{\phi_N^2}{8}(\kappa_1' - \kappa_1) & \frac{m_{0,1}+m_{0,2}}{2} & \frac{m_{0,1}-m_{0,2}}{2} \\ \frac{\phi_N^2}{8}(\kappa_1' - \kappa_1) & -\frac{\phi_N}{2} g_N + \delta m_s - \frac{\phi_N^2}{8}(\kappa_1' + \kappa_1) & -\frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} \\ \frac{m_{0,1}+m_{0,2}}{2} & -\frac{m_{0,1}-m_{0,2}}{2} & \frac{\phi_N}{2} g_M + \delta m_s - \frac{\phi_N^2}{8}(\kappa_2' + \kappa_2) & \frac{\phi_N^2}{8}(\kappa_2' - \kappa_2) \\ \frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} & \frac{\phi_N^2}{8}(\kappa_2' - \kappa_2) & -\frac{\phi_N}{2} g_M + \delta m_s - \frac{\phi_N^2}{8}(\kappa_2' + \kappa_2) \end{pmatrix}, \tag{4.88}$$

$$M_\Lambda = \left(\begin{array}{c:c:c:c} \begin{array}{c}\frac{\phi_N+2\sqrt{2}\phi_S}{6}g_N \\ -\frac{\phi_N^2+6\phi_S^2}{24}(\kappa_1'+\kappa_1) \\ +\delta m_s\end{array} & \frac{\phi_N^2+6\phi_S^2}{24}(\kappa_1'-\kappa_1) & \frac{m_{0,1}+m_{0,2}}{2} & \frac{m_{0,1}-m_{0,2}}{2} \\ \hdashline \frac{\phi_N^2+6\phi_S^2}{24}(\kappa_1'-\kappa_1) & \begin{array}{c}-\frac{\phi_N+2\sqrt{2}\phi_S}{6}g_N \\ -\frac{\phi_N^2+6\phi_S^2}{24}(\kappa_1'+\kappa_1) \\ +\delta m_s\end{array} & -\frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} \\ \hdashline \frac{m_{0,1}+m_{0,2}}{2} & -\frac{m_{0,1}-m_{0,2}}{2} & \begin{array}{c}\frac{\phi_N+2\sqrt{2}\phi_S}{6}g_M \\ -\frac{\phi_N^2+6\phi_S^2}{24}(\kappa_2'+\kappa_2) \\ +\delta m_s\end{array} & \frac{\phi_N^2+6\phi_S^2}{24}(\kappa_2'-\kappa_2) \\ \hdashline \frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} & \frac{\phi_N^2+6\phi_S^2}{24}(\kappa_2'-\kappa_2) & \begin{array}{c}-\frac{\phi_N+2\sqrt{2}\phi_S}{6}g_M \\ -\frac{\phi_N^2+6\phi_S^2}{24}(\kappa_2'+\kappa_2) \\ +\delta m_s\end{array} \end{array}\right), \tag{4.89}$$

$$M_\Xi = \left(\begin{array}{c:c:c:c} \begin{array}{c}\frac{\phi_S}{\sqrt{2}}g_N + 2\delta m_s \\ -\frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_1'+\kappa_1)\end{array} & \frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_1'-\kappa_1) & \frac{m_{0,1}+m_{0,2}}{2} & \frac{m_{0,1}-m_{0,2}}{2} \\ \hdashline \frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_1'-\kappa_1) & \begin{array}{c}-\frac{\phi_S}{\sqrt{2}}g_N + 2\delta m_s \\ -\frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_1'+\kappa_1)\end{array} & -\frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} \\ \hdashline \frac{m_{0,1}+m_{0,2}}{2} & -\frac{m_{0,1}-m_{0,2}}{2} & \begin{array}{c}\frac{\phi_S}{\sqrt{2}}g_M + 2\delta m_s \\ -\frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_2'+\kappa_2)\end{array} & \frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_2'-\kappa_2) \\ \hdashline \frac{m_{0,1}-m_{0,2}}{2} & -\frac{m_{0,1}+m_{0,2}}{2} & \frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_2'-\kappa_2) & \begin{array}{c}-\frac{\phi_S}{\sqrt{2}}g_M + 2\delta m_s \\ -\frac{\phi_N\phi_S}{4\sqrt{2}}(\kappa_2'+\kappa_2)\end{array} \end{array}\right), \tag{4.90}$$

where we have included the correction

$$\delta m_s \simeq m_s - m_{u/d} \simeq 100 \text{ MeV} \tag{4.91}$$

(by hand) to pay attention to the mass difference between the strange and non-strange quarks (which are not considered by our effective model).

4.4.2. Yukawa interaction terms with a pseudoscalar meson

The Lagrangian (4.75) introduces a couple of interaction terms. In this work, we restrict the study to Yukawa terms that describe the interaction of baryons and pseudoscalar mesons (since decays into the latter are most often observed in experiment). After SSB we find

$$\begin{aligned}
\mathcal{L}_{\bar{B}PB} =& - i\bar{\boldsymbol{N}}\gamma_5\hat{\Pi}^N(\boldsymbol{\pi}\cdot\boldsymbol{\tau})\boldsymbol{N} + \bar{\boldsymbol{N}}\gamma_5\hat{\Pi}^N_\partial\gamma_\mu[(\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\boldsymbol{N} \\
& - i\bar{\boldsymbol{\Lambda}}\gamma_5\hat{\Pi}^{\Sigma\Lambda}(\boldsymbol{\pi}\cdot\boldsymbol{\Sigma}) + \bar{\boldsymbol{\Lambda}}\gamma_5\hat{\Pi}^{\Sigma\Lambda}_\partial\gamma_\mu[(\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\Sigma}] - i(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\pi})\gamma_5\hat{\Pi}^{\Sigma\Lambda}\boldsymbol{\Lambda} \\
& + [\bar{\boldsymbol{\Sigma}}\cdot(\partial^\mu\boldsymbol{\pi})]\gamma_5\gamma_\mu\hat{\Pi}^{\Sigma\Lambda}_\partial\boldsymbol{\Lambda} \\
& - i[i\bar{\boldsymbol{\Sigma}}\gamma_5\hat{\Pi}^{\Sigma\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\Sigma})] + i\bar{\boldsymbol{\Sigma}}\gamma_5\gamma_\mu\hat{\Pi}^{\Sigma\Sigma}_\partial\cdot[(\partial^\mu\boldsymbol{\pi})\times\boldsymbol{\Sigma}] \\
& - i\bar{\boldsymbol{\Xi}}\gamma_5\hat{\Pi}^{\Xi}\ \boldsymbol{\pi}\cdot\boldsymbol{\tau}\ \boldsymbol{\Xi} \\
& - i\bar{\boldsymbol{N}}\gamma_5\hat{\mathrm{H}}^N_N\eta_N\boldsymbol{N} - i\bar{\boldsymbol{N}}\gamma_5\hat{\mathrm{H}}^N_S\eta_S\boldsymbol{N} + \bar{\boldsymbol{N}}\gamma_5\hat{\mathrm{H}}^N_{N,\partial}\gamma_\mu(\partial^\mu\eta_N)\boldsymbol{N} \\
& - i\bar{\boldsymbol{\Lambda}}\gamma_5\hat{\mathrm{H}}^\Lambda_N\eta_N\boldsymbol{\Lambda} - i\bar{\boldsymbol{\Lambda}}\gamma_5\hat{\mathrm{H}}^\Lambda_S\eta_S\boldsymbol{\Lambda} + \bar{\boldsymbol{\Lambda}}\gamma_5\hat{\mathrm{H}}^\Lambda_{N,\partial}\gamma_\mu(\partial^\mu\eta_N)\boldsymbol{\Lambda} + \bar{\boldsymbol{\Lambda}}\gamma_5\hat{\mathrm{H}}^\Lambda_{S,\partial}\gamma_\mu(\partial^\mu\eta_S)\boldsymbol{\Lambda} \\
& - i\bar{\boldsymbol{\Sigma}}\gamma_5\hat{\mathrm{H}}^\Sigma_N\eta_N\boldsymbol{\Sigma} - i\bar{\boldsymbol{\Sigma}}\gamma_5\hat{\mathrm{H}}^\Sigma_S\eta_S\boldsymbol{\Sigma} + \bar{\boldsymbol{\Sigma}}\gamma_5\hat{\mathrm{H}}^\Sigma_{N,\partial}\gamma_\mu(\partial^\mu\eta_N)\boldsymbol{\Sigma} \\
& - i\bar{\boldsymbol{\Xi}}\gamma_5\hat{\mathrm{H}}^\Xi_N\eta_N\boldsymbol{\Xi} - i\bar{\boldsymbol{\Xi}}\gamma_5\hat{\mathrm{H}}^\Xi_S\eta_S\boldsymbol{\Xi} + \bar{\boldsymbol{\Xi}}\gamma_5\hat{\mathrm{H}}^\Xi_{S,\partial}\gamma_\mu(\partial^\mu\eta_S)\boldsymbol{\Xi} \\
& - i\bar{\boldsymbol{N}}\gamma_5\hat{K}^{N\Lambda}\boldsymbol{K}\boldsymbol{\Lambda} + \bar{\boldsymbol{N}}\gamma_5\hat{K}^{N\Lambda}_\partial\gamma_\mu(\partial^\mu\boldsymbol{K})\boldsymbol{\Lambda} - i\bar{\boldsymbol{N}}\gamma_5\hat{K}^{N\Sigma}\boldsymbol{\Sigma}\cdot\boldsymbol{\tau}\boldsymbol{K} \\
& - i\bar{\boldsymbol{\Lambda}}\gamma_5\hat{K}^{\Lambda\Xi}\boldsymbol{K}\boldsymbol{\Xi} + \bar{\boldsymbol{\Lambda}}\gamma_5\hat{K}^{\Lambda\Xi}_\partial\gamma_\mu(\partial^\mu\boldsymbol{K})\boldsymbol{\Xi} \\
& - i\boldsymbol{K}^T\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\tau}\hat{K}^{\Sigma\Xi}\boldsymbol{\Xi} + (\partial^\mu\boldsymbol{K}^T)\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\tau}\gamma_5\gamma_\mu\hat{K}^{\Sigma\Xi}_\partial\boldsymbol{\Xi}\ ,
\end{aligned} \tag{4.92}$$

where the following matrices incorporate the coupling constants:

$$\hat{\Pi}^N = \frac{Z_\pi}{2}\left(\begin{array}{c|c|c|c} g_N - \frac{\phi_S}{2\sqrt{2}}(\kappa_1' + \kappa_1) & \frac{\phi_S}{2\sqrt{2}}(\kappa_1' - \kappa_1) & 0 & 0 \\ \hline \frac{\phi_S}{2\sqrt{2}}(\kappa_1' - \kappa_1) & -g_N - \frac{\phi_S}{2\sqrt{2}}(\kappa_1' + \kappa_1) & 0 & 0 \\ \hline 0 & 0 & -g_M + \frac{\phi_S}{2\sqrt{2}}(\kappa_2' + \kappa_2) & -\frac{\phi_S}{2\sqrt{2}}(\kappa_2' - \kappa_2) \\ \hline 0 & 0 & -\frac{\phi_S}{2\sqrt{2}}(\kappa_2' - \kappa_2) & g_M + \frac{\phi_S}{2\sqrt{2}}(\kappa_2' + \kappa_2) \end{array}\right), \tag{4.93}$$

$$\hat{\Pi}^{\Sigma\Lambda} = \frac{Z_\pi}{2\sqrt{3}}\begin{pmatrix} g_N - \frac{\phi_N}{2}\kappa_1' & \frac{\phi_N}{2}\kappa_1' & 0 & 0 \\ \frac{\phi_N}{2}\kappa_1' & -g_N - \frac{\phi_N}{2}\kappa_1' & 0 & 0 \\ 0 & 0 & -g_M + \frac{\phi_N}{2}\kappa_2 & \frac{\phi_N}{2}\kappa_2 \\ 0 & 0 & \frac{\phi_N}{2}\kappa_2 & g_M + \frac{\phi_N}{2}\kappa_2 \end{pmatrix}, \tag{4.94}$$

$$\hat{\Pi}^{\Sigma\Sigma} = \frac{Z_\pi}{2}\begin{pmatrix} g_N - \frac{\phi_N}{2}\kappa_1 & -\frac{\phi_N}{2}\kappa_1 & 0 & 0 \\ -\frac{\phi_N}{2}\kappa_1 & -g_N - \frac{\phi_N}{2}\kappa_1 & 0 & 0 \\ 0 & 0 & -g_M + \frac{\phi_N}{2}\kappa_2' & -\frac{\phi_N}{2}\kappa_2' \\ 0 & 0 & -\frac{\phi_N}{2}\kappa_2' & g_M + \frac{\phi_N}{2}\kappa_2' \end{pmatrix}, \tag{4.95}$$

$$\hat{\Pi}^{\Xi} = \frac{Z_\pi}{4\sqrt{2}}\begin{pmatrix} -\phi_S(\kappa_1' - \kappa_1) & \phi_S(\kappa_1' + \kappa_1) & 0 & 0 \\ \phi_S(\kappa_1' + \kappa_1) & -\phi_S(\kappa_1' - \kappa_1) & 0 & 0 \\ 0 & 0 & -\phi_S(\kappa_2' - \kappa_2) & \phi_S(\kappa_2' + \kappa_2) \\ 0 & 0 & \phi_S(\kappa_2' + \kappa_2) & -\phi_S(\kappa_2' - \kappa_2) \end{pmatrix}, \tag{4.96}$$

$$\hat{\mathrm{H}}_N^N = \frac{Z_{\eta_N}}{2}\left(\begin{array}{c:c:c:c} g_N - \frac{\phi_S}{2\sqrt{2}}(\kappa_1' + \kappa_1) & \frac{\phi_S}{2\sqrt{2}}(\kappa_1' - \kappa_1) & 0 & 0 \\ \hdashline \frac{\phi_S}{2\sqrt{2}}(\kappa_1' - \kappa_1) & -g_N - \frac{\phi_S}{2\sqrt{2}}(\kappa_1' + \kappa_1) & 0 & 0 \\ \hdashline 0 & 0 & -g_M + \frac{\phi_S}{2\sqrt{2}}(\kappa_2' + \kappa_2) & -\frac{\phi_S}{2\sqrt{2}}(\kappa_2' - \kappa_2) \\ \hdashline 0 & 0 & -\frac{\phi_S}{2\sqrt{2}}(\kappa_2' - \kappa_2) & g_M + \frac{\phi_S}{2\sqrt{2}}(\kappa_2' + \kappa_2)\ , \end{array}\right) , \tag{4.97}$$

$$\hat{\mathrm{H}}_S^N = \frac{Z_{\eta_S}}{2\sqrt{2}}\begin{pmatrix} -\frac{\phi_N}{2}(\kappa_1' - \kappa_1) & \frac{\phi_N}{2}(\kappa_1' + \kappa_1) & 0 & 0 \\ \frac{\phi_N}{2}(\kappa_1' + \kappa_1) & -\frac{\phi_N}{2}(\kappa_1' - \kappa_1) & 0 & 0 \\ 0 & 0 & -\frac{\phi_N}{2}(\kappa_2' - \kappa_2) & \frac{\phi_N}{2}(\kappa_2' + \kappa_2) \\ 0 & 0 & \frac{\phi_N}{2}(\kappa_2' + \kappa_2) & -\frac{\phi_N}{2}(\kappa_2' - \kappa_2) \end{pmatrix} , \tag{4.98}$$

$$\hat{\mathrm{H}}_N^\Lambda = \frac{Z_{\eta_N}}{6}\begin{pmatrix} g_N - \frac{\phi_N}{2}\kappa_1' & \frac{\phi_N}{2}\kappa_1' & 0 & 0 \\ \frac{\phi_N}{2}\kappa_1' & -g_N - \frac{\phi_N}{2}\kappa_1' & 0 & 0 \\ 0 & 0 & -g_M + \frac{\phi_N}{2}\kappa_2 & \frac{\phi_N}{2}\kappa_2 \\ 0 & 0 & \frac{\phi_N}{2}\kappa_2 & g_M + \frac{\phi_N}{2}\kappa_2 \end{pmatrix} , \tag{4.99}$$

$$\hat{\mathrm{H}}_S^\Lambda = \frac{\sqrt{2}Z_{\eta_S}}{3}\begin{pmatrix} g_N - \frac{\phi_S}{\sqrt{2}}\kappa_1' & \frac{\phi_S}{\sqrt{2}}\kappa_1' & 0 & 0 \\ \frac{\phi_S}{\sqrt{2}}\kappa_1' & -g_N - \frac{\phi_S}{\sqrt{2}}\kappa_1' & 0 & 0 \\ 0 & 0 & -g_M + \frac{\phi_S}{\sqrt{2}}\kappa_2 & \frac{\phi_S}{\sqrt{2}}\kappa_2 \\ 0 & 0 & \frac{\phi_S}{\sqrt{2}}\kappa_2 & g_M + \frac{\phi_S}{\sqrt{2}}\kappa_2 \end{pmatrix} , \tag{4.100}$$

$$\hat{\mathrm{H}}_N^{\Sigma} = \frac{Z_{\eta_N}}{2} \begin{pmatrix} g_N - \frac{\phi_N}{2}\kappa_1' & \frac{\phi_N}{2}\kappa_1' & 0 & 0 \\ \frac{\phi_N}{2}\kappa_1' & -g_N - \frac{\phi_N}{2}\kappa_1' & 0 & 0 \\ 0 & 0 & -g_M + \frac{\phi_N}{2}\kappa_2 & \frac{\phi_N}{2}\kappa_2 \\ 0 & 0 & \frac{\phi_N}{2}\kappa_2 & g_M + \frac{\phi_N}{2}\kappa_2 \end{pmatrix} , \quad (4.101)$$

$$\hat{\mathrm{H}}_S^{\Sigma} = 0 \, , \quad (4.102)$$

$$\hat{\mathrm{H}}_N^{\Xi} = \frac{Z_{\eta_N}}{2\sqrt{2}} \begin{pmatrix} -\frac{\phi_S}{2}(\kappa_1' - \kappa_1) & \frac{\phi_S}{2}(\kappa_1' + \kappa_1) & 0 & 0 \\ \frac{\phi_S}{2}(\kappa_1' + \kappa_1) & -\frac{\phi_S}{2}(\kappa_1' - \kappa_1) & 0 & 0 \\ 0 & 0 & -\frac{\phi_S}{2}(\kappa_2' - \kappa_2) & \frac{\phi_S}{2}(\kappa_2' + \kappa_2) \\ 0 & 0 & \frac{\phi_S}{2}(\kappa_2' + \kappa_2) & -\frac{\phi_S}{2}(\kappa_2' - \kappa_2) \end{pmatrix} , \quad (4.103)$$

$$\hat{\mathrm{H}}_S^{\Xi} = \frac{Z_{\eta_S}}{\sqrt{2}} \left(\begin{array}{c:c:c:c} g_N - \frac{\phi_N}{4}(\kappa_1' + \kappa_1) & \frac{\phi_N}{4}(\kappa_1' - \kappa_1) & 0 & 0 \\ \hdashline \frac{\phi_N}{4}(\kappa_1' - \kappa_1) & -g_N - \frac{\phi_N}{4}(\kappa_1' + \kappa_1) & 0 & 0 \\ \hdashline 0 & 0 & -g_M + \frac{\phi_N}{4}(\kappa_2' + \kappa_2) & -\frac{\phi_N}{4}(\kappa_2' - \kappa_2) \\ \hdashline 0 & 0 & -\frac{\phi_N}{4}(\kappa_2' - \kappa_2) & g_M + \frac{\phi_N}{4}(\kappa_2' + \kappa_2) \end{array}\right) , \quad (4.104)$$

$$\hat{K}^{N\Sigma} = \frac{Z_K}{4} \begin{pmatrix} -\frac{\phi_N}{2}(\kappa_1' - \kappa_1) & \frac{\phi_N}{2}(\kappa_1' + \kappa_1) & 0 & 0 \\ \frac{\phi_N}{2}(\kappa_1' + \kappa_1) & -\frac{\phi_N}{2}(\kappa_1' - \kappa_1) & 0 & 0 \\ 0 & 0 & -\frac{\phi_N}{2}(\kappa_2' - \kappa_2) & \frac{\phi_N}{2}(\kappa_2' + \kappa_2) \\ 0 & 0 & \frac{\phi_N}{2}(\kappa_2' + \kappa_2) & -\frac{\phi_N}{2}(\kappa_2' - \kappa_2) \end{pmatrix} , \quad (4.105)$$

$$\hat{K}^{N\Lambda} = \frac{Z_K}{\sqrt{3}} \left(\begin{array}{c|c|c|c} \begin{array}{c} -g_N \\ +\frac{\phi_S}{2\sqrt{2}}(\kappa_1' + \kappa_1) \\ -\frac{\phi_N}{8}(\kappa_1' - \kappa_1) \end{array} & \begin{array}{c} -\frac{\phi_S}{2\sqrt{2}}(\kappa_1' - \kappa_1) \\ +\frac{\phi_N}{8}(\kappa_1' + \kappa_1) \end{array} & 0 & 0 \\ \hline \begin{array}{c} -\frac{\phi_S}{2\sqrt{2}}(\kappa_1' - \kappa_1) \\ +\frac{\phi_N}{8}(\kappa_1' + \kappa_1) \end{array} & \begin{array}{c} g_N \\ +\frac{\phi_S}{2\sqrt{2}}(\kappa_1' + \kappa_1) \\ -\frac{\phi_N}{8}(\kappa_1' - \kappa_1) \end{array} & 0 & 0 \\ \hline 0 & 0 & \begin{array}{c} g_M \\ -\frac{\phi_S}{2\sqrt{2}}(\kappa_2' + \kappa_2) \\ -\frac{\phi_N}{8}(\kappa_2' - \kappa_2) \end{array} & \begin{array}{c} \frac{\phi_S}{2\sqrt{2}}(\kappa_2' - \kappa_2) \\ +\frac{\phi_N}{8}(\kappa_2' + \kappa_2) \end{array} \\ \hline 0 & 0 & \begin{array}{c} \frac{\phi_S}{2\sqrt{2}}(\kappa_2' - \kappa_2) \\ +\frac{\phi_N}{8}(\kappa_2' + \kappa_2) \end{array} & \begin{array}{c} -g_M \\ -\frac{\phi_S}{2\sqrt{2}}(\kappa_2' + \kappa_2) \\ -\frac{\phi_N}{8}(\kappa_2' - \kappa_2) \end{array} \end{array}\right), \tag{4.106}$$

$$\hat{K}^{\Lambda\Xi} = \frac{Z_K}{2\sqrt{2}} \left(\begin{array}{c|c|c|c} \begin{array}{c} g_N \\ -\frac{\phi_N}{4}(\kappa_1' + \kappa_1) \end{array} & \frac{\phi_N}{4}(\kappa_1' - \kappa_1) & 0 & 0 \\ \hline \frac{\phi_N}{4}(\kappa_1' - \kappa_1) & \begin{array}{c} -g_N \\ -\frac{\phi_N}{4}(\kappa_1' + \kappa_1) \end{array} & 0 & 0 \\ \hline 0 & 0 & \begin{array}{c} -g_M \\ +\frac{\phi_N}{4}(\kappa_2' + \kappa_2) \end{array} & -\frac{\phi_N}{4}(\kappa_2' - \kappa_2) \\ \hline 0 & 0 & -\frac{\phi_N}{4}(\kappa_2' - \kappa_2) & \begin{array}{c} g_M \\ +\frac{\phi_N}{4}(\kappa_2' + \kappa_2) \end{array} \end{array}\right), \tag{4.107}$$

$$\hat{K}^{\Sigma\Xi} = \frac{Z_K}{2} \left(\begin{array}{c|c|c|c} g_N - \frac{\phi_N}{4}(\kappa_1' + \kappa_1) & \frac{\phi_N}{4}(\kappa_1' - \kappa_1) & 0 & 0 \\ \hline \frac{\phi_N}{4}(\kappa_1' - \kappa_1) & -g_N - \frac{\phi_N}{4}(\kappa_1' + \kappa_1) & 0 & 0 \\ \hline 0 & 0 & -g_M + \frac{\phi_N}{4}(\kappa_2' + \kappa_2) & -\frac{\phi_N}{4}(\kappa_2' - \kappa_2) \\ \hline 0 & 0 & -\frac{\phi_N}{4}(\kappa_2' - \kappa_2) & g_M + \frac{\phi_N}{4}(\kappa_2' + \kappa_2) \end{array} \right) , \tag{4.108}$$

and

$$\hat{\Pi}_\partial^N = \frac{Z_\pi w_{a_1}}{2} \hat{C} , \tag{4.109}$$

$$\hat{\Pi}_\partial^{\Sigma\Lambda} = \frac{Z_\pi w_{a_1}}{2\sqrt{3}} \hat{C} , \tag{4.110}$$

$$\hat{\Pi}_\partial^{\Sigma\Sigma} = \frac{Z_\pi w_{a_1}}{2} \hat{C} , \tag{4.111}$$

$$\hat{\mathrm{H}}_{N,\partial}^N = \frac{Z_{\eta_N} w_{f_{1N}}}{2} \hat{C} , \tag{4.112}$$

$$\hat{\mathrm{H}}_{N,\partial}^\Lambda = \frac{Z_{\eta_N} w_{f_{1N}}}{6} \hat{C} , \tag{4.113}$$

$$\hat{\mathrm{H}}_{S,\partial}^\Lambda = \frac{\sqrt{2} Z_{\eta_S} w_{f_{1S}}}{3} \hat{C} , \tag{4.114}$$

$$\hat{\mathrm{H}}_{N,\partial}^\Sigma = \frac{Z_{\eta_N} w_{f_{1N}}}{2} \hat{C} , \tag{4.115}$$

$$\hat{\mathrm{H}}_{S,\partial}^\Xi = \frac{Z_{\eta_S} w_{f_{1S}}}{\sqrt{2}} \hat{C} , \tag{4.116}$$

$$\hat{K}_\partial^{N\Lambda} = -\frac{Z_K w_{K_1}}{\sqrt{3}} \hat{C} , \tag{4.117}$$

$$\hat{K}_\partial^{\Lambda\Xi} = -\frac{Z_K w_{K_1}}{\sqrt{2}} \hat{C} , \tag{4.118}$$

$$\hat{K}_\partial^{\Sigma\Xi} = -\frac{Z_K w_{K_1}}{2} \hat{C} , \tag{4.119}$$

with

$$\hat{C} = \begin{pmatrix} c_N & c_{AN} & 0 & 0 \\ c_{AN} & c_N & 0 & 0 \\ 0 & 0 & -c_M & -c_{AM} \\ 0 & 0 & -c_{AM} & -c_M \end{pmatrix} . \tag{4.120}$$

Chapter 5.

Two-flavor study

Since a full three-flavor analysis of the eLSM (4.75) with four baryon octets, i.e., 32 baryonic resonances is very complex, in this work we investigate the simpler $N_f = 2$. After the reduction of the Lagrangian (4.75) to $N_f = 2$, four nucleonic doublets remain, see Sec. 5.1. These fields mix to form the nucleon N and the resonances $N(1440)$, $N(1535)$, and $N(1650)$, see Sec. 5.2. Susequently, we investigate decay widths in Sec. 5.3 and calculate axial coupling constants in Sec. 5.4. The fitting of the parameters to experimental data as well as results are presented in Sec. 5.5.

5.1. Lagrangian for $N_f = 2$

The two-flavor Lagrangian follows from the three-flavor Lagrangian (4.75) by setting all hadronic fields with non-vanishing strangeness to zero.
In the case of the baryon matrices (4.66)find that only their (1 3) and (2 3) elements remain non-zero, see Eq. (3.55):

$$B_N \to \begin{pmatrix} 0 & 0 & \multirow{2}{*}{$\boldsymbol{N}_N$} \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_{N_*} \to \begin{pmatrix} 0 & 0 & \multirow{2}{*}{$\boldsymbol{N}_{N_*}$} \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

$$B_M \to \begin{pmatrix} 0 & 0 & \multirow{2}{*}{$\boldsymbol{N}_M$} \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_{M_*} \to \begin{pmatrix} 0 & 0 & \multirow{2}{*}{$\boldsymbol{N}_{M_*}$} \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.2}$$

where $\boldsymbol{N}_N$, $\boldsymbol{N}_{N_*}$, $\boldsymbol{N}_M$, and $\boldsymbol{N}_{M_*}$ are isodoublets whose left- and right-handed components transform under $SU(2)_L \times SU(2)_R$ as

$$\boldsymbol{N}_{NL} \to U_L \boldsymbol{N}_{NL} \ , \quad \boldsymbol{N}_{NR} \to U_R \boldsymbol{N}_{NR} \ , \tag{5.3}$$
$$\boldsymbol{N}_{N_*L} \to U_L \boldsymbol{N}_{N_*L} \ , \quad \boldsymbol{N}_{N_*R} \to U_R \boldsymbol{N}_{N_*R} \ , \tag{5.4}$$
$$\boldsymbol{N}_{ML} \to U_R \boldsymbol{N}_{ML} \ , \quad \boldsymbol{N}_{MR} \to U_L \boldsymbol{N}_{MR} \ , \tag{5.5}$$
$$\boldsymbol{N}_{M_*L} \to U_R \boldsymbol{N}_{M_*L} \ , \quad \boldsymbol{N}_{M_*R} \to U_L \boldsymbol{N}_{M_*R} \ , \tag{5.6}$$

where now $U_{L(R)} \in SU(2)_{L(R)}$.

In the mesonic sector, we reduce the matrices Φ [Eq. (2.57)], L^μ, and R^μ [Eqs. (2.65),(2.66)] by applying the same procedure. However, in the case of Φ we carefully have to consider first the condensations (2.83) before we reduce the matrix, because otherwise we would lose the VEV ϕ_S of the field σ_S. We obtain the following reductions of the meson matrices:

$$\begin{aligned}\Phi \;\to\; & \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{(\sigma_N+\phi_N+a_0^0)+i(\eta_N+\pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & 0 \\ a_0^- + i\pi^- & \frac{(\sigma_N+\phi_N-a_0^0)+i(\eta_N-\pi^0)}{\sqrt{2}} & 0 \\ 0 & 0 & \phi_S \end{pmatrix} \\ \equiv & \begin{pmatrix} \left(\Phi_{N_f=2}\right) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \frac{1}{\sqrt{2}}\phi_S \end{pmatrix}, \end{aligned} \tag{5.7}$$

$$L^\mu \;\to\; \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\omega_N^\mu+\rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu+a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} + a_1^{\mu +} & 0 \\ \rho^{\mu -} + a_1^{\mu -} & \frac{\omega_N^\mu-\rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu-a_1^{\mu 0}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \left(L^\mu_{N_f=2}\right) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}, \tag{5.8}$$

$$R^\mu \;\to\; \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\omega_N^\mu+\rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu+a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu +} - a_1^{\mu +} & 0 \\ \rho^{\mu -} - a_1^{\mu -} & \frac{\omega_N^\mu-\rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu-a_1^{\mu 0}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \left(R^\mu_{N_f=2}\right) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}, \tag{5.9}$$

where the 2×2 meson matrices ($N_f = 2$) are given in Eqs. (2.137) to (2.139).

Inserting the reduced matrices (5.1), (5.2) and (5.7) to (5.9) into the Lagrangian (4.75) yields the following two-flavor Lagrangian describing the nucleon and nucleonic resonances and their interactions with mesons:

$$\begin{aligned}\mathcal{L}_{\mathrm{bar},N} =& \bar{\boldsymbol{N}}_{NR} i\gamma_\mu D^\mu_{NR} \boldsymbol{N}^{NR} + \bar{\boldsymbol{N}}_{NL} i\gamma_\mu D^\mu_{NL} \boldsymbol{N}^{NL} \\ &+ \bar{\boldsymbol{N}}_{N_*R} i\gamma_\mu D^\mu_{NR} \boldsymbol{N}^{N_*R} + \bar{\boldsymbol{N}}_{N_*L} i\gamma_\mu D^\mu_{NL} \boldsymbol{N}^{N_*L} \\ &+ \bar{\boldsymbol{N}}_{MR} i\gamma_\mu D^\mu_{ML} \boldsymbol{N}_{MR} + \bar{\boldsymbol{N}}_{ML} i\gamma_\mu D^\mu_{MR} \boldsymbol{N}_{ML} \\ &+ \bar{\boldsymbol{N}}_{M_*R} i\gamma_\mu D^\mu_{ML} \boldsymbol{N}_{M_*R} + \bar{\boldsymbol{N}}_{M_*L} i\gamma_\mu D^\mu_{MR} \boldsymbol{N}_{M_*L} \\ &+ c_{AN} \big(\bar{\boldsymbol{N}}_{NR} i\gamma_\mu R^\mu \boldsymbol{N}_{N_*R} + \bar{\boldsymbol{N}}_{N_*R} i\gamma_\mu R^\mu \boldsymbol{N}_{NR} \\ &\qquad - \bar{\boldsymbol{N}}_{NL} i\gamma_\mu L^\mu \boldsymbol{N}_{N_*L} - \bar{\boldsymbol{N}}_{N_*L} i\gamma_\mu L^\mu \boldsymbol{N}_{NL} \big) \\ &+ c_{AM} \big(\bar{\boldsymbol{N}}_{M} i\gamma_\mu L^\mu \boldsymbol{N}_{M_*R} + \bar{\boldsymbol{N}}_{M_*R} i\gamma_\mu L^\mu \boldsymbol{N}_{MR} \\ &\qquad - \bar{\boldsymbol{N}}_{ML} i\gamma_\mu R^\mu \boldsymbol{N}_{M_*L} - \bar{\boldsymbol{N}}_{M_*L} i\gamma_\mu R^\mu \boldsymbol{N}_{ML} \big) \end{aligned}$$

$$
\begin{aligned}
&-g_N\big(\bar{\boldsymbol{N}}_{NL}\Phi\boldsymbol{N}_{NR}+\bar{\boldsymbol{N}}_{NR}\Phi^\dagger\boldsymbol{N}_{NL}+\bar{\boldsymbol{N}}_{N_*L}\Phi\boldsymbol{N}_{N_*R}+\bar{\boldsymbol{N}}_{N_*L}\Phi^\dagger\boldsymbol{N}_{N_*R}\big)\\
&-g_M\big(\bar{\boldsymbol{N}}_{ML}\Phi^\dagger\boldsymbol{N}_{MR}+\bar{\boldsymbol{N}}_{MR}\Phi\boldsymbol{N}_{ML}+\bar{\boldsymbol{N}}_{M_*L}\Phi^\dagger\boldsymbol{N}_{M_*R}+\bar{\boldsymbol{N}}_{M_*L}\Phi\boldsymbol{N}_{M_*R}\big)\\
&-\frac{m_{0,1}+m_{0,2}}{2}\,\big(\bar{\boldsymbol{N}}_{NL}\boldsymbol{N}_{MR}+\bar{\boldsymbol{N}}_{NR}\boldsymbol{N}_{ML}+\bar{\boldsymbol{N}}_{N_*L}\boldsymbol{N}_{M_*R}+\bar{\boldsymbol{N}}_{N_*R}\boldsymbol{N}_{M_*L}\\
&\qquad\qquad+\bar{\boldsymbol{N}}_{ML}\boldsymbol{N}_{NR}+\bar{\boldsymbol{N}}_{MR}\boldsymbol{N}_{NL}+\bar{\boldsymbol{N}}_{M_*L}\boldsymbol{N}_{N_*R}+\bar{\boldsymbol{N}}_{M_*R}\boldsymbol{N}_{N_*L}\big)\\
&-\frac{m_{0,1}-m_{0,2}}{2}\,\big(\bar{\boldsymbol{N}}_{NL}\boldsymbol{N}_{M_*R}-\bar{\boldsymbol{N}}_{NR}\boldsymbol{N}_{M_*L}-\bar{\boldsymbol{N}}_{ML}\boldsymbol{N}_{N_*R}+\bar{\boldsymbol{N}}_{MR}\boldsymbol{N}_{N_*L}\\
&\qquad\qquad-\bar{\boldsymbol{N}}_{N_*L}\boldsymbol{N}_{MR}+\bar{\boldsymbol{N}}_{N_*R}\boldsymbol{N}_{ML}+\bar{\boldsymbol{N}}_{M_*L}\boldsymbol{N}_{NR}-\bar{\boldsymbol{N}}_{M_*R}\boldsymbol{N}_{NL}\big)\\
&-\frac{\kappa_1'+\kappa_1}{2}\frac{\phi_S}{\sqrt{2}}\,\big(-\bar{\boldsymbol{N}}_{NL}\Phi\boldsymbol{N}_{NR}-\bar{\boldsymbol{N}}_{NR}\Phi^\dagger\boldsymbol{N}_{NL}+\bar{\boldsymbol{N}}_{N_*L}\Phi\boldsymbol{N}_{N_*R}+\bar{\boldsymbol{N}}_{N_*R}\Phi^\dagger\boldsymbol{N}_{N_*L}\big)\\
&-\frac{\kappa_1'-\kappa_1}{2}\frac{\phi_S}{\sqrt{2}}\,\big(\bar{\boldsymbol{N}}_{NL}\Phi\boldsymbol{N}_{N_*R}-\bar{\boldsymbol{N}}_{NR}\Phi^\dagger\boldsymbol{N}_{N_*L}-\bar{\boldsymbol{N}}_{N_*L}\Phi\boldsymbol{N}_{NR}+\bar{\boldsymbol{N}}_{N_*R}\Phi^\dagger\boldsymbol{N}_{NL}\big)\\
&-\frac{\kappa_2'+\kappa_2}{2}\frac{\phi_S}{\sqrt{2}}\,\big(-\bar{\boldsymbol{N}}_{ML}\Phi^\dagger\boldsymbol{N}_{MR}-\bar{\boldsymbol{N}}_{MR}\Phi\boldsymbol{N}_{ML}+\bar{\boldsymbol{N}}_{M_*L}\Phi^\dagger\boldsymbol{N}_{M_*R}+\bar{\boldsymbol{N}}_{M_*R}\Phi\boldsymbol{N}_{M_*L}\big)\\
&-\frac{\kappa_2'-\kappa_2}{2}\frac{\phi_S}{\sqrt{2}}\,\big(\bar{\boldsymbol{N}}_{ML}\Phi^\dagger\boldsymbol{N}_{M_*R}-\bar{\boldsymbol{N}}_{MR}\Phi\boldsymbol{N}_{M_*L}-\bar{\boldsymbol{N}}_{M_*L}\Phi^\dagger\boldsymbol{N}_{MR}+\bar{\boldsymbol{N}}_{M_*R}\Phi\boldsymbol{N}_{ML}\big)\;,
\end{aligned}
\tag{5.10}
$$

where we have suppressed the subscript "$N_f = 2$". After SSB the Lagrangian as a function of the physical mesonic field reads

$$
\begin{aligned}
\mathcal{L}_{\text{bar},N}=\;&\bar{\boldsymbol{N}}_N i\gamma^\mu\partial_\mu\boldsymbol{N}_N+\bar{\boldsymbol{N}}_{N_*}i\gamma^\mu\partial_\mu\boldsymbol{N}_{N_*}+\bar{\boldsymbol{N}}_M i\gamma^\mu\partial_\mu\boldsymbol{N}_M+\bar{\boldsymbol{N}}_{M_*}i\gamma^\mu\partial_\mu\boldsymbol{N}_{M_*}\\
&+c_N\Big(\bar{\boldsymbol{N}}_N\gamma_\mu\left\{\left[\omega^\mu-\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu-\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\boldsymbol{N}_N\\
&+\bar{\boldsymbol{N}}_{N_*}\gamma_\mu\left\{\left[\omega^\mu-\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu-\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\boldsymbol{N}_{N_*}\Big)\\
&+c_M\Big(\bar{\boldsymbol{N}}_M\gamma_\mu\left\{\left[\omega^\mu+\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu+\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\boldsymbol{N}_M\\
&+\bar{\boldsymbol{N}}_{M_*}\gamma_\mu\left\{\left[\omega^\mu+\gamma^5\left(f_1^\mu+Zw\partial^\mu\eta_N\right)\right]T^0+\left[\boldsymbol{\rho}^\mu+\gamma^5\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}\right)\right]\cdot\boldsymbol{T}\right\}\boldsymbol{N}_{M_*}\Big)\\
&+c_{A_N}\Big\{\bar{\boldsymbol{N}}_N\gamma_\mu\left[\left(-f_1^\mu-Zw\partial^\mu\eta_N+\gamma^5\omega^\mu\right)T^0+\left(-\boldsymbol{a}_1^\mu-Zw\partial^\mu\boldsymbol{\pi}+\gamma^5\boldsymbol{\rho}^\mu\right)\cdot\boldsymbol{T}\right]\boldsymbol{N}_{N_*}\\
&+\bar{\boldsymbol{N}}_{N_*}\gamma_\mu\left[\left(-f_1^\mu-Zw\partial^\mu\eta_N+\gamma^5\omega^\mu\right)T^0+\left(-\boldsymbol{a}_1^\mu-Zw\partial^\mu\boldsymbol{\pi}+\gamma^5\boldsymbol{\rho}^\mu\right)\cdot\boldsymbol{T}\right]\boldsymbol{N}_N\Big\}\\
&+c_{A_M}\Big\{\bar{\boldsymbol{N}}_M\gamma_\mu\left[\left(f_1^\mu+Zw\partial^\mu\eta_N+\gamma^5\omega^\mu\right)T^0+\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}+\gamma^5\boldsymbol{\rho}^\mu\right)\cdot\boldsymbol{T}\right]\boldsymbol{N}_{M_*}\\
&+\bar{\boldsymbol{N}}_{M_*}\gamma_\mu\left[\left(f_1^\mu+Zw\partial^\mu\eta_N+\gamma^5\omega^\mu\right)T^0+\left(\boldsymbol{a}_1^\mu+Zw\partial^\mu\boldsymbol{\pi}+\gamma^5\boldsymbol{\rho}^\mu\right)\cdot\boldsymbol{T}\right]\boldsymbol{N}_M\Big\}\\
&-g_N\Big\{\bar{\boldsymbol{N}}_N\left[\left(\sigma_N+\phi_N+i\gamma^5Z\eta_N\right)T^0+\left(\boldsymbol{a}_0+i\gamma^5Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\boldsymbol{N}_N\\
&+\bar{\boldsymbol{N}}_{N_*}\left[\left(\sigma_N+\phi_N+i\gamma^5Z\eta_N\right)T^0+\left(\boldsymbol{a}_0+i\gamma^5Z\boldsymbol{\pi}\right)\cdot\boldsymbol{T}\right]\boldsymbol{N}_{N_*}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- g_M \Big\{ \bar{\boldsymbol{N}}_M \left[\left(\sigma_N + \phi_N - i\gamma^5 Z \eta_N\right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z \boldsymbol{\pi}\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_M \\
&+ \bar{\boldsymbol{N}}_{M_*} \left[\left(\sigma_N + \phi_N - i\gamma^5 Z \eta_N\right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z \boldsymbol{\pi}\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_{M_*} \Big\} \\
&- \frac{\kappa_1' + \kappa_1}{2\sqrt{2}} \phi_S \Big\{ -\bar{\boldsymbol{N}}_N \left[\left(\sigma_N + \phi_N + i\gamma^5 Z \eta_N\right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z \boldsymbol{\pi}\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_N \\
&+ \bar{\boldsymbol{N}}_{N_*} \left[\left(\sigma_N + \varphi_N + i\gamma^5 Z \eta_N\right) T^0 + \left(\boldsymbol{a}_0 + i\gamma^5 Z \boldsymbol{\pi}\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_{N_*} \Big\} \\
&- \frac{\kappa_1' - \kappa_1}{2\sqrt{2}} \phi_S \Big\{ \ \bar{\boldsymbol{N}}_N \left[\left(i Z \eta_N + \gamma^5 (\sigma_N + \phi_N)\right) T^0 + \left(i Z \boldsymbol{\pi} + \gamma^5 \boldsymbol{a}_0\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_{N_*} \\
&- \bar{\boldsymbol{N}}_{N_*} \left[\left(i Z \eta_N + \gamma^5 (\sigma_N + \phi_N)\right) T^0 + \left(i Z \boldsymbol{\pi} + \gamma^5 \boldsymbol{a}_0\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_N \Big\} \\
&- \frac{\kappa_2' + \kappa_2}{2\sqrt{2}} \phi_S \Big\{ -\bar{\boldsymbol{N}}_M \left[\left(\sigma_N + \phi_N - i\gamma^5 Z \eta_N\right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z \boldsymbol{\pi}\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_M \\
&+ \bar{\boldsymbol{N}}_{M_*} \left[\left(\sigma_N + \varphi_N - i\gamma^5 Z \eta_N\right) T^0 + \left(\boldsymbol{a}_0 - i\gamma^5 Z \boldsymbol{\pi}\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_{M_*} \Big\} \\
&- \frac{\kappa_2' - \kappa_2}{2\sqrt{2}} \phi_S \Big\{ -\bar{\boldsymbol{N}}_M \left[\left(i Z \eta_N - \gamma^5 (\sigma_N + \phi_N)\right) T^0 + \left(i Z \boldsymbol{\pi} - \gamma^5 \boldsymbol{a}_0\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_{M_*} \\
&+ \bar{\boldsymbol{N}}_{M_*} \left[\left(i Z \eta_N - \gamma^5 (\sigma_N + \phi_N)\right) T^0 + \left(i Z \boldsymbol{\pi} - \gamma^5 \boldsymbol{a}_0\right) \cdot \boldsymbol{T} \right] \boldsymbol{N}_M \Big\} \\
&- \frac{m_{0,1} + m_{0,2}}{2} \left(\bar{\boldsymbol{N}}_N \boldsymbol{N}_M + \bar{\boldsymbol{N}}_{N_*} \boldsymbol{N}_{M_*} + \bar{\boldsymbol{N}}_M \boldsymbol{N}_N + \bar{\boldsymbol{N}}_{M_*} \boldsymbol{N}_{N_*} \right) \\
&- \frac{m_{0,2} - m_{0,1}}{2} \left(\bar{\boldsymbol{N}}_N \gamma^5 \boldsymbol{N}_{M_*} + \bar{\boldsymbol{N}}_{N_*} \gamma^5 \boldsymbol{N}_M - \bar{\boldsymbol{N}}_M \gamma^5 \boldsymbol{N}_{N_*} - \bar{\boldsymbol{N}}_{M_*} \gamma^5 \boldsymbol{N}_N \right) , \qquad (5.11)
\end{aligned}
$$

where the baryonic isodoublets $\boldsymbol{N}_N$, $\boldsymbol{N}_{N_*}$, $\boldsymbol{N}_M$, and $\boldsymbol{N}_{M_*}$ mix to form physical fields.

Interestingly, if we would study four two-flavor baryon doublets by simply expanding the two-flavor studies presented in Sec. 2.3 to include four baryonic isodoublets, the resulting Lagrangian [a combination of Eqs. (2.140) and (2.171)] would be indeed similar to Eq. (5.11). However, it would contain more parameters. The Lagrangian (5.11), obtained as a reduction of the more general $N_f = 3$ Lagrangian (4.75) is more restrictive due to the more complex parity and charge-conjugation transformations of the baryonic fields, compare Tab. 3.4. Some terms which in principle would have different coupling constants in the model constructed in the $N_f = 2$ case are in Eq. (5.11) parametrized by the same coupling, because as we have seen in the $N_f = 3$ study, they transform into each other under parity or charge-conjugation transformations.

5.2. Mass matrix

The mass matrices of the three-flavor Lagrangian are already given in Sec. 4.4.1. Within the reduced two-flavor Lagrangian (5.11) only the terms in Eq. (4.85) remain. In matrix form they are given by

$$
\mathcal{L}_{\text{mass},N} = -\bar{\boldsymbol{N}} M \boldsymbol{N} \ , \qquad (5.12)
$$

where $M \equiv M_N$ (we omit the "N" subscript for the sake for the sake of clarity) is the 4×4 mass matrix given in Eq. (4.87) and $\boldsymbol{N}$ is the vector of isodoublets (and $\bar{\boldsymbol{N}}$ its Dirac-adjoint version), see Eq. (4.80)]:

$$\boldsymbol{N} = (\boldsymbol{N}_N, \gamma_5 \boldsymbol{N}_{N_*}, \boldsymbol{N}_M, \gamma_5 \boldsymbol{N}_{M_*})^T \ , \tag{5.13}$$

$$\bar{\boldsymbol{N}} = (\bar{\boldsymbol{N}}_N, -\bar{\boldsymbol{N}}_{N_*}\gamma_5, \bar{\boldsymbol{N}}_M, -\bar{\boldsymbol{N}}_{M_*}\gamma_5) \ . \tag{5.14}$$

Due to the non-diagonal form of the mass matrix (4.85), the fields $\boldsymbol{N}_N(x)$, $\boldsymbol{N}_{N_*}(x)$, $\boldsymbol{N}_M(x)$, and $\boldsymbol{N}_{M_*}(x)$ cannot be assigned to observable particles or resonances. Instead they mix to form the physical fields physical fields $N_{939}(x)$, $N_{1535}(x)$, $N_{1440}(x)$, and $N_{1650}(x)$ describing the nucleon $N(939)$ and the resonances $N(1535)$, $N(1440)$, and $N(1650)$.

To diagonalize the mass matrix M, we consider the eigenvalue problem

$$M\boldsymbol{u}_k = m_k \boldsymbol{u}_k \ , \qquad \forall k \in \{1, 2, 3, 4 = \dim M\} \tag{5.15}$$

$$\Leftrightarrow \qquad M^i{}_j u^j_k = m_k u^i_k \ , \tag{5.16}$$

where $\boldsymbol{u}_k$ are the four eigenvectors and m_k are the four eigenvalues of M. Note, written in terms of components $i, j \in \{1, 2, 3, 4\}$ a sum over j (but not over k) is understood. Multiplying Eq. (5.16) with $\boldsymbol{u}_l$ from the left-hand side and using that the eigenvectors are orthogonal, i.e., $\boldsymbol{u}_l \cdot \boldsymbol{u}_k = \delta_{lk}$, we find

$$u_{l,i} M^i{}_j u^j_k = m_k u_{l,i} u^i_k \equiv m_k \delta_{kl} \ , \tag{5.17}$$

which shows that the transformation matrix

$$U \equiv (U_{li}) = (u_{l,i}) \tag{5.18}$$

diagonalizes M:

$$\begin{aligned} UMU^\dagger &= \mathrm{diag}(m_1, m_2, m_3, m_4) \\ &\equiv \mathrm{diag}(m_{939}, -m_{1535}, m_{1440}, -m_{1650}) \ . \end{aligned} \tag{5.19}$$

In the second line, we have identified the masses of the physical fields. The minus signs occur due to the definitions (5.13) and (5.14), which entail that the negative-parity states correspond to the negative eigenvalues of M, compare Eqs. (2.152) and (2.153).
After this discussion it is clear that the Lagrangian (5.12) is diagonalized by

$$\begin{aligned} \mathcal{L}_{\mathrm{mass},N} &= -\bar{\boldsymbol{N}} U^\dagger U M U^\dagger U \boldsymbol{N} \\ &= -\bar{\boldsymbol{N}}^{\mathrm{phys}} \mathrm{diag}(m_1, m_2, m_3, m_4) \boldsymbol{N}^{\mathrm{phys}} \ , \end{aligned} \tag{5.20}$$

where the physical fields are contained in

$$\boldsymbol{N}^{\mathrm{phys}} = U\boldsymbol{N} \ \equiv \ \left(N_{939}, \gamma^5 N_{1535}, N_{1440}, \gamma^5 N_{1650}\right)^T \ . \tag{5.21}$$

The masses of these fields [corresponding up to some signs to eigenvalues m_1 to m_4 of M, see Eq. (5.19)] are result from the root of the equation

$$\det\Big[M - m_i \mathbb{1}\Big]=0\ . \tag{5.22}$$

Since the analytical approach is very elaborate, we will determine the solutions numerically, see Sec. 5.5.

However, before closing this section, it is interesting to discuss some limiting cases, where analytical solutions are possible:

(1) Without SSB:
In the (chiral) limit $\phi_N, \phi_S \to 0$, the mass matrix reads

$$M_{\phi_{N/S}=0} \equiv \begin{pmatrix} 0 & 0 & \bar{M} & \mu \\ 0 & 0 & -\mu & -\bar{M} \\ \bar{M} & -\mu & 0 & 0 \\ \mu & -\bar{M} & 0 & 0 \end{pmatrix}\ , \tag{5.23}$$

where $\bar{M} \equiv (m_{0,1} + m_{0,2})/2$ and $\mu \equiv (m_{0,1} - m_{0,2})/2$. In this case, Eq. (5.22) can be analytically solved:

$$\begin{aligned} \det\Big[M_{\phi_{N/S}=0} - m\mathbb{1}\Big] &= 0 \\ \Leftrightarrow \qquad m^4 - 2\mu^2 m^2 - 2\bar{M}^2 m^2 - 2\bar{M}^2\mu^2 + \bar{M}^4 + \mu^4 &= 0 \\ \Leftrightarrow \qquad m^4 - 2(\bar{M}^2 + \mu^2)m^2 + (\bar{M}^2 - \mu^2)^2 &= 0\ , \end{aligned} \tag{5.24}$$

which yields the following eigenvalues:

$$m_{1/2} = \pm(\bar{M} + \mu) \equiv \pm m_{0,1}\ , \tag{5.25}$$
$$m_{3/4} = \pm(\bar{M} - \mu) \equiv \pm m_{0,2}\ . \tag{5.26}$$

As expected, a twofold mass degeneration occurs corresponding to the two distinct sets of chiral partners. The chiral partners of one set have a common mass of $m_{0,1}$, while the other set has the mass $m_{0,2}$ ($\neq m_{0,1}$ in general).

In order to assign physical fields (mass eigenstates) to these sets of chiral partners, we we compute the transformation matrix U, which is composed of the eigenvectors of M, see Eq. (5.18). We determine the four eigenvectors by solving Eq. (5.16) for the specific cases (i) $m = m_1 = m_{0,1}$, (ii) $m == m_2 = -m_{0,1}$, (iii) $m = m_3 = m_{0,2}$, and (iv) $m = m_4 = -m_{0,2}$. Using the results and with Eq. (5.18) we finally obtain the following transform matrix:

$$U = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \equiv U^\dagger \ . \tag{5.27}$$

As we see, the mass eigenstates, i.e., the physical fields (5.21) are uniform mixtures of $\boldsymbol{N}_N, \gamma_5\boldsymbol{N}_{N_*}, \boldsymbol{N}_M$, and $\gamma_5\boldsymbol{N}_{M_*}$. Namely, the chiral partners with mass $m_{0,1}$ are described by the linear combinations

$$\boldsymbol{N}_N - \gamma_5\boldsymbol{N}_{N_*} + (\boldsymbol{N}_M + \gamma_5\boldsymbol{N}_{M_*}) \ , \tag{5.28}$$

$$-\boldsymbol{N}_N + \gamma_5\boldsymbol{N}_{N_*} + (\boldsymbol{N}_M + \gamma_5\boldsymbol{N}_{M_*}) \ , \tag{5.29}$$

and the eigenstates with mass $m_{0,2}$ read

$$\boldsymbol{N}_N + \gamma_5\boldsymbol{N}_{N_*} + (\boldsymbol{N}_M - \gamma_5\boldsymbol{N}_{M_*}) \ , \tag{5.30}$$

$$\boldsymbol{N}_N + \gamma_5\boldsymbol{N}_{N_*} - (\boldsymbol{N}_M - \gamma_5\boldsymbol{N}_{M_*}) \ . \tag{5.31}$$

In this way, it is not possible to decide whether $N(1535)$ or $N(1650)$ is the chiral partner of the nucleon. Instead, in Sec. 5.5 we compute the eigenvalues as a function of ϕ_N in order to trace the mass for $\phi_N \to 0$. This allows to check whether the mass of $N(1535)$ or that of $N(1650)$ merge the nucleon mass in the chiral limit.

(2) Decoupling:
In the case $m_{0,1} = -m_{0,2}$ and $\kappa_{1(2)} = \kappa'_{1(2)}$ the mass matrix reads (4.87) reads

$$M_{\text{decoupled}} = \left(\begin{array}{c|c|c|c} \frac{g_N\phi_N}{2} - \frac{\kappa_1}{2\sqrt{2}}\phi_N\phi_S & 0 & 0 & m_{0,1} \\ \hline 0 & -\frac{g_N\phi_N}{2} - \frac{\kappa_1}{2\sqrt{2}}\phi_N\phi_S & -m_{0,1} & 0 \\ \hline 0 & -m_{0,1} & \frac{g_M\phi_N}{2} - \frac{\kappa_2}{2\sqrt{2}}\phi_N\phi_S & 0 \\ \hline m_{0,1} & 0 & 0 & -\frac{g_M\phi_N}{2} - \frac{\kappa_2}{2\sqrt{2}}\phi_N\phi_S \end{array}\right) \ . \tag{5.32}$$

In this case, the mixing of each two fields decouples from the others: $\boldsymbol{N}_N$ and $\boldsymbol{N}_{M_*}$ mix separately from $\boldsymbol{N}_{N_*}$ and $\boldsymbol{N}_M$. Thus, the diagonalization of these two sets can be performed independently. [In this case, the Lagrangian can be compared to the previously presented two-flavor Lagrangian containing only the nucleon and its chiral partner, see Eq. (2.145) and following equations.] However, if we determine the eigenvalues in the chiral limit ($\phi_N, \phi_S \to 0$) by evaluating

$$\begin{aligned} & \det\Big[M_{\text{decoupled}} - m\mathbb{1}\Big] = 0 \\ \Leftrightarrow \quad & m^4 - 2m_{0,1}^2 m^2 + m_{0,1}^2 = 0 \\ \Leftrightarrow \quad & (m^2 - m_{0,1}^2)^2 = 0 \ , \end{aligned} \tag{5.33}$$

we see that all mass eigenstates have the same mass $m_{0,1}$. Thus, even in this scenario with decoupled sets of fields, we cannot conclude which resonance is the chiral partner of the nucleon.

(3) Dilatation invariance:

Finally, we explain why we have to include the terms parametrized by κ_1, κ_1', κ_2, and κ_2' in the Lagrangian (4.1), although they are not dilatation-invariant. If we take $\kappa_1 = 0 = \kappa_2 = \kappa_1' = \kappa_2'$, the mass matrix (4.87) becomes

$$M_{\kappa=0} = \frac{1}{2}\begin{pmatrix} g_N\phi_N & 0 & m_{0,1}+m_{0,2} & m_{0,1}-m_{0,2} \\ 0 & -g_N\phi_N & m_{0,2}-m_{0,1} & -m_{0,1}-m_{0,2} \\ m_{0,1}+m_{0,2} & m_{0,2}-m_{0,1} & g_M\phi_N & 0 \\ m_{0,1}-m_{0,2} & -m_{0,1}-m_{0,2} & 0 & -g_M\phi_N \end{pmatrix} . \tag{5.34}$$

In order to obtain the mass eigenvalues, we have to solve

$$\begin{aligned} & \det\Big[M_{\kappa=0} - m\mathbb{1}\Big] = 0 \\ \Leftrightarrow \quad & m^4 - \left[m_{0,1}^2 + m_{0,2}^2 + \frac{\phi_N^2}{4}(g_N^2 + g_M^2)\right] m^2 + \left(m_{0,1}m_{0,2} - \frac{\phi_N^2}{4}g_N g_M\right)^2 = 0 \ . \end{aligned} \tag{5.35}$$

This yields

$$\begin{aligned} m^2 &= \frac{1}{2}(m_{0,1}^2 + m_{0,2}^2) + \frac{\phi_N^2}{8}(g_N^2 + g_M^2) \\ &\pm \frac{1}{2}\sqrt{(m_{0,1}^2 - m_{0,2}^2)^2 + \frac{\phi_N^2}{16}(g_N^2 - g_M^2)^2 + \frac{\phi_N^2}{2}(m_{0,1}^2 + m_{0,2}^2)(g_N^2 + g_M^2) + 2\phi_N^2 m_{0,1}m_{0,2}g_N g_M} \\ &= \left(\sqrt{\frac{1}{4}(m_{0,1}+m_{0,2})^2 + \frac{1}{16}(g_N - g_M)^2\phi_N^2} \pm \sqrt{\frac{1}{4}(m_{0,1}-m_{0,2})^2 + \frac{1}{16}(g_N + g_M)^2\phi_N^2}\right)^2 . \end{aligned} \tag{5.36}$$

Denoting $\Omega_{1/2} = \sqrt{\frac{1}{4}(m_{0,1} \pm m_{0,2})^2 + \frac{1}{16}(g_N \mp g_M)^2\phi_N^2}$, we obtain the following eigenvalues:

$$\begin{aligned} m_1 &= \Omega_1 + \Omega_2 = -m_2 \ , \\ m_3 &= \Omega_1 - \Omega_2 = -m_4 \ . \end{aligned} \tag{5.37}$$

Thus due to Eq. (5.19), the masses of $N(939)$ and $N(1535)$ as well as the masses of $N(1440)$ and $N(1650)$ would be degenerate.

We may avoid the mass degeneration in the dilatation-invariant scenario by including the mass difference Δm between the scalar and pseudoscalar diquark. This mass difference is experimentally well known, but not featured in our model. An effective model cannot "see" the inner structure of the baryon fields. Nevertheless, to take this into account we add Δm by hand to the respective terms. This leads to the following mass matrix:

$$M_{\kappa=0+\Delta m} = \frac{1}{2}\begin{pmatrix} g_N\phi_N & 0 & m_{0,1}+m_{0,2} & m_{0,1}-m_{0,2} \\ 0 & -g_N\phi_N - 2\Delta m & m_{0,2}-m_{0,1} & -m_{0,1}-m_{0,2} \\ m_{0,1}+m_{0,2} & m_{0,2}-m_{0,1} & g_M\phi_N & 0 \\ m_{0,1}-m_{0,2} & -m_{0,1}-m_{0,2} & 0 & -g_M\phi_N - 2\Delta m \end{pmatrix} . \tag{5.38}$$

Calculating the eigenvalues of this matrix, we find that this correction indeed resolves the issue of the mass degeneration. However, it predicts that the mass difference between the chiral partners of each set should be equal (and identical to Δm). This is experimentally not observed [18]:

$$596 \text{ MeV} \simeq m_{N(1535)} - m_{N(939)} \neq m_{N(1650)} - m_{N(1440)} \simeq 225 \text{ MeV} \tag{5.39}$$

$$716 \text{ MeV} \simeq m_{N(1650)} - m_{N(939)} \neq m_{N(1535)} - m_{N(1440)} \simeq 105 \text{ MeV} \ . \tag{5.40}$$

Thus, the only possibility to avoid a mass degeneracy but still keeping chiral symmetry is to introduce the terms proportional to κ_1, κ_1', κ_2, and κ_2' into the Lagrangian (4.1).

5.3. Decay widths

The Lagrangian (5.11) includes various interactions between the nucleon, nucleon resonances and (pseudo)scalar mesons. Due to existing experimental data [18], we are especially interested in the decays

$$N(1535) \longrightarrow N\pi \ , \qquad N(1535) \longrightarrow N\eta \ , \tag{5.41}$$

$$N(1650) \longrightarrow N\pi \ , \qquad N(1650) \longrightarrow N\eta \ , \tag{5.42}$$

$$N(1440) \longrightarrow N\pi \ . \tag{5.43}$$

In the following we first investigate the decays involving a pion in Sec. 5.3.1. Then, in Sec. 5.3.2 we give the expression for the decay widths in the η channel.

5.3.1. Pion channel

The Yukawa-coupling terms of two nucleonic fields (4.80) and a pion are given in Eq. (4.92) (the first two terms). The diagonalized version is given by:

$$\begin{aligned}\mathcal{L}_{\boldsymbol{N}\boldsymbol{N}\boldsymbol{\pi}} &= -i\bar{\boldsymbol{N}}U^\dagger\gamma_5 U\hat{\Pi}^N U^\dagger(\boldsymbol{\pi}\cdot\boldsymbol{\tau})U\boldsymbol{N} + \bar{\boldsymbol{N}}U^\dagger\;\gamma_5 U\hat{\Pi}^N_\partial U^\dagger\gamma_\mu[(\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]U\boldsymbol{N}\\ &= -i\bar{\boldsymbol{N}}^{\text{phys}}\gamma_5 U\hat{\Pi}^N U^\dagger(\boldsymbol{\pi}\cdot\boldsymbol{\tau})\;\boldsymbol{N}^{\text{phys}} + \bar{\boldsymbol{N}}^{\text{phys}}\gamma_5 U\hat{\Pi}^N_\partial U^\dagger\gamma_\mu[(\partial^\mu\boldsymbol{\pi})\cdot\boldsymbol{\tau}]\boldsymbol{N}^{\text{phys}}\;, \quad (5.44)\end{aligned}$$

where U is the rotation matrix (5.18). The terms describing the decay of a nucleon resonance, either $N(1440)$, $N(1535)$, or $N(1650)$ into the nucleon N and a pion read

$$\begin{aligned}&-i(\bar{N}_{939})^i_\alpha\gamma_5^{\alpha\beta}\hat{\Pi}^{N\prime}_{13}\pi_a\tau^a_{ij}(N_{1440})^j_\beta + (\bar{N}_{939})^i_\alpha\gamma_5^{\alpha\delta}\hat{\Pi}^{N\prime}_{\partial,13}\gamma_\mu^{\delta\beta}(\partial^\mu\pi_a)\cdot\tau^a_{ij}(N_{1440})^j_\beta\\ &-i(\bar{N}_{939})^i_\alpha\hat{\Pi}^{N\prime}_{12}\pi_a\tau^a_{ij}(N_{N_{1535}})^j_\alpha - (\bar{N}_{939})^i_\alpha\hat{\Pi}^{N\prime}_{\partial,12}\gamma_\mu^{\alpha\beta}(\partial^\mu\pi_a)\cdot\tau^a_{ij}(N_{N_{1535}})^j_\beta\\ &-i(\bar{N}_{939})^i_\alpha\hat{\Pi}^{N\prime}_{14}\pi_a\tau^a_{ij}(N_{N_{1650}})^j_\alpha - (\bar{N}_{939})^i_\alpha\hat{\Pi}^{N\prime}_{\partial,14}\gamma_\mu^{\alpha\beta}(\partial^\mu\pi_a)\cdot\tau^a_{ij}(N_{N_{1650}})^j_\beta\;, \qquad (5.45)\end{aligned}$$

where $i, j \in \{1, 2\}$ are fundamental isospin indices, $a \in \{1, 2, 3\}$ are adjoint isospin indices and $\alpha, \beta, \delta \in \{1, 2, 3, 4\}$ are Dirac indices. Furthermore, we have introduced

$$\hat{\Pi}^{N\prime} = U\hat{\Pi}^N U^\dagger\;, \qquad (5.46)$$

$$\hat{\Pi}^{N\prime}_\partial = U\hat{\Pi}^N_\partial U^\dagger\;, \qquad (5.47)$$

which are the rotated versions of the matrices (4.93) and (4.109) including the coupling constants.

Considering the decay of a certain resonance

$$N_{\text{excited}} \in \{N(1440)\;,\; N(1535)\;,\; N(1650)\} \qquad (5.48)$$

we see that the corresponding interaction terms are of the same form as in the introductory example in Eq. (1.38). Performing similar steps as leading to Eq. (1.40), we obtain

$$\mathcal{L}_{N_{\text{excited}}\to N\pi} = -i\bar{N}_{939}[\tau_a\Upsilon(\Pi^{N\prime} \pm \Pi^{N\prime}_\partial \not{p}_\pi)]\pi_a N_{\text{excited}}\;, \qquad (5.49)$$

where the coupling constants (no matrices, note the absent ˆ)

$$\Pi^{N\prime}\;,\;\Pi^{N\prime}_\partial \quad \in \mathbb{R} \qquad (5.50)$$

are the respective elements of the matrices $\hat{\Pi}^{N\prime}$ and $\hat{\Pi}^{N\prime}_\partial$, see Eqs. (4.93) and (4.109), and p^μ_π is the four-momentum of the pion with $\boldsymbol{p}_\pi$ pointing into the vertex. The upper and lower sign in Eq. (5.49) is valid if N_{excited} describes a positive- or negative-parity resonance, respectively. The matrix Υ is either equal to γ_5 if N_* corresponds to a positive-parity resonance or equal to 1 if N_* is a negative-parity resonance, see Eq. (1.39).

Compared to the Lagrangian in Eq. (1.40), the interaction terms in Eq. (5.49) contain an additional Pauli matrix τ_a which pays attention to possible isospin combinations.

Identifying λ_{iso} in Eq. (1.55) as $\tau_a^{ji}\tau_a^{ji}/2 = 6/2 = 3$, we can use the formula in Eq. (1.57) and obtain the following decay width:

$$\Gamma_{N_{\text{excited}}\to N\pi} = 3\frac{p_f^2}{4\pi m_{N_{\text{excited}}}}[\Pi^{N\prime} - \Pi_\partial^{N\prime}(m_N \pm m_{N_{\text{excited}}})]^2(E_N \mp m_N) \ , \tag{5.51}$$

where p_f is given by

$$p_f = \frac{1}{2m_{N_{\text{excited}}}}\sqrt{(m_{N_{\text{excited}}}^2 - m_N^2 - m_\pi^2)^2 - 4m_N^2 m_\pi^2} \ , \tag{5.52}$$

see Eq. (1.48).

5.3.2. η channel

The interaction terms describing the decay of a nucleonic resonance into the nucleon and an η meson read

$$\mathcal{L}_{NN\eta} = -i\bar{\boldsymbol{N}}\gamma_5\hat{\text{H}}_N^N\eta_N\boldsymbol{N} + \bar{\boldsymbol{N}}\gamma_5\hat{\text{H}}_{N,\partial}^N\gamma_\mu(\partial^\mu\eta_N)\boldsymbol{N} \ , \tag{5.53}$$

compare Eq. (4.92). The η meson is a mixture of η_N and η_S, see Eq. (2.29). However, it is assumed that the amplitude of the decay $N_{\text{excited}} \to N\eta_S$ is massively suppressed. This means that to good approximation

$$\Gamma_{N_{\text{excited}}\to N\eta} \simeq \cos^2\theta_P\Gamma_{N_{\text{excited}}\to N\eta_N} \ . \tag{5.54}$$

Taking this into account and using Eq. (1.57) (with $\lambda_{\text{iso}} = 1$), we obtain the following decay width:

$$\Gamma_{N_{\text{excited}}\to N\eta} = \cos^2\theta_P\frac{p_f^2}{4\pi m_{N_{\text{excited}}}}\{[\text{H}_N^{N\prime} - \text{H}_{N\partial}^{N\prime}(m_N \pm m_{N_{\text{excited}}})]\}^2(E_N \mp m_N) \ , \tag{5.55}$$

where $\hat{\text{H}}_{N(\partial)}^{N\prime} = U\hat{\text{H}}_{N(\partial)}^{N}U^\dagger$ are the rotated versions of the matrices (4.97) and (4.112) and the coupling constants

$$\text{H}^{N\prime} \ , \ \text{H}_\partial^{N\prime} \quad \in \mathbb{R} \tag{5.56}$$

are the respective elements of the matrices $\hat{\text{H}}^{N\prime}$ and $\hat{\text{H}}_\partial^{N\prime}$. Again, the upper (lower) sign in Eq. (5.55) is valid for a positive-(negative-)parity N_* and the magnitude of the three-momentum of the decay products is given in Eq. (1.48).

5.4. Axial Coupling Constants

The Lagrangian (5.10) is invariant under transformations of the chiral group, $SU(2)_L \times SU(2)_R$, which is isomorphic $SU(2)_V \times SU(2)_A$. In this section, we investigate the related axial-vector current resulting from the symmetry under the axial-vector transformation (1.143).

While we have already discussed the axial-vector current of quarks (1.141) in Sec. 1.3.4, now we derive the axial-vector currents of the nucleon and nucleonic resonances. These currents are of the form

$$J_A^\mu = \theta^a J_{a,A}^\mu \qquad \text{with} \qquad J_{a,A}^\mu = g_{A,ij} \boldsymbol{N}_i \gamma^\mu \gamma_5 \frac{\tau_a}{2} \boldsymbol{N}_j \qquad \forall\, i,j \in \{N, N_*, M, M_*\} \ , \tag{5.57}$$

where $g_{A,ij}$ are the axial coupling constants of the bare fields $\boldsymbol{N}_i$ and $\boldsymbol{N}_j$.

Due to Noether's theorem (1.58), the symmetry of the Lagrangian (5.11) under axial-vector transformations leads to the following axial-vector current:

$$J_A^\mu = \frac{\partial \mathcal{L}_{\text{bar},N}}{\partial(\partial_\mu \boldsymbol{N}_i)} \delta \boldsymbol{N}_i + \frac{\partial \mathcal{L}_{\text{bar},N}}{\partial(\partial_\mu \pi^a)} \delta \pi^a \ , \tag{5.58}$$

where $\delta \boldsymbol{N}_i$ and $\delta \pi^a$ are the infinitesimal variations of $\boldsymbol{N}_i$ and π^a under axial-vector transformations.

Using the transformations of the quark and antiquark fields (1.147) to (1.149), we obtain the axial-vector transformation of the pion field represented by the pseudoscalar bilinear form in the right expression of Eq. (2.36):

$$\begin{aligned}
\pi^a(x) \equiv i\sqrt{2}\bar{q}(x)\gamma_5 \frac{\tau^a}{2} q(x) \ \stackrel{A}{\to}\ & i\sqrt{2}\bar{q}(x) \left(1 - i\theta_A^b \gamma_5 \frac{\tau_b}{2}\right) \gamma_5 \frac{\tau^a}{2} \left(1 - i\theta_A^c \frac{\tau_c}{2}\right) q(x) + \mathcal{O}(\theta_A^2) \\
&= i\sqrt{2}\bar{q}(x)\gamma_5 \frac{\tau^a}{2} q(x) + \sqrt{2}\bar{q}(x)\theta_A^b \frac{\tau^b\tau^a + \tau^a\tau^b}{4} q(x) + \mathcal{O}(\theta_A^2) \\
&= i\sqrt{2}\bar{q}(x)\gamma_5 \frac{\tau^a}{2} q(x) + \sqrt{2}\theta_A^a \bar{q}(x) \frac{1}{2} q(x) + \mathcal{O}(\theta_A^2) \\
&\equiv \pi^a(x) + \theta^a \sigma_N(x) + \mathcal{O}(\theta_A^2)
\end{aligned} \tag{5.59}$$

where we have used the anticommutator of Pauli matrices $\{\tau^a, \tau^b\} = 2\delta^{ab}$, and identified the scalar non-strange $\sigma_N(x)$ as the scalar bilinear form in the left expression Eq. (2.36).

After SSB (2.83) and taking into account the consequential renormalization (2.93) of the pion field, the infinitesimal variation of $\pi^a(x)$ under axial transformations is given by

$$\delta \pi^a = \theta^a \left(\frac{\sigma_N}{Z} + \frac{\phi_N}{Z} \right) \ , \tag{5.60}$$

where $Z = Z_\pi$.

The axial-vector transformations of $\boldsymbol{N}_N$, $\boldsymbol{N}_{N_*}$, $\boldsymbol{N}_M$, and $\boldsymbol{N}_{M_*}$ follow from the $SU(2)_L \times SU(2)_R$ transformations (5.3) to (5.6) by taking the group parameters as

$$\theta_L = -\theta_R = \theta_A \ , \qquad \text{i.e.,} \qquad U_{L/R} \ \to\ \exp(\mp i\theta_A^a T_a) \ , \tag{5.61}$$

see Eq. (1.154). In this way, we obtain

$$
\begin{aligned}
\boldsymbol{N}_{N_{(*)}} \xrightarrow{A} & [1 - i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\boldsymbol{N}_{N_{(*)}L} + [1 + i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\boldsymbol{N}_{N_{(*)}R} \\
&= [1 - i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\frac{1-\gamma_5}{2}\boldsymbol{N}_{N_{(*)}} + [1 + i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\frac{1+\gamma_5}{2}\boldsymbol{N}_{N_{(*)}} \\
&= \boldsymbol{N}_{N_{(*)}} - i\theta_A^a T_a \gamma_5 \boldsymbol{N}_{N_{(*)}} + \mathcal{O}(\theta_A^2) \ , \\
\boldsymbol{N}_{M_{(*)}} \xrightarrow{A} & [1 + i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\boldsymbol{N}_{M_{(*)}L} + [1 - i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\boldsymbol{N}_{M_{(*)}R} \\
&= [1 + i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\frac{1-\gamma_5}{2}\boldsymbol{N}_{M_{(*)}} + [1 - i\theta_A^a T_a + \mathcal{O}(\theta_A^2)]\frac{1+\gamma_5}{2}\boldsymbol{N}_{M_{(*)}} \\
&= \boldsymbol{N}_{M_{(*)}} + i\theta_A^a T_a \gamma_5 \boldsymbol{N}_{M_{(*)}} + \mathcal{O}(\theta_A^2) \ ,
\end{aligned} \tag{5.62}
$$

where we have used the chiral projection operators (1.129). Thus, the infinitesimal variation of the two-flavor baryon fields under axial-vector transformations is given by

$$
\delta \boldsymbol{N}_{N(*)} = -iT_a\gamma_5\boldsymbol{N}_{N(*)} \ , \qquad \delta \boldsymbol{N}_{M(*)} = iT_a\gamma_5\boldsymbol{N}_{M(*)} \tag{5.63}
$$

Using Eqs. (5.60) and (5.63). the axial-vector current (5.57) finally reads

$$
\begin{aligned}
J^\mu_{a,A} =& g_{A,NN}\bar{\boldsymbol{N}}_N\gamma^\mu\gamma^5\frac{\tau^a}{2}\boldsymbol{N}_N + g_{A,N_*N_*}\bar{\boldsymbol{N}}_{N_*}\gamma^\mu\gamma^5\frac{\tau^a}{2}\boldsymbol{N}_{N_*} \\
&+ g_{A,MM}\bar{\boldsymbol{N}}_M\gamma^\mu\gamma^5\frac{\tau^a}{2}\boldsymbol{N}_M + g_{A,M_*M_*}\bar{\boldsymbol{N}}_{M_*}\gamma^\mu\gamma^5\frac{\tau^a}{2}\boldsymbol{N}_{M_*} \\
&+ g_{A,NN_*}\bar{\boldsymbol{N}}_N\gamma^\mu\frac{\tau^a}{2}\boldsymbol{N}_{N_*} + g_{A,NN_*}\bar{\boldsymbol{N}}_{N_*}\gamma^\mu\frac{\tau^a}{2}\boldsymbol{N}_N \\
&+ g_{A,MM_*}\bar{\boldsymbol{N}}_M\gamma^\mu\frac{\tau^a}{2}\boldsymbol{N}_{M_*} + g_{A,MM_*}\bar{\boldsymbol{N}}_{M_*}\gamma^\mu\frac{\tau^a}{2}\boldsymbol{N}_M \ ,
\end{aligned} \tag{5.64}
$$

where

$$
g_{A,NN} = g_{A,N_*N_*} = 1 - \frac{c_N}{g_1}\left(1 - \frac{1}{Z^2}\right) \ , \tag{5.65}
$$

$$
g_{A,MM} = g_{A,M_*M_*} = -1 + \frac{c_M}{g_1}\left(1 - \frac{1}{Z^2}\right) \ , \tag{5.66}
$$

are the axial coupling constants of $\boldsymbol{N}_N$, $\boldsymbol{N}_{N_*}$, $\boldsymbol{N}_M$, $\boldsymbol{N}_{M_*}$, and

$$
g_{A,NN_*} = -\frac{c_{A_N}}{g_1}\left(1 - \frac{1}{Z^2}\right) \ , \tag{5.67}
$$

$$
g_{A,MM_*} = \frac{c_{A_M}}{g_1}\left(1 - \frac{1}{Z^2}\right) \tag{5.68}
$$

are the 'mixed' axial coupling constants of $\boldsymbol{N}_N$ with $\boldsymbol{N}_{N_*}$ and $\boldsymbol{N}_M$ with $\boldsymbol{N}_{M_*}$.

The axial coupling constants of the physical fields have to be extracted from the respective terms of the axial current after the transformation (5.21) has been performed. We directly implement this in our numerical code for the fitting procedure and thus we do not give the explicit expressions here.

5.5. Fit and results

This section is subdivided into a discussion of the fitting procedure of the parameters to experimental data in Sec. 5.5.1, the presentation of the numerical results for physical quantities in Sec. 5.5.2, the discussion of the mass matrices including an identification of chiral partners in Sec. 5.5.3, and conclusions in Sec. 5.5.4.

5.5.1. Parameter fit

The two-flavor baryonic eLSM (5.10) contains the following twelve parameters:

$$m_{0,1},\ m_{0,2},\ c_N,\ c_M,\ c_{A_N},\ c_{A_M},\ g_N,\ g_M,\ \kappa_1,\ \kappa_2,\ \kappa_1',\ \kappa_2'\ . \tag{5.69}$$

We fit these parameters to thirteen physical quantities which are known from experiment: the four masses of the nucleon and the resonances, five partial decay widths, and four axial coupling constants.

For the masses the four baryonic states we use the following experimental values as listed in Ref. [81]:

$$m_N^{\text{exp}} = (938.9 \pm 0.000041)\ \text{MeV}\ , \tag{5.70}$$
$$m_{N(1440)}^{\text{exp}} = (1430 \pm 20)\ \text{MeV}\ , \tag{5.71}$$
$$m_{N(1535)}^{\text{exp}} = (1535 \pm 10)\ \text{MeV}\ , \tag{5.72}$$
$$m_{N(1650)}^{\text{exp}} = (1655 \pm 15)\ \text{MeV}\ . \tag{5.73}$$

The errors of these values range from $4 \cdot 10^{-6}$ to 1.4 %, i.e. the masses are known to (in some cases very) high precision. We do not expect that our model describes the baryon masses such precisely, since it does not contain effects arising from isospin breaking. For this reason, we assume a 5% uncertainty of the masses instead:

$$\delta m_N^{5\%} = 47\ \text{MeV}\ , \tag{5.74}$$
$$\delta m_{N(1440)}^{5\%} = 72\ \text{MeV}\ , \tag{5.75}$$
$$\delta m_{N(1535)}^{5\%} = 77\ \text{MeV}\ , \tag{5.76}$$
$$\delta m_{N(1650)}^{5\%} = 83\ \text{MeV}\ . \tag{5.77}$$

This is a strategy which was already followed in the fit the mesonic sector in Ref. [3].

Furthermore, we fit to the partial decay widths of the baryonic resonances into a nucleon and a pseudoscalar meson:

$$\Gamma^{\text{exp}}_{N(1535)\to N\pi} = (0.068 \pm 0.019)\text{ GeV} , \tag{5.78}$$
$$\Gamma^{\text{exp}}_{N(1535)\to N\eta} = (0.063 \pm 0.018)\text{ GeV} , \tag{5.79}$$
$$\Gamma^{\text{exp}}_{N(1650)\to N\pi} = (0.105 \pm 0.037)\text{ GeV} , \tag{5.80}$$
$$\Gamma^{\text{exp}}_{N(1650)\to N\eta} = (0.015 \pm 0.008)\text{ GeV} , \tag{5.81}$$
$$\Gamma^{\text{exp}}_{N(1440)\to N\pi} = (0.195 \pm 0.087)\text{ GeV} , \tag{5.82}$$

which are obtained from the total width and the branching ratios in Ref. [81].

Finally we use the axial coupling constant of the nucleon as given in Ref. [81]:

$$g_A^{N,\text{exp}} = 1.267 \pm 0.003 , \tag{5.83}$$

and the following axial coupling constants which are known from lattice results, see Ref. [68]:

$$g_A^{N(1440),\text{exp}} = 1.2 \pm 0.2 , \tag{5.84}$$
$$g_A^{N(1535),\text{exp}} = 0.2 \pm 0.3 , \tag{5.85}$$
$$g_A^{N(1650),\text{exp}} = 0.55 \pm 0.2 . \tag{5.86}$$

The eLSM Lagrangian contains also some parameters which are already determined by the mesonic eLSM studies in Ref. [3], see Tab. 2.3 and Eqs. (2.89) to (2.92), (2.97):

$$\phi_N = 165\text{ MeV} , \tag{5.87}$$
$$\phi_S = 126\text{ MeV} , \tag{5.88}$$
$$Z = \frac{\phi_N}{f_\pi} = 1.79 , \tag{5.89}$$
$$w = \frac{g_1 \phi_N}{m_{a_1}^2} = 0.000636\text{ MeV}^{-1} , \tag{5.90}$$

where the pion decay constant is $f_\pi = 92.2$ MeV [18], $g_1 = 5.84 \pm 0.02$ [Tab. 2.3], and $m_{a_1} = (1230 \pm 40)$ MeV [18] is the mass of $a_1(1260)$.

To fit the twelve parameters to these experimental values, we use a standard χ^2 procedure, i.e., we minimize the function

$$\chi^2 = \left(\frac{m_{939} - m_N^{\text{exp}}}{\delta m_N^{5\%}}\right)^2 + \ldots + \left(\frac{\Gamma_{N(1535)\to N\pi} - \Gamma^{\text{exp}}_{N(1535)\to N\pi}}{\delta\Gamma^{\text{exp}}_{N(1535)\to N\pi}}\right)^2 + \ldots + \left(\frac{g_A^N - g_A^{N,\text{exp}}}{\delta g_A^{N,\text{exp}}}\right)^2 + \ldots , \tag{5.91}$$

	minimum 1	minimum 2	minimum 3
$m_{0,1}$ [GeV]	0.1393 ± 0.0026	0.14 ± 0.11	-1.078 ± 0.017
$m_{0,2}$ [GeV]	-0.2069 ± 0.0027	-0.18 ± 0.12	0.894 ± 0.019
c_N	-2.071 ± 0.023	-2.83 ± 0.39	-33.6 ± 2.2
c_M	12.4 ± 1.3	11.7 ± 1.8	-19.1 ± 3.1
c_{A_N}	-1.00 ± 0.23	0.03 ± 0.40	-2.68 ± 0.80
c_{A_M}	-51.0 ± 2.8	80 ± 41	-71.7 ± 6.5
g_N	15.485 ± 0.012	15.24 ± 0.36	10.58 ± 0.24
g_M	17.96 ± 0.17	18.26 ± 0.52	13.07 ± 0.33
κ_1 [GeV^{-1}]	37.80 ± 0.26	59.9 ± 8.5	32.4 ± 4.2
κ_1' [GeV^{-1}]	57.12 ± 0.29	29.8 ± 6.6	55.2 ± 4.0
κ_2 [GeV^{-1}]	-20.7 ± 2.5	32 ± 13	-20 ± 13
κ_2' [GeV^{-1}]	41.5 ± 3.2	-8 ± 13	48.9 ± 4.5
χ^2	10.3	10.7	10.3

Table 5.1. Parameter values of the baryonic eLSM.

where the dots refer to the terms containing the remaining masses, decay widths, and axial coupling constants.

We find that three reasonable minima exist, which are all almost equally deep. The corresponding parameter values are listed in Tab. 5.1. An interesting difference between the first two minima and the third one, are the fundamentally different values of $m_{0,1}$ and $m_{0,2}$. In the cases of minima 1 and 2 they are small, but for minimum 3 they are in the range of the vacuum mass of the nucleon. This indicates that for the first two minima all masses arise almost exclusively from chiral symmetry breaking. In contrast, in the third minimum the masses contain sizable contributions from other source, such as for instance a gluon condensate. [Note, indeed the value of this contribution obtained in the third minimum is in accordance with the results of a recent study of Ref. [62]. Here, a three-flavor chiral model with baryons including also the decuplet baryons, but no (axial-)vector meson fields, predicts a chirally invariant contribution to the nucleon mass in the range of $500 - 800$ MeV. Also their calculated upper bounds for the axial coupling constants fit well to our results.]

	minimum 1	minimum 2	minimum 3	exp/lattice
m_N [GeV]	0.9389 ± 0.0010	0.9389 ± 0.0010	0.9389 ± 0.0010	0.9389 ± 0.001
$m_{N(1440)}$ [GeV]	1.430 ± 0.071	1.432 ± 0.073	1.429 ± 0.074	1.43 ± 0.07
$m_{N(1535)}$ [GeV]	1.561 ± 0.065	1.585 ± 0.069	1.559 ± 0.069	1.53 ± 0.08
$m_{N(1650)}$ [GeV]	1.658 ± 0.076	1.619 ± 0.071	1.663 ± 0.081	1.65 ± 0.08
$\Gamma_{N(1440)\to N\pi}$ [GeV]	0.195 ± 0.087	0.195 ± 0.088	0.196 ± 0.087	0.195 ± 0.087
$\Gamma_{N(1535)\to N\pi}$ [GeV]	0.072 ± 0.019	0.073 ± 0.019	0.072 ± 0.019	0.068 ± 0.019
$\Gamma_{N(1535)\to N\eta}$ [GeV]	0.0055 ± 0.0025	0.0062 ± 0.0024	0.0055 ± 0.0027	0.063 ± 0.018
$\Gamma_{N(1650)\to N\pi}$ [GeV]	0.112 ± 0.033	0.114 ± 0.033	0.112 ± 0.033	0.105 ± 0.037
$\Gamma_{N(1650)\to N\eta}$ [GeV]	0.0117 ± 0.0038	0.0109 ± 0.0038	0.0119 ± 0.0038	0.015 ± 0.008
g_A^N	1.2670 ± 0.0025	1.2670 ± 0.0025	1.2670 ± 0.0025	1.267 ± 0.003
$g_A^{N(1440)}$	1.20 ± 0.20	1.19 ± 0.20	1.21 ± 0.21	1.2 ± 0.2
$g_A^{N(1535)}$	0.20 ± 0.30	0.21 ± 0.30	0.20 ± 0.31	0.2 ± 0.3
$g_A^{N(1650)}$	0.55 ± 0.20	0.55 ± 0.20	0.55 ± 0.20	0.55 ± 0.2

Table 5.2. Numerical results for physical quantities and experimental/lattice data.

5.5.2. Numerical results for experimental quantities

Using the parameter values in Tab. 5.1, we compute experimental quantities, such as masses decay widths, and axial coupling constants. The numerical results are given in Tab. 5.2. The theoretical errors of these values are calculated using Gaussian error propagation, i.e., for a function $f(x_i)$ depending on n parameters x_i whose errors are given by Δx_i we calculate the error function as

$$\Delta f = \sqrt{\left(\sum_{i=1}^{n} \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \Delta x_i\right)^2} \,. \tag{5.92}$$

Considering the results given in Tab. 5.2, we see that the model (for all three solutions) describes most of the physical quantities in good agreement with experiment. Indeed, the value of χ^2 for all three minima is completely dominated by only one deviation: the $N(1535) \to N\eta$ decay width. The theoretical prediction is by about an order of magnitude smaller than the experimental value. This decay width already poses a problem in the earlier studies of Ref. [12] (see Sec. 2.3.3). They speculated that the inclusion of more multiplets could solve the problem, but as we have shown now, this does not solve the problem.

A possible approach to the issue with the resonance $N(1535)$ can be found in Ref. [82], where a sizable amount of $s\bar{s}$ within $N(1535)$ is discussed. However, in these studies $N(1535)$ is assumed to be a five-quark object, which is in contrast to recent lattice-QCD calculations [83] predicting that $N(1535)$ has a dominant three-quark core. A three-quark substructure of $N(1535)$ is also assumed in Ref. [68] which predicts axial coupling constants in agreement with the non-relativistic quark model.

In this work, we investigate a further possibility: the chiral anomaly in the baryonic sector, see Refs. [6, 22]. The inclusion of such a term into eLSM will lead to an enhanced coupling to the resonances η and η', see Chap. 6.

5.5.3. Mass matrices and chiral partners

In the following, we discuss the numerical results for the mass matrix (4.87) and the transformation matrix (5.18) for the three different scenarios (minimum 1 to 3).

Minimum 1: With the parameters corresponding to minimum 1, we obtain the following numerical results for the parameter combinations occurring in the mass matrix (4.87):

$$\begin{aligned}
(m_{0,1}+m_{0,2})/2 &= -33.782 \text{ MeV} , & (m_{0,1}-m_{0,2})/2 &= 173.127 \text{ MeV} , \\
g_N\phi_N/2 &= 1274.42 \text{ MeV} , & g_M\phi_N/2 &= 1478.31 \text{ MeV} , \\
(\kappa_1'+\kappa_1)\phi_N\phi_S/(4\sqrt{2}) &= -348.585 \text{ MeV} , & (\kappa_1'-\kappa_1)\phi_N\phi_S/(4\sqrt{2}) &= 70.9497 \text{ MeV} , \\
(\kappa_2'+\kappa_2)\phi_N\phi_S/(4\sqrt{2}) &= -76.2258 \text{ MeV} , & (\kappa_2'-\kappa_2)\phi_N\phi_S/(4\sqrt{2}) &= 228.307 \text{ MeV} .
\end{aligned} \tag{5.93}$$

Thus, the mass matrix reads

$$M_{\text{min1}} = \begin{pmatrix} 0.926 & 0.071 & -0.034 & 0.173 \\ 0.071 & -1.623 & -0.173 & 0.034 \\ -0.034 & -0.173 & 1.402 & 0.228 \\ 0.173 & 0.034 & 0.228 & -1.555 \end{pmatrix} \text{GeV} . \tag{5.94}$$

The transformation matrix U is defined via the eigenvectors of this matrix, see Eq. (5.18). The numerical result is given by

$$\begin{pmatrix} N_{939} \\ \gamma^5 N_{1535} \\ N_{1440} \\ \gamma^5 N_{1650} \end{pmatrix} = \begin{pmatrix} \mathbf{-0.996} & -0.025 & -0.046 & -0.074 \\ 0.075 & \mathbf{-0.492} & 0.039 & \mathbf{-0.867} \\ -0.050 & -0.057 & \mathbf{0.995} & 0.073 \\ 0.010 & \mathbf{0.869} & 0.086 & \mathbf{-0.488} \end{pmatrix} \begin{pmatrix} \boldsymbol{N}_N \\ \gamma^5 \boldsymbol{N}_{N_*} \\ \boldsymbol{N}_M \\ \gamma^5 \boldsymbol{N}_{M_*} \end{pmatrix} , \tag{5.95}$$

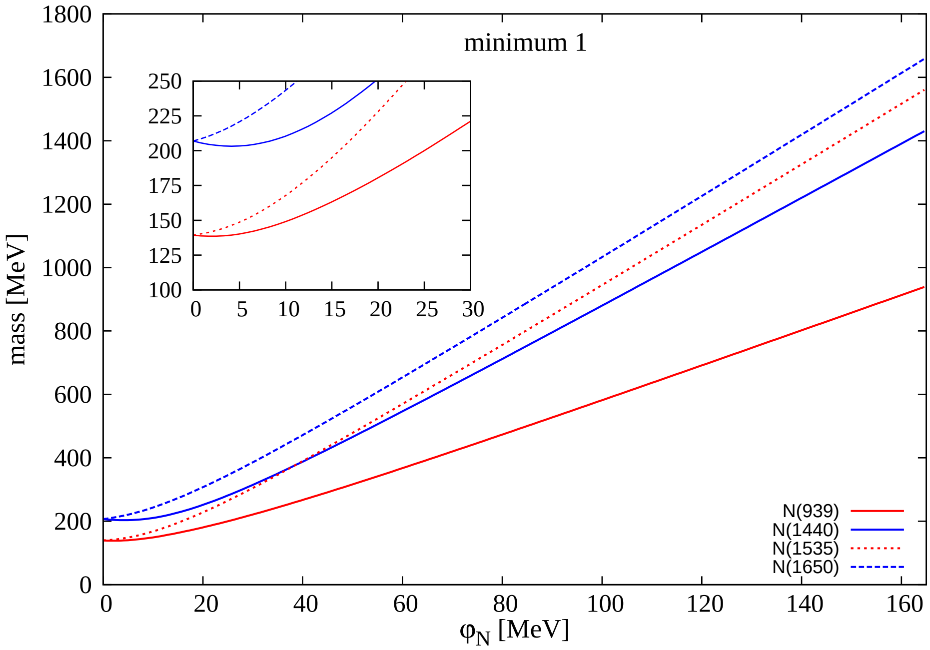


Figure 5.1. Masses as a function of ϕ_N for minimum 1 [16].

revealing that to first approximation, we can assign

$$N_{939} \approx \boldsymbol{N}_N \ , \quad N_{1650} \approx \boldsymbol{N}_{N_*} \ , \quad N_{1440} \approx \boldsymbol{N}_M \ , \quad N_{1535} \approx \boldsymbol{N}_{M_*} \ . \tag{5.96}$$

Nevertheless, all four eigenstates are mixtures of the field, but most notable, the two negative-parity states N_{1535} and N_{1650} mix strongly (the mixing angle is about 30°).

Finally, it is interesting to trace all masses when chiral symmetry is restored ($\phi_N, \phi_S \to 0$), since the determination of which states become degenerate in mass allows to deduce which states form chiral partners.
Since ϕ_S only appears with a factor ϕ_N in the mass matrix (4.87), it is sufficient to compute the masses (i.e., the eigenvalues of M_{min1}) as a function of ϕ_N, while keeping ϕ_S at its vacuum value. Tracing the masses for $\phi_N \to 0$, we obtain the result shown in Fig. 5.1. Since the masses of $N(939)$ and $N(1535)$ merge for $\phi_N \to 0$, we conclude that

$$N(939) \text{ and } N(1535) \tag{5.97}$$

are chiral partners with a common mass $m_{0,1} = 139$ MeV when chiral symmetry is restored. Following the same argument,

$$N(1440) \text{ and } N(1650) \tag{5.98}$$

form chiral partners with a mass $|m_{0,2}| = 207$ MeV in the chiral limit.

Minimum 2: With the parameters corresponding to minimum 2, we obtain

$$\begin{aligned} (m_{0,1}+m_{0,2})/2 &= -17.034 \text{ MeV} , & (m_{0,1}-m_{0,2})/2 &= 160.686 \text{ MeV} , \\ g_N\phi_N/2 &= 1253.94 \text{ MeV} , & g_M\phi_N/2 &= 1502.71 \text{ MeV} , \\ (\kappa_1'+\kappa_1)\phi_N\phi_S/(4\sqrt{2}) &= 329.237 \text{ MeV} , & (\kappa_1'-\kappa_1)\phi_N\phi_S/(4\sqrt{2}) &= -110.532 \text{ MeV} , \\ (\kappa_2'+\kappa_2)\phi_N\phi_S/(4\sqrt{2}) &= 87.2649 \text{ MeV} , & (\kappa_2'-\kappa_2)\phi_N\phi_S/(4\sqrt{2}) &= -146.129 \text{ MeV} . \end{aligned} \tag{5.99}$$

and the mass matrix (4.87) reads

$$M_{\text{min2}} = \begin{pmatrix} 0.925 & -0.111 & -0.017 & 0.161 \\ -0.111 & -1.583 & -0.161 & 0.017 \\ -0.017 & -0.161 & 1.415 & -0.146 \\ 0.161 & 0.017 & -0.146 & -1.590 \end{pmatrix} \text{GeV} . \tag{5.100}$$

The transformation matrix (5.18) is given by

$$\begin{pmatrix} N_{939} \\ \gamma^5 N_{1535} \\ N_{1440} \\ \gamma^5 N_{1650} \end{pmatrix} = \begin{pmatrix} \mathbf{-0.996} & 0.046 & -0.039 & -0.061 \\ -0.002 & \mathbf{0.806} & 0.072 & \mathbf{0.587} \\ -0.038 & -0.052 & \mathbf{0.997} & -0.051 \\ 0.076 & \mathbf{0.588} & -0.007 & \mathbf{-0.805} \end{pmatrix} \begin{pmatrix} \boldsymbol{N}_N \\ \gamma^5 \boldsymbol{N}_{N_*} \\ \boldsymbol{N}_M \\ \gamma^5 \boldsymbol{N}_{M_*} \end{pmatrix} . \tag{5.101}$$

As for minimum 1, a significant mixing of the negative-parity states is present, but here we predominantly assign

$$N_{939} \approx \boldsymbol{N}_N , \quad N_{1535} \approx \boldsymbol{N}_{N_*} , \quad N_{1440} \approx \boldsymbol{N}_M , \quad N_{1650} \approx \boldsymbol{N}_{M_*} . \tag{5.102}$$

Again, we decide which states form chiral partners tracing the masses for $\phi_N \to 0$ and we obtain the result as shown in Fig. 5.2. The conclusion is equivalent to the case of minimum 1:

$$N(939) \text{ and } N(1535) \tag{5.103}$$

are chiral partners. their common mass is $m_{0,1} = 144$ MeV as $\phi_N \to 0$. The second pair of chiral partners is

$$N(1440) \text{ and } N(1650) \tag{5.104}$$

with a mass $|m_{0,2}| = 178$ MeV, when chiral symmetry is restored.

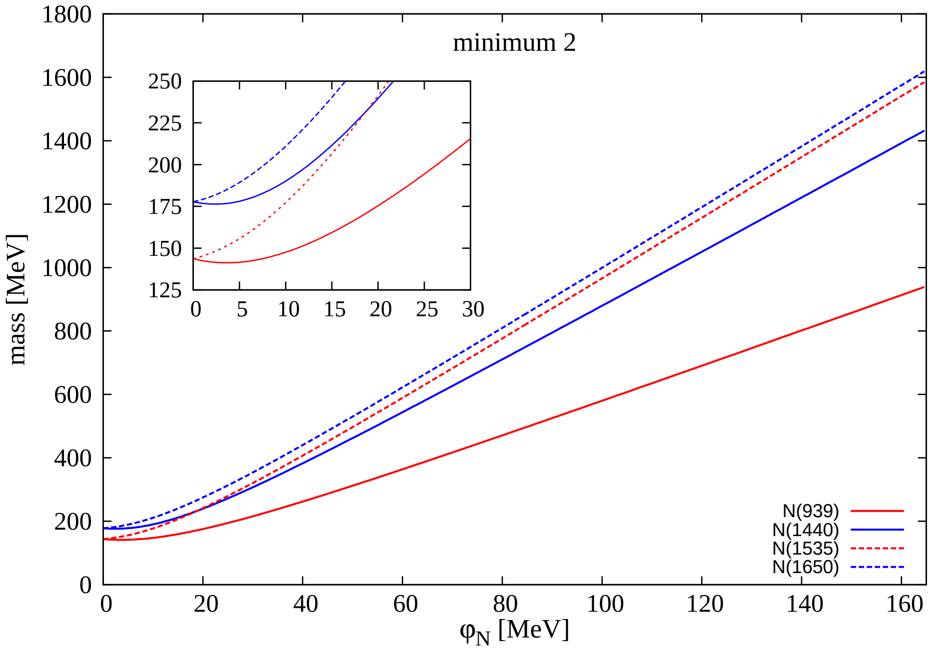


Figure 5.2. Masses as a function of ϕ_N for minimum 2 [16].

Minimum 3: In this case, the parameter values are given by

$$
\begin{aligned}
(m_{0,1}+m_{0,2})/2 &= -92.041 \text{ MeV} \, , & (m_{0,1}-m_{0,2})/2 &= -985.689 \text{ MeV} \, , \\
g_N\phi_N/2 &= 870.339 \text{ MeV} \, , & g_M\phi_N/2 &= 1075.52 \text{ MeV} \, , \\
(\kappa_1'+\kappa_1)\phi_N\phi_S/(4\sqrt{2}) &= 321.802 \text{ MeV} \, , & (\kappa_1'-\kappa_1)\phi_N\phi_S/(4\sqrt{2}) &= 83.9592 \text{ MeV} \, , \\
(\kappa_2'+\kappa_2)\phi_N\phi_S/(4\sqrt{2}) &= 105.62 \text{ MeV} \, , & (\kappa_2'-\kappa_2)\phi_N\phi_S/(4\sqrt{2}) &= 253.352 \text{ MeV} \, ,
\end{aligned}
\tag{5.105}
$$

which lead to the following mass matrix (4.87):

$$
M_{\text{min3}} = \begin{pmatrix} 0.549 & 0.084 & -0.092 & -0.986 \\ 0.084 & -1.192 & 0.986 & 0.092 \\ -0.092 & 0.986 & 0.970 & 0.253 \\ -0.986 & 0.092 & 0.253 & -1.181 \end{pmatrix} \text{GeV} \, . \tag{5.106}
$$

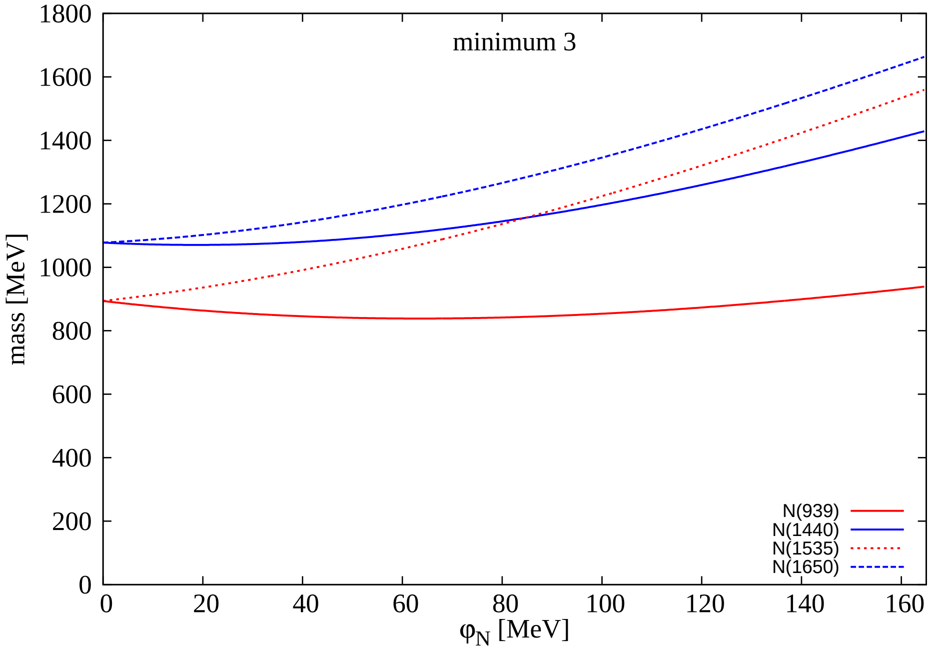


Figure 5.3. Masses as a function of ϕ_N for minimum 3 [16].

Considering the transformation matrix (5.18), we find that all states mix strongly with each other:

$$\begin{pmatrix} N_{939} \\ \gamma^5 N_{1535} \\ N_{1440} \\ \gamma^5 N_{1650} \end{pmatrix} = \begin{pmatrix} \mathbf{-0.865} & -0.163 & \mathbf{-0.312} & \mathbf{0.358} \\ 0.140 & \mathbf{0.830} & \mathbf{-0.359} & \mathbf{0.404} \\ \mathbf{-0.292} & \mathbf{0.327} & \mathbf{0.875} & 0.207 \\ \mathbf{-0.384} & \mathbf{0.422} & -0.093 & \mathbf{-0.816} \end{pmatrix} \begin{pmatrix} \boldsymbol{N}_N \\ \gamma^5 \boldsymbol{N}_{N_*} \\ \boldsymbol{N}_M \\ \gamma^5 \boldsymbol{N}_{M_*} \end{pmatrix} . \tag{5.107}$$

Once more, we decide which states form chiral partners by computing the masses as a function of ϕ_N, while keeping ϕ_S at its vacuum value. The result is shown in Fig. 5.3. Again we conclude that

$$N(939) \text{ and } N(1535) \tag{5.108}$$

are chiral partners. Their common mass is $m_{0,2} = 894$ MeV for restored chiral symmetry. The other pair of chiral partners is

$$N(1440) \text{ and } N(1650) \tag{5.109}$$

with a mass $|m_{0,1}| = 1078$ MeV in the chiral limit.

5.5.4. Conclusions

A fit of the parameters to masses, decay widths, and axial coupling constants of the nucleon $N(939)$ and the resonances $N(1440)$, $N(1535)$, and $N(1650)$, yields three acceptable sets of parameter values which minimize the χ^2 function.

With the exception of the decay $N(1535) \to N\eta$, physical quantities are described in good agreement with experiments and lattice calculations for all three minima, see Tab. 5.2. Concerning the issue of the $N(1535) \to N\eta$ decay width, we investigate the influence of the axial $U(1)_A$ anomaly in the following Chap. 6. As we show, it enhances the coupling of some excited baryons [inter alia the $N(1535)$] to the η meson.

Investigating the masses for $\phi_N \to 0$ (i.e., to the point where chiral symmetry is restored) shows that the pairs $N(939)$, $N(1535)$, as well as $N(1440)$, $N(1650)$ should be identified as chiral partners.

Chapter 6.

Anomaly term and pseudoscalar-glueball interactions in the baryonic sector

The two-flavor study has revealed that the eLSM in Eq. (4.1) or (4.75) produces too small values for the decay width of $N(1535) \to N\eta$ when compared to experiments, see Chap. 5. In this chapter, we propose an anomaly term in the baryonic sector to solve this problem. To this end, it is useful to first investigate a simple model based on flavor symmetry only, see Sec. 6.1. Then, in Sec. 6.2 we construct anomalous terms for the case $N_f = 2$ and $N_f = 3$. Finally, since the mathematical structure of such terms can be easily used to couple the pseudoscalar glueball to baryons, we discuss the corresponding Lagrangian in Sec. 6.3.

6.1. A simple model based on flavor symmetry only

It is reasonable to assume that the three lightest quarks are degenerate in mass. In any chiral model, this leads to a residual $U(3)_V$ flavor symmetry. For this reason, a simple model which is based on this symmetry alone (i.e., without the full chiral symmetry and without terms parametrizing the axial anomaly) already provides instructive insights into the phenomenology of hadrons. In this section we elaborate such a model investigating the decays of a negative-parity octet-baryon resonance into a ground-state baryon and a pseudoscalar meson.

6.1.1. Motivation

The theoretical decay width of $N(1535) \to N\eta$, predicted by the eLSM in Eq. (4.75), is too small by about one order of magnitude when compared to the experimental value, see Tab. 5.2.
Interestingly, the experimental decay width $\Gamma^{\exp}_{N(1535)\to N\eta} \simeq (65 \pm 25)$ MeV [18] is as large as the decay width of $N(1535) \to N\pi$, $\Gamma_{N(1535)\to N\pi} = (67.5 \pm 19)$ MeV [18]. This is

intuitively very surprising, because flavor symmetry implies that

$$\frac{\Gamma_{N(1535)\to N\eta}}{\Gamma_{N(1535)\to N\pi}} \approx \frac{1}{3}\cos^2\theta_P \approx 0.17 \ , \tag{6.1}$$

where the factor three counts the three isospin states of the pion and $\cos^2\theta_P^2$ with $\theta_P \simeq -44.6°$ [3] takes into account the mixing (2.29) of the pseudoscalar isosinglets η_N and η_S, see Eq. (5.54). Evidently, the predictions of flavor symmetry contradict the experimental findings. Note, the ratio (6.1) would be further reduced by phase space. Flavor symmetry holds in any chiral model, thus Eq. (6.1) explains why the decay width of $N(1535) \to N\eta$ was too small in all solutions which we obtained in Chap. 5.

This implementation of flavor symmetry can be enlarged to $N_f = 3$, in order to check if further decays are underestimated by flavor symmetry. Thus, in the following subsections, we construct an effective Lagrangian based on flavor symmetry and consider the decays of the whole excited baryon octet into the ground-state baryons and a pseudoscalar meson.

6.1.2. Physical content and mathematical representation

The nonet of pseudoscalar mesons is represented by the matrix P, see Eq. (2.36):

$$P = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\eta_N+\pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_N-\pi^0}{\sqrt{2}} & K^0 \\ K^- & \bar{K}^0 & \eta_S \end{pmatrix} \ , \tag{6.2}$$

where η_N and η_S mix to form the physical $\eta = \eta(547)$ and $\eta' = \eta'(958)$, see Eq. (2.29). In order to describe the baryons, we introduce the two matrices

$$O \equiv \begin{pmatrix} \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} \quad \text{and} \quad O_* \equiv \begin{pmatrix} \frac{\Lambda_*}{\sqrt{6}} + \frac{\Sigma^0_*}{\sqrt{2}} & \Sigma^+_* & p_* \\ \Sigma^-_* & \frac{\Lambda_*}{\sqrt{6}} - \frac{\Sigma^0_*}{\sqrt{2}} & n_* \\ \Xi^-_* & \Xi^0_* & -\frac{2\Lambda_*}{\sqrt{6}} \end{pmatrix} . \tag{6.3}$$

The ground-state positive-parity baryons $\{N(939),\ \Lambda(1116),\ \Sigma(1193),\ \Xi(1338)\}$ are represented by O. An octet of excited negative-parity baryons is contained in O_*. We can assign the latter matrix to any baryon octet with the quantum numbers $J^P = \frac{1}{2}^-$, see below.

The flavor transformations are given by

$$P \xrightarrow{V} U_V P U_V^\dagger \ , \tag{6.4}$$

$$O \xrightarrow{V} U_V O U_V^\dagger \ , \tag{6.5}$$

$$O_* \xrightarrow{V} U_V O_* U_V^\dagger \ , \tag{6.6}$$

the parity transformations read

$$P(x) \xrightarrow{P} P^\dagger(t, -\mathbf{x}) \;, \tag{6.7}$$
$$O(x) \xrightarrow{P} \gamma^0 O(t, -\mathbf{x}) \;, \tag{6.8}$$
$$O_*(x) \xrightarrow{P} -\gamma^0 O_*(t, -\mathbf{x}) \;, \tag{6.9}$$

and charge-conjugation transformations are given by

$$P \xrightarrow{C} P^T \;, \tag{6.10}$$
$$O \xrightarrow{C} -C\bar{O}^T \;, \tag{6.11}$$
$$O_* \xrightarrow{C} C\bar{O}_*^T \;, \tag{6.12}$$

see Tabs. 2.2 and 4.2. Note, the flavor transformations follow from the chiral transformations by setting $\theta_L^a = \theta_R^a = \theta_V^a$, see Eq. (1.153).

6.1.3. The Lagrangian

A flavor, parity, and charge-conjugation invariant Lagrangian which describes the interaction of O_*, O, and P is given by

$$\mathcal{L}_V = i\lambda_V \,\mathrm{Tr}(\bar{O}PO_* - \bar{O}_*PO) \;. \tag{6.13}$$

The coupling constant λ_V is dimensionless and scales as

$$\lambda_V \propto \sqrt{N_c} \tag{6.14}$$

in the large-N_c limit. The derivation of the large-N_c scaling is the same as the one leading to Eq. (4.18). The corresponding diagram is depicted in Fig. 4.3, describing a standard decay by creating a quark-antiquark pair from the vacuum.

One can construct further flavor- and CP-invariant terms as

$$i\beta_V \,\mathrm{Tr}(\bar{O}O_*P - \bar{O}_*OP) \;, \tag{6.15}$$
$$i\gamma_V \,\mathrm{Tr}(\bar{O}O_* - \bar{O}_*O)\,\mathrm{Tr}\, P \;, \tag{6.16}$$

but they involve gluons in the intermediate state, which entails that they are large-N_c suppressed. They scale as

$$\beta_V, \gamma_V \propto N_c^{-1/2} \;, \tag{6.17}$$

where the large-N_c analysis of the term $i\beta_V \,\mathrm{Tr}(\bar{O}O_*P - \bar{O}_*OP)$ corresponds to a diagram as depicted in Fig. 4.4. The discussion is equivalent to the one related to Eqs. (4.21) and

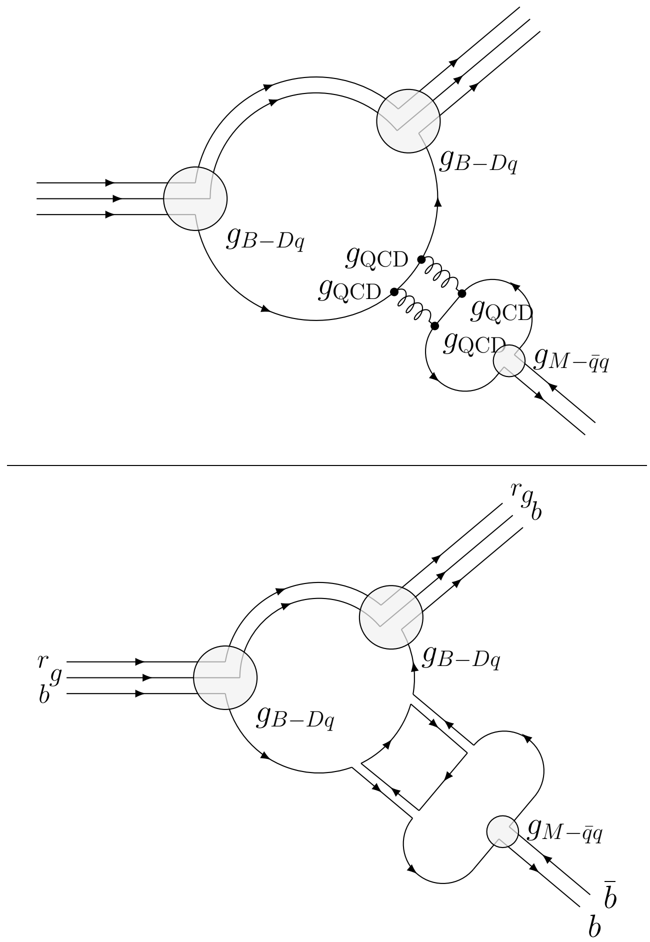


Figure 6.1. Baryon-meson interaction with intermediate gluons, where the first diagram represents the flavor flow, and the second one represents the color flow - γ_V term.

(4.24). To proof the large-N_c scaling of the term to γ_V in Eq. (6.16), we investigate its flavor structure:

$$
\begin{aligned}
i\gamma_V \operatorname{Tr}[\bar{O}O^*] \operatorname{Tr}(P) &= i\gamma_V \bar{O}_{AB} O_{BA} P_{CC} \\
&\sim i\gamma_V \left(\bar{q}_B \bar{D}_A\right) \left(D_A q_B\right) \left(\bar{q}_C q_C\right) \ , \qquad (6.18)
\end{aligned}
$$

which corresponds to the interaction where the meson arises from (at least) two gluons as depicted in Fig. 6.1. As Eq. (6.18) shows the flavor of the meson is completely independent from the quarks of the baryons. Among other couplings, this term includes the coupling term $\bar{N}^* N \eta_S$, i.e., it allows for the coupling of a strange-antistrange pair to the baryons via the exchange of a white di-gluon configuration. The exchange of the two gluons can be intuitively understood in the pseudoscalar channel: first, the pseudoscalar di-gluon configuration $G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$ couples to the baryons as $\left(\bar{q}_B \bar{D}_A\right) \left(D_A q_B\right) G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$., then, the two gluons interact with the singlet configuration $\eta_0 \equiv \bar{q}^b_C q^b_C$ via $\eta_0 \ G^a_{\mu\nu}\tilde{G}^{a,\mu\nu}$. In the second diagram of Fig. (6.1) we represent the gluons in the double-quark line notation, see Ref. [41]. This allows to calculate the large-N_c scaling of the diagram as

$$\left(g_{B\text{-}Dq}\,\frac{1}{m_D}\,g_{M\text{-}\bar{q}q}\,g^4_{QCD}\,g_{B\text{-}Dq}\right)\,N_c^2\propto\frac{1}{\sqrt{N_c}}\;, \tag{6.19}$$

where we took into account two circulating colors: one in the main part of the diagram (as it was the case in Figs. 4.2 and 4.3), and another one in the loop which arises from the di-gluon exchange. Hence, the diagram in Fig. 6.1 is suppressed by a factor N_c, i.e., γ_V scales as $1/\sqrt{N_c}$.

Thus, we set β_V and γ_V to zero in the following discussion of this section.

After the evaluation of the trace in Eq. (6.13), we find the following terms describing decays into the pseudoscalar mesons π, K, η, and η':

$$\begin{aligned}\mathcal{L}_V=&\frac{i\lambda_V}{2\sqrt{3}}\bar{\Lambda}(\boldsymbol{\pi}\cdot\boldsymbol{\Sigma}_*)+\frac{i\lambda_V}{2\sqrt{3}}(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\pi})\Lambda_*+\frac{i\lambda_V}{2}\bar{\boldsymbol{\Sigma}}\cdot(\boldsymbol{\pi}\times\boldsymbol{\Sigma}_*)+\frac{i\lambda_V}{2}\bar{\boldsymbol{N}}(\boldsymbol{\pi}\cdot\boldsymbol{\tau})\boldsymbol{N}_*\\&+\frac{i\lambda_V}{6}\bar{\Lambda}(\eta_N+i2\sqrt{2}\eta_S)\Lambda_*+\frac{i\lambda_V}{2}\bar{\boldsymbol{\Sigma}}\eta_N\boldsymbol{\Sigma}_*+\frac{i\lambda_V}{4\sqrt{2}}\bar{\boldsymbol{\Xi}}\eta_S\boldsymbol{\Xi}_*+\frac{i\lambda_V}{2}\bar{\boldsymbol{N}}\eta_N\boldsymbol{N}_*\\&-\frac{i\lambda_V}{\sqrt{3}}\bar{\Lambda}\boldsymbol{K}\cdot\boldsymbol{\Xi}_*+\frac{i\lambda_V}{2\sqrt{3}}\bar{\boldsymbol{N}}\cdot\boldsymbol{K}\Lambda_*-\frac{i\lambda_V}{2}\boldsymbol{K}^T(\bar{\boldsymbol{\Sigma}}\cdot\boldsymbol{\tau})\boldsymbol{\Xi}_*\\&+\frac{i\lambda_V}{\sqrt{3}}\bar{\Lambda}(\bar{\boldsymbol{K}}\cdot\boldsymbol{N}_*)-\frac{i\lambda_V}{2\sqrt{3}}\bar{\boldsymbol{\Xi}}\cdot\bar{\boldsymbol{K}}\Lambda_*-\frac{i\lambda_V}{2}\bar{\boldsymbol{\Xi}}(\boldsymbol{\Sigma}_*\cdot\boldsymbol{\tau})\bar{\boldsymbol{K}}\;.\end{aligned} \tag{6.20}$$

6.1.4. Decay Widths

The general structure of a Lagrangian describing the decay of a resonance B_* into a ground-state baryon B and a pseudoscalar meson $P=\pi,\eta,\bar{K}$ reads

$$\mathcal{L}=ig_{PBB_*}\bar{B}PB_*\;. \tag{6.21}$$

The coupling constant g_{PBB_*} is obtained from the respective terms of Eq. (6.20). We have listed them in Tab. 6.1. According to Eq. (1.57) the corresponding tree-level decay width is

$$\Gamma_{B_*\to BP}=\lambda_{\text{iso}}^{PBB_*}\frac{p_f}{4\pi m_{B_*}}g^2_{PBB_*}(E_B+m_B)\;, \tag{6.22}$$

where E_B is the relativistic baryon energy in the rest frame of the decaying B_* and for p_f see Eq. (1.48). The computation of the squared averaged amplitude is similar to the calculations of Sec. 5.3. The factor $\lambda_{\text{iso}}^{PBB_*}$ includes the isospin of the involved particles. The corresponding values are also listed in Tab. 6.1.

In the case $P=\eta$ one has to take into account that the physical η is a mixture η_N and η_S, see Eq. (2.29). In this case, the decay amplitude is a mixture of the amplitudes in the strange and the non-strange sector:

$$\mathcal{M}_{B_*\to B\eta}=\cos\theta_P\mathcal{M}_{B_*\to B\eta_N}+\sin\theta_P\mathcal{M}_{B_*\to B\eta_S}\;. \tag{6.23}$$

Decay	g_{PBB_*}	$\lambda_{\text{iso}}^{PBB_*}$	Additive anomalous contribution [Eq. (6.44)] to g_{PBB_*}
$\Sigma_* \to \Lambda\pi$	$\frac{i\lambda_V}{2\sqrt{3}}$	1	
$\Lambda_* \to \Sigma\pi$	$\frac{i\lambda_V}{2\sqrt{3}}$	3	
$\Sigma_* \to \Sigma\pi$	$\frac{i\lambda_V}{2}$	2	
$N_* \to N\pi$	$\frac{i\lambda_V}{2}$	3	
$\Lambda_* \to \Lambda\eta$	$\frac{i\lambda_V}{6}\cos\theta_P$	1	$+\frac{i\tilde{\lambda}_A}{\sqrt{6}}(\sqrt{2}\cos\theta_P + \sin\theta_P)$
$\Sigma_* \to \Sigma\eta$	$\frac{i\lambda_V}{2}\cos\theta_P$	1	$+\frac{i\tilde{\lambda}_A}{\sqrt{6}}(\sqrt{2}\cos\theta_P + \sin\theta_P)$
$\Xi_* \to \Xi\eta$	$\frac{i\lambda_V}{4\sqrt{2}}\cos\theta_P$	1	$+\frac{i\tilde{\lambda}_A}{\sqrt{6}}(\sqrt{2}\cos\theta_P + \sin\theta_P)$
$N_* \to N\eta$	$\frac{i\lambda_V}{2}\cos\theta_P$	1	$+\frac{i\tilde{\lambda}_A}{\sqrt{6}}(\sqrt{2}\cos\theta_P + \sin\theta_P)$
$\Xi_* \to \Lambda\bar{K}$	$-\frac{i2\lambda_V}{2\sqrt{3}}$	1	
$\Lambda_* \to N\bar{K}$	$\frac{i\lambda_V}{2\sqrt{3}}$	1	
$\Sigma_* \to N\bar{K}$	$\frac{i\lambda_V}{2}$	2	

Table 6.1. Coupling constants g_{PBB_*} and γ^{PBB_*} factors accounting for isospin.

The coupling constant $g_{\eta BB_*}$ in Tab. 6.1 is defined as

$$g_{\eta BB_*} = g_{\eta_N BB_*} \cos\theta_P + g_{\eta_S BB_*} \sin\theta_P \ . \tag{6.24}$$

6.1.5. Results

We consider two distinct models of the type (6.13). In the first one, we assume that O_* represents the baryon states $\{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$, while in the second one, it is assigned to the heavier states $\{N(1650),\ \Lambda(1800),\ \Sigma(1750),\ \Xi(?)\}$.

In both cases, we determine the parameter λ_V by fitting the decay width of $N_* \to N\pi$,

$$\Gamma_{N_* \to N\pi} = \frac{3p_f}{4\pi m_{N_*}} \left(\frac{\lambda_V}{2}\right)^2 (E_N + m_N) \ , \tag{6.25}$$

to experimental data.

(1) Model with $O_* \equiv \{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$

For the first assignment, we fit the decay width of $N(1535) \to N\pi$ to the experimentally well determined value (67.5 ± 19) MeV [18]. We obtain

$$\lambda_V = \lambda_V^{N(1535)} = 1.37 \pm 0.19 \ . \tag{6.26}$$

Using this parameter value, we compute the remaining decay widths featured by the Lagrangian (6.20) and obtain the results which are listed in Tab. 6.2. Since the assignment $\Sigma^* \equiv \Sigma(1560)$ is alternatively suggested by the quark-model review in Ref. [18], Tab. 6.2 also contains results with this assignment.

The model can reproduce most of the decay widths, but with two important exceptions. The theoretical predictions of the decay widths $N(1535) \to N\eta$ and $\Lambda(1670) \to \Lambda\eta$ are too small by about one order of magnitude when compared to experiment. It is interesting that in both the η meson is involved. We suspect that flavor symmetry alone is not enough to describe the decays of $\{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$ into a ground-state baryon and an η meson. Evidently, an additional element is missing when we consider the η meson. (Indeed, as we shell see in Sec. 6.2, a term parametrizing the chiral anomaly can correct the results in both channels.)

(2) Model with $O_* \equiv \{N(1650),\ \Lambda(1800),\ \Sigma(1750),\ \Xi(?)\}$

For the second assignment, we fit the decay width of $N(1650) \to N\pi$ to the experimentally determined value (84 ± 23) Mev [18] and obtain

$$\lambda_V = \lambda_V^{N(1650)} = 1.45 \pm 0.20 \ . \tag{6.27}$$

	Flavor model [MeV]	Experiment [18] [MeV]
$\Gamma_{N(1535)\to N\pi}$	67.5 ± 19	67.5 ± 19
$\Gamma_{N(1535)\to N\eta}$	$\mathbf{4.3 \pm {}^{1.3}_{1.1}}$	$\mathbf{40 - 91}$
$\Gamma_{\Lambda(1670)\to N\bar{K}}$	$6.0 \pm {}^{1.8}_{1.6}$	$5 - 15$
$\Gamma_{\Lambda(1670)\to \Sigma\pi}$	$21.3 \pm {}^{6.4}_{5.6}$	$6.25 - 27.5$
$\Gamma_{\Lambda(1670)\to \Lambda\eta}$	$\mathbf{0.6 \pm {}^{1.8}_{1.6}}$	$\mathbf{2.5 - 12.5}$
$\Gamma_{\Sigma(1620)\to N\bar{K}}$	$32 \pm {}^{26}_{20}$	–
$\Gamma_{\Sigma(1620)\to \Lambda\pi}$	$21.7 \pm {}^{6.5}_{5.7}$	–
$\Gamma_{\Sigma(1620)\to \Sigma\pi}$	$39 \pm {}^{12}_{10}$	–
$\Gamma_{\Sigma(1620)\to \Sigma\eta}$	kin. not allowed	–
$\Gamma_{\Sigma(1560)\to N\bar{K}}$	$27 \pm {}^{22}_{17}$	–
$\Gamma_{\Sigma(1560)\to \Lambda\pi}$	$19.8 \pm {}^{6.0}_{5.2}$	–
$\Gamma_{\Sigma(1560)\to \Sigma\pi}$	$34 \pm {}^{10}_{9.1}$	–
$\Gamma_{\Sigma(1560)\to \Sigma\eta}$	kin. not allowed	–

Table 6.2. Results from the flavor model with $O_* \equiv \{N(1535),\, \Lambda(1670),\, \Sigma(1620),\, \Xi(?)\}$.

	Flavor model [MeV]	Experiment [18] [MeV]
$\Gamma_{N(1650)\to N\pi}$	84 ± 23	84 ± 23
$\Gamma_{N(1650)\to N\eta}$	$8.7 \pm {}^{2.6}_{2.2}$	$15.4 - 37.5$
$\Gamma_{N(1650)\to \Lambda K}$	$13.2 \pm {}^{3.9}_{3.4}$	$5.5 - 25.5$
$\Gamma_{\Lambda(1800)\to N\bar{K}}$	$8.2 \pm {}^{2.4}_{2.1}$	$50 - 160$
$\Gamma_{\Lambda(1800)\to \Sigma\pi}$	$28.2 \pm {}^{8.2}_{7.2}$	seen
$\Gamma_{\Lambda(1800)\to \Lambda\eta}$	$3.09 \pm {}^{0.91}_{0.79}$	$2 - 44$
$\Gamma_{\Sigma(1750)\to N\bar{K}}$	$23.0 \pm {}^{6.7}_{5.9}$	$6 - 64$
$\Gamma_{\Sigma(1750)\to \Lambda\pi}$	$28.2 \pm {}^{8.2}_{7.2}$	seen
$\Gamma_{\Sigma(1750)\to \Sigma\pi}$	$53 \pm {}^{16}_{14}$	< 12.8
$\Gamma_{\Sigma(1750)\to \Sigma\eta}$	$2.37 \pm {}^{0.69}_{0.60}$	$9 - 88$
$\Gamma_{\Sigma(1620)\to N\bar{K}}$	$18.1 \pm {}^{5.3}_{4.6}$	–
$\Gamma_{\Sigma(1620)\to \Lambda\pi}$	$24.2 \pm {}^{7.2}_{6.2}$	–
$\Gamma_{\Sigma(1620)\to \Sigma\pi}$	$44 \pm {}^{13}_{11}$	–
$\Gamma_{\Sigma(1620)\to \Sigma\eta}$	kin. not allow	–

Table 6.3. Results from the flavor model with $O_* \equiv \{N(1650), \Lambda(1800), \Sigma(1750), \Xi(?)\}$.

As before, we calculate the remaining decay widths. The results are summarized in Tab. 6.3. For the second assignment, the model is able to describe the $\Lambda(1800) \to \Lambda\eta$ decay width in accordance with data.
The decay width $N(1650) \to N\eta$ comes out too small by at least a factor of 1.4 (when we take the maximum theoretical and minimum experimental value) when compared to the $N\eta$ branching fraction from Ref. [18]. However, it seems that the experimental data in Ref. [18] took only the analysis of Ref. [84] into account, while the much smaller result of Ref. [85] has not been used (although quoted in the list of experiments which should be included in the average). Indeed, in the previous 2014 PDG edition [81] the decay width was estimated between 5.5 and 25.5 MeV, which is in accordance with our result.
The decay width of $\Sigma(1750) \to \Sigma\eta$ is underestimated by the model by at least a factor of 3. However, the experimental data rely on a single experiment performed over 40 years ago [86]. We think that a reasonable comparison would need a new experimental analysis of this quantity.

In conclusion, the investigations of model (2) do not lead to an firmly evidence that the decays with an η meson in the final state should be enhanced.

The results show, however, two mismatches when considering the decay widths of $\Lambda(1800) \to N\bar{K}$ and $\Sigma(1750) \to \Sigma\pi$.
Nevertheless, while the decay width for $\Lambda(1800) \to N\bar{K}$ comes out too small when compared to Ref. [18], it is in accordance with a recent partial-wave analysis of $\bar{K}N$ scattering [87], where $\Gamma_{\Lambda(1800)\to N\bar{K}} = 33 \pm 20$ MeV is estimated.
The result for the decay width of $\Sigma(1750) \to \Sigma\pi$ is too large when compared to the data [18], which, however, is only the information of a (rather small) upper limit. In the most recent and complete analysis of the decays of Ref. [87] [which is cited in Ref. [18] but not included in the summary table] this decay is clearly seen. The resulting decay width of this analysis has large errors, but the central value of 58 MeV coincides with our result. This reference also estimates the decay width for $\Sigma(1750) \to \Lambda\pi$ to be of the order of 20 MeV, which is also compatible with our results.

(3) Conclusions

While most of the results from a model based only on flavor symmetry are in rough agreement with experimental data, the model cannot properly describe the decays of $\{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$ into an η meson and a ground-state baryon. As an explanation of this mismatch, in the following section we investigate a possible solution: the inclusion of the $U(1)_A$ anomaly.

6.2. Anomaly term in the mirror assignment

In this section we study the effects of the chiral $U(1)_A$ anomaly on the decays of baryons. We investigate a two- and a three-flavor eLSM where baryons are treated in the mirror

assignment, see Sec. 2.3 and Chap. 4. The $U(1)_A$ anomaly is parametrized by introducing a term which is $SU(N_f)_V \times SU(N_f)_A$ symmetric, but explicitly breaks the axial $U(1)_A$ symmetry.
Such a term may explain the large decay width of $N(1535) \to N\eta$ as follows: the anomaly is related to quantum fluctuations which couple $N(1535)$ to N via the emission of two gluons ($I = 0$, $J^{PC} = 0^{-+}$), which in turn couple to the quark-antiquark pairs $\bar{u}u$, $\bar{d}d$, and $\bar{s}s$ with the same intensity. Hence, the di-gluon couples almost exclusively to η and η' and thus explains why the anomaly increases the interaction. [Note, a recent study of the effect of the anomaly on various mesons is performed in Ref. [75].]

6.2.1. The case $N_f = 2$

We investigate the basic ideas of an anomaly term in the baryonic sector for the two-flavor model presented in Sec. 2.3.2. In this model, two nucleon fields, Ψ_1 and Ψ_2, are introduced in the mirror assignment (2.134). These fields mix to form the nucleon and its chiral partner, see Eq. (2.155). The striking point of the mirror assignment is the possibility to construct a chirally invariant mass term, see Eq. (2.142). Similarly to this term, also the combination

$$\bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2 = \bar{\Psi}_{2,L}\Psi_{1,R} + \bar{\Psi}_{2,R}\Psi_{1,L} - \bar{\Psi}_{1,R}\Psi_{2,L} - \bar{\Psi}_{1,L}\Psi_{2,R} \tag{6.28}$$

is chirally invariant,

$$\begin{aligned}\bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2 &\overset{\chi}{\to} \bar{\Psi}_{2R}U_L^\dagger U_L\Psi_{1L} + \bar{\Psi}_{2L}U_R^\dagger U_R\Psi_{1R} - \bar{\Psi}_{1R}U_L^\dagger U_L\Psi_{2L} - \bar{\Psi}_{1L}U_R^\dagger U_R\Psi_{2R} \\ &= \bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2 \ .\end{aligned} \tag{6.29}$$

However, using Eqs. (2.129) and (2.131), the term shows a negative parity transformation (charge conjugation is positive),

$$\begin{aligned}\bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2 &\overset{P}{\to} -\bar{\Psi}_2\gamma_0\gamma_0\Psi_1 - \bar{\Psi}_1\gamma_0(-\gamma_0\Psi_2) = -(\bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2) \ , \\ \bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2 &\overset{C}{\to} -\Psi_2^T CC\bar{\Psi}_1^T - \Psi_1^T C(-C\bar{\Psi}_2^T) \\ &= \Psi_2^T\bar{\Psi}_1^T - \Psi_1^T\bar{\Psi}_2^T \\ &= -(\bar{\Psi}_1\Psi_2 - \bar{\Psi}_2\Psi_1)^T \\ &= \bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2 \ ,\end{aligned} \tag{6.30}$$

which forbids to directly add it to the effective Lagrangian (2.140). However we can construct a positive-parity invariant by multiplying the expression with another chirally invariant negative-parity term:

$$\det\Phi - \det\Phi^\dagger \ , \tag{6.31}$$

where Φ includes the (pseudo)scalar mesons, see Eq. (2.137). The transformation behavior of Φ is formally identical to the three-flavor version given in Tab. 2.2. We find the following behavior under parity and charge-conjugation transformations of the term in Eq. (6.31):

$$\begin{aligned}\det\Phi - \det\Phi^\dagger &\xrightarrow{P} \det\Phi^\dagger - \det\Phi = -(\det\Phi - \det\Phi^\dagger)\ ,\\ \det\Phi - \det\Phi^\dagger &\xrightarrow{C} \det\Phi^T - \det(\Phi^T)^* = \det\Phi - \det\Phi^\dagger\ ,\end{aligned} \tag{6.32}$$

where $\det A^T = \det A$ $\forall$ square matrices A. We already came across such a determinant term. Namely, the anomaly term (2.79) in the mesonic sector of the eLSM has the same structure. In general, anomalous terms like $\det\Phi + \det\Phi^\dagger$, see Refs. [1, 88] and references therein, as well as $\left(\det\Phi - \det\Phi^\dagger\right)^2$, see Eq. (2.79) or Ref. [3] and references therein, are used in the purely mesonic sector. The determinant ensures the invariance under chiral $SU(2)_L \times SU(2)_R$ transformations, but explicitly breaks the $U(1)_A$ symmetry. Namely, the chiral transformation of the determinant in Eq. (6.31) reads:

$$\begin{aligned}\det\Phi \quad\rightarrow\quad \det\Phi' &= \det(U_L\Phi U_R^\dagger)\\ &= \det U_L \det\Phi \det U_R^\dagger\\ &= \det[e^{i(\theta_R^0-\theta_L^0)T_0}\Phi]\\ &= \det(e^{2i\theta_A^0 T_0}\Phi)\\ &= e^{2i\theta_A^0}\det\Phi\\ &\neq \det\Phi\ .\end{aligned} \tag{6.33}$$

Since $\det U_L = \det U_R = 1$ for $U_{L(R)} \in SU(2)_{L(R)}$ the determinant is invariant under $SU(2)_L \times SU(2)_R$ transformations. In the last line of Eq. (6.33) we used Eq. (1.144) to identify the remaining complex phase as $U(1)_A$. Since the generator is defined as $T_0 = \mathbb{1}_{N_f\times N_f}/\sqrt{2N_f}$, for $N_f = 2$ we find $\det T_0 = \det(\mathbb{1}_{2\times 2}/2) = 1$. This proves that the determinant explicitly breaks the $U(1)_A$ symmetry. Indeed, determinants are usually employed to model the axial anomaly. They generate an additional mass difference between η and π, as discussed in Ref. [57].

As a side remark, it is also possible to write down the following parity-even and chirally invariant anomalous term:

$$\begin{aligned}&(\det\Phi + \det\Phi^\dagger)(\bar\Psi_2\gamma^5\Psi_1 - \bar\Psi_1\gamma^5\Psi_2)\\ &\qquad = \frac{1}{2}(\sigma_N^2 - \eta_N^2 - \boldsymbol{a}_0^2 + \boldsymbol{\pi}^2)(\bar\Psi_2\gamma^5\Psi_1 - \bar\Psi_1\gamma^5\Psi_2)\ .\end{aligned} \tag{6.34}$$

After SSB ($\sigma_N \rightarrow \sigma_N + \phi_N$) an anomalous contribution to m_0 proportional to ϕ_N^2 appears.

We model the axial anomaly in the baryonic sector by considering the term

$$\mathcal{L}_A^{N_f=2} = \lambda_A^{N_f=2}(\det\Phi - \det\Phi^\dagger)(\bar\Psi_2\Psi_1 - \bar\Psi_1\Psi_2)\ , \tag{6.35}$$

It couples baryons to mesons, where the parameter λ_A has dimension [energy^{-1}]. The expansion of σ_N around this minimum of the potential density, Eq. (2.83), gives

$$\det\Phi - \det\Phi^\dagger = -i\left[(\sigma_N + \phi_N)\eta_N - \boldsymbol{a_0}\cdot\boldsymbol{\pi}\right]\ , \tag{6.36}$$

where we have used

$$\det\Phi = \det\begin{pmatrix} \frac{\sigma_N+a_{0,3}+i(\eta_N+\pi_3)}{2} & \frac{a_{0,1}+\pi_2-i(a_{0,2-\pi_1})}{2} \\ \frac{a_{0,1}-\pi_2+i(a_{0,2+\pi_1})}{2} & \frac{\sigma_N-a_{0,3}+i(\eta_N-\pi_3)}{2} \end{pmatrix}$$
$$= \frac{1}{4}[\ \sigma_N\sigma_N + 2i\sigma_N\eta_N - \eta_N\eta_N - a_{0,1}a_{0,1} - a_{0,2}a_{0,2} - a_{0,3}a_{0,3}$$
$$- 2i(a_{0,1}\pi_1 + a_{0,2}\pi_2 + a_{0,3}\pi_3) + \pi_1\pi_1 + \pi_2\pi_2 + \pi_3\pi_3]$$
$$= \frac{1}{4}(\sigma_N^2 - \eta_N^2 - \boldsymbol{a_0}^2 + \boldsymbol{\pi}^2 + 2i\sigma_N\eta_N - 2i\boldsymbol{a_0}\cdot\boldsymbol{\pi}) \ ,$$
$$\det\Phi^\dagger = \det\begin{pmatrix} \frac{\sigma_N+a_{0,3}-i(\eta_N+\pi_3)}{2} & \frac{a_{0,1}-\pi_2-i(a_{0,2+\pi_1})}{2} \\ \frac{a_{0,1}+\pi_2+i(a_{0,2-\pi_1})}{2} & \frac{\sigma_N-a_{0,3}-i(\eta_N-\pi_3)}{2} \end{pmatrix}$$
$$= \frac{1}{4}(\sigma_N^2 - \eta_N^2 - \boldsymbol{a_0}^2 + \boldsymbol{\pi}^2 - 2i\sigma_N\eta_N + 2i\boldsymbol{a_0}\cdot\boldsymbol{\pi}) \ . \tag{6.37}$$

Thus, the Lagrangian (6.35) includes a direct coupling of the meson η_N to the baryonic combination $\bar{\Psi}_2\Psi_1 - \bar{\Psi}_1\Psi_2$.
Since the eLSM contains (axial-)vector mesons as well, we have to consider the shifts and renormalizations as given in Eq. (2.160). Then, as a function of the physical fields (2.155) the anomaly term reads

$$\mathcal{L}_A^{N_f=2} = -i\lambda_A^{N_f=2}\frac{Z}{\cosh\delta}[(\sigma_N+\phi_N)\eta_N - \boldsymbol{a_0}\cdot\boldsymbol{\pi}]$$
$$\times(\bar{N}\gamma_5 N + \bar{N}_*\gamma_5 N_* - \sinh\delta\bar{N}N_* + \sinh\delta\bar{N}_*N) \ , \tag{6.38}$$

where $Z = Z_\pi = Z_{\eta_N}$ is the renormalization constant. In particular, we observe a term which contributes to the decay width of $N_* \to N\eta$:

$$i\lambda_A^{N_f=2}\tanh\delta\phi_N Z\bar{N}\eta_N N_* + \text{ h.c. } . \tag{6.39}$$

The orginial two-flavor eLSM with baryons in the mirror assignment [12] was not able to reproduce the decay width of $N_* \to N\eta$, see Sec. 2.3. It produces far too small values if we choose $N_* \equiv N(1535)$ as the chiral partner of the nucleon. Now, incorporating the anomaly, this decay width is enhanced:

$$\Gamma_{N_*\to N\eta} = \Gamma^{\text{without anomaly}}_{N_*\to N\eta}$$
$$+ \cos^2\theta_P\frac{p_f}{2\pi}\frac{m_N}{m_{N_*}}\frac{Z^2}{2}\lambda_A\varphi_N\left\{\left[\lambda_A\varphi_N\tanh^2\delta - \frac{1}{2}(\hat{g}_1-\hat{g}_2)\frac{\sinh\delta}{\cosh^2\delta}\right]\left(\frac{E_N}{m_N}+1\right)\right.$$
$$\left.-\frac{1}{2}w(c_1+c_2)\frac{\sinh\delta}{\cosh^2\delta}\left(\frac{m_{N_*}^2-m_N^2-m_\eta^2}{2m_N}+E_\eta\right)\right\} \ , \tag{6.40}$$

where $\Gamma^{\text{without anomaly}}_{N_*\to N\eta}$ is the decay width resulting from the model without anomaly term in the baryonic sector, see Eq. (C.12) in the Appendix. The constants $\hat{g}_1$, $\hat{g}_2$, c_1, c_2, and Z

were already determined in Eqs. (2.180) and (2.182) [see also Ref. [14]]. The pseudoscalar mixing angle (2.29) is $\theta_P = -44.6°$ [3].

The new constant λ_A influences only the η decay and therefore it can be chosen such that it correctly describes the decay width of $N_* \to N\eta$. In particular for $N_* \equiv N(1535)$, we have to choose

$$\lambda_A^{N_f=2} = 0.006 \text{ MeV}^{-1} \quad \text{or} \quad \lambda_A^{N_f=2} = -0.011 \text{ MeV}^{-1} \tag{6.41}$$

in order to reproduce the decay width of $N(1535) \to N\eta$.

6.2.2. The case $N_f = 3$

In the three-flavor case of the eLSM, four octet-baryon fields, N_1, N_2, M_1, and M_2, are present, see Chap. 4. Their transformation properties are summarized in Tab. 3.4. The (pseudo)scalar mesons are represented by Φ which is now a 3×3 matrix, Eq. (2.57). As in the two-flavor case, the determinant of Φ is invariant under $SU(3)_L \times SU(3)_R$ transformations, but explicitly breaks the $U(1)_A$ symmetry,

$$\det \Phi \to \det \left(e^{2i\theta_A^0 T_0} \Phi \right) = e^{3i\theta_A^0} \det \Phi \neq \det \Phi \ , \tag{6.42}$$

see Eq. (2.80). Thus, we again employ the determinant to model the axial anomaly. Again, we exploit the negative-parity term $\det \Phi - \det \Phi^\dagger$ and multiply it by an appropriate combination of baryon fields. A parity, charge-conjugation, and $SU(3)_L \times SU(3)_R$ invariant term which models the axial anomaly in the baryonic sector of the three-flavor eLSM is constructed as

$$\begin{aligned}\mathcal{L}_A^{N_f=3} =& \lambda_{A1}(\det \Phi - \det \Phi^\dagger) \operatorname{Tr}(\bar{M}_{1R} N_{1L} - \bar{N}_{1L} M_{1R} - \bar{M}_{2L} N_{2R} + \bar{N}_{2R} M_{2L}) \\ &+ \lambda_{A2}(\det \Phi - \det \Phi^\dagger) \operatorname{Tr}(\bar{M}_{1L} N_{1R} - \bar{N}_{1R} M_{1L} - \bar{M}_{2R} N_{2L} + \bar{N}_{2L} M_{2R}) \ ,\end{aligned} \tag{6.43}$$

where the parameters λ_{A1} and λ_{A2} have dimension [energy^{-2}]. We write this Lagrangian as a function of the parity eigenstates in Eqs. (4.58) to (4.61):

$$\begin{aligned}\mathcal{L}_A^{N_f=3} =& \frac{\lambda_{A1} + \lambda_{A2}}{2} (\det \Phi - \det \Phi^\dagger) \operatorname{Tr}(\bar{B}_{M_*} B_N - \bar{B}_N B_{M_*} - \bar{B}_{N_*} B_M + \bar{B}_M B_{N_*}) \\ &- \frac{\lambda_{A1} - \lambda_{A2}}{2} (\det \Phi - \det \Phi^\dagger) \operatorname{Tr}(\bar{B}_N \gamma_5 B_M + \bar{B}_M \gamma_5 B_N + \bar{B}_{N_*} \gamma_5 B_{M_*} + \bar{B}_{M_*} \gamma_5 B_{N_*}) \ .\end{aligned} \tag{6.44}$$

The second line of this Lagrangian describes interactions of the (pseudo)scalar mesons with two baryons of equal parity. However, decays involving the η meson are kinematically forbidden.

The first line of Eq. (6.44), parametrizes contributions from the anomaly to terms which couple mesons to B_N and B_{N_*} (as well as B_M and B_{N_*}). This allows for the possibility of enhancing decays type

$$B_{M_*} \to N\eta \ . \tag{6.45}$$

Assigning $B_{M_*} \equiv \{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$ as the chiral partners of the ground-state baryons, the anomalous terms give rise to an enhanced decay of $N(1535)$ into $N\eta$. In contrast, the anomaly terms do not contain terms where mesons couple to B_N and B_{N_*} or B_M and B_{M_*}, i.e. upon assuming $B_{N_*} \equiv \{N(1650), \Lambda(1800), \Sigma(1750), \Xi(?)\}$, the anomaly does not enhance the decay of $N(1650)$ into $N\eta$.

In Sec. 5.5.3, we have shown that $N(1535)$ always represents the chiral partner of the nucleon. However, for different choices of the values of the coupling constants in the Lagrangian (5.10), $N(1535)$ can be (predominantly) a state of the multiplet B_{M_*} or of the multiplet B_{N_*}. In both cases the model describes masses and decay widths (apart from the decay $N(1535) \to N\eta$) equally well.
However, if we assign $B_{N_*} \equiv \{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$, the anomaly would no enhance the decay width of $N(1535) \to N\eta$. For this reason restrict the following discussion to the assignment $B_{M_*} \equiv \{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$ as the chiral partner of the ground-state baryon field B_N (and B_{N_*} is the chiral partner of B_M). (Interestingly, this might also restrict the number of meaningful results in Sec. 5.5.2: a proper description of the decay width $N(1535) \to N\eta$ via the anomaly might force us to discard the scenario in which B_{M_*} predominantely describes $\{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$.)

Furthermore, in order to keep the discussion simple we neglect the mixing between B_N, B_M, B_{N_*}, and B_{M_*} in remainder of this chapter. (A detailed study of mixing should be performed in future work.) We assign the octets as follows:

$$B_N \equiv \{N(939), \Lambda(1116), \Sigma(1193), \Xi(1318)\} , \tag{6.46}$$
$$B_M \equiv \{N(1440), \Lambda(1600), \Sigma(1660), \Xi(1690)\} , \tag{6.47}$$
$$B_{M*} \equiv \{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\} , \tag{6.48}$$
$$B_{N*} \equiv \{N(1650), \Lambda(1800), \Sigma(1750), \Xi(?)\} . \tag{6.49}$$

Then, the Lagrangian (6.44) corresponds to the two-flavor version of Eq. (6.35) upon setting $\Psi_1 = B_N$, $\Psi_2 = B_{M_*}$ and identifying:

$$\lambda_A^{N_f=3} = \frac{\lambda_{A1} + \lambda_{A2}}{2} . \tag{6.50}$$

We now discuss the consequences of the chiral anomaly in detail. Using the matrix form (2.57) of the (pseudo)scalar meson field Φ, we find

$$\begin{aligned} \det\Phi - \det\Phi^\dagger = & \frac{i}{2\sqrt{2}}\Big[Z_{\eta_S}\phi_N^2\eta_S + 2Z_{\eta_N}\phi_N\phi_S\eta_N\Big] \\ & +\frac{i}{2\sqrt{2}}\Big[2Z_{\eta_S}\phi_N\sigma_N\eta_S + 2Z_{\eta_N}(\phi_N\sigma_S\eta_N + \phi_S\sigma_N\eta_N) \\ & - \sqrt{2}Z_K Z_{K_0^*}\phi_N(K_0^{*+}K^- + K^+K_0^{*-} + K_0^{*0}\bar{K}^0 + K^0\bar{K}_0^{*0}) \\ & - 2Z_\pi\phi_S(a_0^0\pi^0 + a_0^+\pi^- + \pi^+a_0^-)\Big] + \end{aligned}$$

$$
\begin{aligned}
+\frac{i}{2\sqrt{2}}\Big[\; & Z_{\eta_S}\sigma_N\sigma_N\eta_S + 2Z_{\eta_N}\sigma_N\sigma_S\eta_N \\
& - \sqrt{2}\sigma_N Z_K Z_{K_0^*}(K_0^{*+}K^- + K^+K_0^{*-} + K_0^{*0}\bar{K}^0 + K^0\bar{K}_0^{*0}) \\
& - 2\sigma_S Z_\pi (a_0^0\pi^0 + a_0^+\pi^- + \pi^+a_0^-) \\
& + Z_\pi^2 Z_{\eta_S}(\pi^0\pi^0\eta_S + 2\pi^+\pi^-\eta_S) \\
& + \sqrt{2}Z_\pi\pi^0(Z_{K_0^*}^2 K_0^{*+}K_0^{*-} - Z_K^2K^+K^- - Z_{K_0^*}^2K_0^{*0}\bar{K}_0^* + Z_K^2K^0\bar{K}^0) \\
& + 2Z_\pi\pi^+(Z_{K_0^*}^2K_0^{*0}K_0^{*-} - Z_K^2K^0K^-) \\
& + 2Z_\pi\pi^-(Z_{K_0^*}^2K_0^{*+}\bar{K}_0^{*0} - Z_K^2K^+\bar{K}^0) \\
& - Z_{\eta_N}^2 Z_{\eta_S}\eta_N\eta_N\eta_S \\
& - \sqrt{2}Z_{\eta_N}\eta_N(Z_{K_0^*}^2K_0^{*+}K_0^{*-} - Z_K^2K^+K^- + Z_{K_0^*}^2K_0^{*0}\bar{K}_0^{*0} - Z_K^2K^0\bar{K}^0) \\
& - Z_{\eta_S}\eta_S(a_0^0a_0^0 + 2a_0^+a_0^-) \\
& + 2Z_KZ_{K_0^*}a_0^+(K_0^{*0}K^- + K^0K_0^{*-}) \\
& + \sqrt{2}Z_KZ_{K_0^*}a_0^0(K_0^{*+}K^- + K^+K_0^{*-} - K_0^{*0}\bar{K}^0 - K^0\bar{K}_0^{*0})\Big] \,,
\end{aligned}
\tag{6.51}
$$

where the effects of SSB have been taken into account. The renormalization factors are given in Eqs. (2.97) to (2.100). Considering only terms which are linear in the fields (i.e., omitting four- and five-point interactions),

$$
\det\Phi - \det\Phi^\dagger = \frac{i}{2\sqrt{2}}\Big[Z_{\eta_S}\phi_N^2\eta_S + 2Z_{\eta_N}\phi_N\phi_S\eta_N\Big] + \dots \,, \tag{6.52}
$$

we see that $\det\Phi - \det\Phi^\dagger$ is proportional to the fields η_S and η_N. Using the definition (4.80) of the vectors combining the four types of particles or resonances which are included in the baryon octet, we write the anomaly Lagrangian in a very compact form:

$$
\begin{aligned}
\mathcal{L}_A^{N_f=3} = & - i\bar{\boldsymbol{N}}\gamma_5\hat{\eta}_N^A\eta_N\boldsymbol{N} - i\bar{\boldsymbol{N}}\gamma_5\hat{\eta}_S^A\eta_S\boldsymbol{N} - i\bar{\boldsymbol{\Lambda}}\gamma_5\hat{\eta}_N^A\eta_N\boldsymbol{\Lambda} - i\bar{\boldsymbol{\Lambda}}\gamma_5\hat{\eta}_S^A\eta_S\boldsymbol{\Lambda} \\
& - i\bar{\boldsymbol{\Sigma}}\gamma_5\hat{\eta}_N^A\eta_N\boldsymbol{\Sigma} - i\bar{\boldsymbol{\Sigma}}\gamma_5\hat{\eta}_S^A\eta_S\boldsymbol{\Sigma} - i\bar{\boldsymbol{\Xi}}\gamma_5\hat{\eta}_N^A\eta_N\boldsymbol{\Xi} - i\bar{\boldsymbol{\Xi}}\gamma_5\hat{\eta}_S^A\eta_S\boldsymbol{\Xi} \,,
\end{aligned}
\tag{6.53}
$$

where the 4×4 matrices $\hat{\eta}_N^A$ and $\hat{\eta}_S^A$ contain the coupling constants:

$$
\hat{\eta}_N^A = \frac{Z_{\eta_N}\phi_N\phi_S}{2\sqrt{2}}\begin{pmatrix} 0 & 0 & \lambda_{A1}-\lambda_{A2} & \lambda_{A1}+\lambda_{A2} \\ 0 & 0 & -(\lambda_{A1}+\lambda_{A2}) & -(\lambda_{A1}-\lambda_{A2}) \\ \lambda_{A1}-\lambda_{A2} & -(\lambda_{A1}+\lambda_{A2}) & 0 & 0 \\ \lambda_{A1}+\lambda_{A2} & -(\lambda_{A1}-\lambda_{A2}) & 0 & 0 \end{pmatrix} , \tag{6.54}
$$

$$\hat{\eta}_S^A = \frac{Z_{\eta_S}\phi_N^2}{4\sqrt{2}} \begin{pmatrix} 0 & 0 & \lambda_{A1}-\lambda_{A2} & \lambda_{A1}+\lambda_{A2} \\ 0 & 0 & -(\lambda_{A1}+\lambda_{A2}) & -(\lambda_{A1}-\lambda_{A2}) \\ \lambda_{A1}-\lambda_{A2} & -(\lambda_{A1}+\lambda_{A2}) & 0 & 0 \\ \lambda_{A1}+\lambda_{A2} & -(\lambda_{A1}-\lambda_{A2}) & 0 & 0 \end{pmatrix} . \tag{6.55}$$

To a good approximation, we can take [the $U(3)_V$-limit]

$$\phi_N = \sqrt{2}\phi_S \quad \text{and} \quad Z_\pi = Z_{\eta_N} = Z_{\eta_S} . \tag{6.56}$$

Then Eq. (6.52) becomes

$$\det\Phi - \det\Phi^\dagger = \frac{iZ_\pi}{2}\sqrt{\frac{3}{2}}\phi_N^2\eta_0 + \ldots \tag{6.57}$$

where η_0 is the isosinglet-pseudoscalar combination

$$\begin{aligned} \eta_0 &= (\sqrt{2}\eta_N + \eta_S)/\sqrt{3} \\ &= \sqrt{2}\,\mathrm{Tr}\,P = \frac{\eta}{\sqrt{3}}\left(\sqrt{2}\cos\theta_P + \sin\theta_P\right) + \frac{\eta'}{\sqrt{3}}\left(\cos\theta_P - \sqrt{2}\sin\theta_P\right) . \end{aligned} \tag{6.58}$$

In the last line we have identified the flavor-singlet field η_0 with $\sqrt{2}\,\mathrm{Tr}\,P$, where P is the matrix containing the pseudoscalar mesons. Finally, we have given η_0 as a function of the physical fields. Using $\theta_P = -44.6°$ [3], one obtains $\eta_0 = 0.18\eta + 0.98\eta'$.

Hence, the anomaly term of Eq. (6.44) causes an enhanced interaction of the baryon fields with the flavor-singlet field $\eta_0 = \sqrt{2}\,\mathrm{Tr}\,P$. We investigate this interaction with baryons in more detail within a simplified scenario. Namely, we consider only one particular term which has a nonzero contribution to decays. Restricting to the fields B_N and B_{M_*} and using Eq. (6.58), the Lagrangian (6.44) becomes

$$\mathcal{L}_A^{N_f=3} = i\tilde{\lambda}_A\,\mathrm{Tr}(\bar{B}_{M_*}B_N - \bar{B}_N B_{M_*})\,\mathrm{Tr}\,P + \ldots , \tag{6.59}$$

where dots refer to interactions with more than one mesonic field and to flavor-breaking terms. The new coupling constant $\tilde{\lambda}_A$ is defined as

$$\tilde{\lambda}_A = \frac{\lambda_{A1}+\lambda_{A2}}{2}\frac{\sqrt{3}Z_\pi}{2}\phi_N^2 . \tag{6.60}$$

Upon the identification $B_N \equiv O, B_{M_*} \equiv O_*$, [Sec. 6.1, first model], the anomalous term (6.59) reads

$$i\tilde{\lambda}_A\,\mathrm{Tr}(\bar{O}O_* - \bar{O}_*O)\,\mathrm{Tr}\,P , \tag{6.61}$$

which is identical to the (large-N_c suppressed) term in Eq. (6.16).

Hence, an improved flavor model [compare Eq. (6.13)] which takes the axial anomaly into account is described by

$$\begin{aligned} \mathcal{L}_V^{\text{improved}} &= \mathcal{L}_V + \mathcal{L}_A^{N_f=3} \\ &= i\lambda_V \operatorname{Tr}(\bar{O}PO_* - \bar{O}_*PO) + i\tilde{\lambda}_A \operatorname{Tr}(\bar{O}O_* - \bar{O}_*O) \operatorname{Tr} P \ . \end{aligned} \tag{6.62}$$

The constant $\lambda_V = \lambda_V^{N(1535)}$ is determined in Eq. (6.26). We fit the new parameter $\tilde{\lambda}_A$ to the decay width $\Gamma^{\text{exp}}_{N(1535)\to N\eta} = (63 \pm 18)$ MeV [18]. Note, the expressions of the decay widths resulting from the improved flavor model (6.62) are formally identical to Eq. (6.22), but the coupling constants get some additional contribution from the anomalous term, as summarized in Tab. 6.1. With $\lambda_V = \lambda_V^{N(1535)}$ given in Eq. (6.26), $\phi_N = 164.6$ MeV [Tab. 2.3] and Z_π as defined in Eq. (2.122), the fit to the decay width of $N(1535) \to N\eta$ fixes the parameter to

$$\tilde{\lambda}_A = 11 \pm 0.6 \ . \tag{6.63}$$

Its error is computed via Gaussian error propagation and upon neglecting the errors in the masses. Note, there is a second solution given by $\tilde{\lambda}_A = -19 \pm 0.6$, and $\lambda_A^{N_f=3} = (\lambda_{A1} + \lambda_{A2})/2 = (-451 \pm 13)\ \text{GeV}^{-2}$. However, using these values, the model predicts $\Gamma_{\Lambda(1670)\to\Lambda\eta} = (51 \ \pm 4)$ MeV, which is unacceptably large when compared to the experimentally observed values ranging between 2.5 and 12.5 MeV [18].

Using the parameter value (6.63), the improved flavor model (6.62) reproduces the decay width of $\Lambda(1670) \to \Lambda\eta$ properly:

$$\Gamma_{\Lambda(1670)\to\Lambda\eta} = (8.7 \ \pm 0.4)\ \text{MeV} \ , \tag{6.64}$$

which should be compared to the present experimental value $\Gamma^{\text{exp}}_{\Lambda(1670)\to\Lambda\eta} = (7.5 \pm 5)$ MeV [18]. The error is calculated as the difference between the maximum and minimum decay width resulting from the calculation using the maximum and minimum values of $\lambda_V^{N(1535)}$ and $\tilde{\lambda}_A$. Besides the decay width of $N(1535) \to N\eta$, in the model with flavor symmetry only Eq. (6.13), this decay could not been correctly described, compare Tab. 6.2. Now, in the improved flavor model (6.62), the anomaly causes an increase which is in good agreement with the experiment.

Unfortunately, further tests of the model are not possible by using decays, since the other decays into the η meson are kinematically forbidden.

Nevertheless, the inclusion of an anomalous term in the baryonic sector has another interesting consequence: it enhances the coupling of the nucleon N and its chiral partner N_* to the η' meson. The expanded form of the Lagrangian (6.62) reads

$$\begin{aligned} \mathcal{L}_V^{\text{improved}} &= i g_{\eta N N_*} \eta \left(\bar{N}_* N - \bar{N} N_*\right) + i g_{\eta' N N_*} \eta' \left(\bar{N}_* N - \bar{N} N_*\right) \\ &\quad + i g_{\pi N N_*} \boldsymbol{\pi} \cdot \left(\bar{N}_* \boldsymbol{\tau} N - \bar{N} \boldsymbol{\tau} N_*\right) + \dots \ , \end{aligned} \tag{6.65}$$

where the coupling constants are given by

$$g_{\eta NN_*} = \frac{i\lambda_V}{2}\cos\theta_P + \frac{i\tilde{\lambda}_A}{\sqrt{6}}(\sqrt{2}\cos\theta_P + \sin\theta_P) \simeq 1.9 \;, \tag{6.66}$$

$$g_{\eta' NN_*} = \frac{i\lambda_V}{2}\sin\theta_P + \frac{i\tilde{\lambda}_A}{\sqrt{6}}(-\sqrt{2}\sin\theta_P + \cos\theta_P) \simeq 7.2 \;, \tag{6.67}$$

$$g_{\pi NN_*} = -\frac{\lambda_V}{2} \simeq -0.7 \;. \tag{6.68}$$

In comparison, if the effects of the axial anomaly are neglected ($\tilde{\lambda}_A = 0$), the first two constants are $g_{\eta NN_*} \simeq |g_{\eta' NN_*}| \simeq 0.5$. This shows that the axial anomaly leads to an increased coupling of N and N_* to η'. This qualitatively confirms the result of the investigations from scattering processes of the type $pn \to pn\eta'$ in Ref. [19], where $g_{\eta' NN_*} \simeq 3.7$ was found. The study of scattering processes in this framework, see Ref. [89], is a possible future extension of this work.

6.3. Interactions of the pseudoscalar glueball with baryons

Gluons interact strongly with quarks and with each other. It is then natural to expect the existence of color singlets formed by gluons alone. These objects are called glueball [90, 47]. Indeed QCD calculations on the lattice [quenched [91] as well as unquenched (i.e., with quark fluctuations) [92, 93, 94]] predict a rich glueball spectrum. The pseudoscalar $J^{PC} = 0^{-+}$ glueball has a lattice-predicted mass of 2.6 GeV [91]. Experimentally, the BESIII experiment [20] can search for these objects and in particular the future PANDA experiment [21] will be able to investigate glueball states.
The mathematical structure of the pseudoscalar glueball coupling resembles very closely that of the axial anomaly. For this reason, we attach this section about the pseudoscalar glueball and its interactions with baryons into the chapter on the axial anomaly.

The pseudoscalar glueball has not been discovered yet. Thus, at present the results of these studies cannot be compared to data, but they can be of use in the future, when more experimental information will be available.

In the framework of the eLSM the coupling of the pseudoscalar glueball to (pseudo)scalar mesons was already studied in Refs. [2, 4, 22, 23]. It is described by the following Lagrangian [22]

$$\mathcal{L}_{\tilde{G}\Phi} = i g_{\tilde{G}\Phi} \tilde{G}(\det\Phi - \det\Phi^\dagger) \;, \tag{6.69}$$

where $\tilde{G}$ is the pseudoscalar glueball field. It is invariant under chiral $U(3)_L \times U(3)_R$ transformations,

$$\tilde{G} \xrightarrow{\chi} \tilde{G} \;,$$

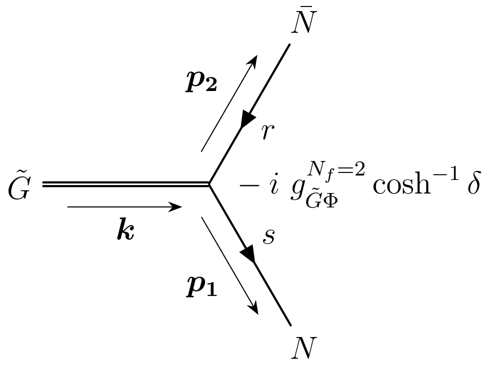


Figure 6.2. Decay of the pseudoscalar glueball into $\bar{N}N$.

and transformations under parity and charge conjugation as follows:

$$\tilde{G}\xrightarrow{P}-\tilde{G}\;,\qquad \tilde{G}\xrightarrow{C}\tilde{G}\;.$$

The parameter $g_{\tilde{G}\Phi}$ is the coupling constant which for $N_f = 2$ has dimension [energy] and for $N_f = 3$ is dimensionless. The term (6.69) is directly linked to the chiral anomaly, see Ref. [47]. Namely, we use the mathematical structure which we have developed in Sec. 6.2 in order to construct the coupling of the pseudoscalar glueball to baryons.

6.3.1. Two-flavor case

Since the pseudoscalar glueball transforms under parity and charge conjugation just as the determinant term in Eq. (6.35), we can simply replace this determinant term by the pseudoscalar glueball field $\tilde{G}$. The following chirally as well as parity and charge-conjugation invariant Lagrangian results:

$$\begin{aligned}\mathcal{L}_{\tilde{G}\Phi}^{N_f=2} &= i g_{\tilde{G}\Phi}^{N_f=2}\tilde{G}(\bar{\Psi}_2\Psi_1-\bar{\Psi}_1\Psi_2)\\ &= -i\,\frac{g_{\tilde{G}\Phi}^{N_f=2}}{\cosh\delta}\,\tilde{G}(\bar{N}\gamma_5 N+\bar{N}_*\gamma_5 N_*-\sinh\delta\bar{N}N_*+\sinh\delta\bar{N}_*N)\;.\end{aligned} \tag{6.70}$$

It describes the interaction of the pseudoscalar glueball with the nucleon N and its chiral partner N_*. In the limit of zero mixing, i.e., $\delta\to\infty$ ($\Psi_1 = N$ and $\Psi_2 = N_*$), the pseudoscalar glueball couples strongly to NN_*. For $\delta<\infty$, also the decay $\tilde{G}\to\bar{N}N$ is present. It is described by the first term in Eq. (6.70) and corresponds to the following Feynman diagram in Fig. 6.2. Using Eq. (1.35), we obtain the following expression for the decay width (in the rest frame of the glueball):

$$\Gamma_{\tilde{G}\to\bar{N}N}=\frac{p_{f,\tilde{G}NN}}{8\pi m_{\tilde{G}}^2}\overline{|\mathcal{M}_{\tilde{G}\to\bar{N}N}|^2}\;, \tag{6.71}$$

where

$$p_{f,\tilde{G}NN}=\sqrt{\frac{m_{\tilde{G}}^2}{4}-m_N^2}\;. \tag{6.72}$$

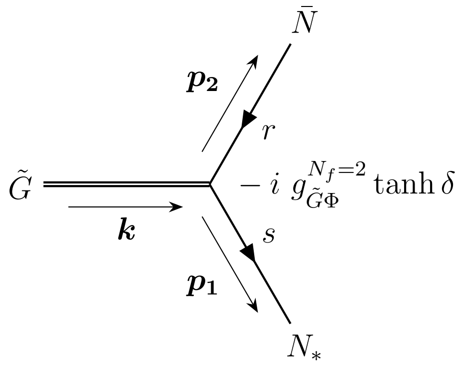


Figure 6.3. Decay of the pseudoscalar glueball into N_*N.

Upon using the Feynman rules at the end of Sec. 1.2.2 and the corresponding Feynman diagram we calculate the averaged squared amplitude as follows

$$\begin{aligned}\overline{|\mathcal{M}_{\tilde{G}\to\bar{N}N}|^2} &= \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2}{\cosh^2\delta}\sum_{r,s}\bar{u}(\boldsymbol{p_1},s)\gamma_5 v(\boldsymbol{p_2},r)\,[\bar{u}(\boldsymbol{p_1},s)\gamma_5 v(\boldsymbol{p_2},r)]^\dagger \\ &= \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2}{\cosh^2\delta}\,\mathrm{Tr}\,[\gamma_5(-\gamma_\mu p_2^\mu + m_N)\gamma_5(\gamma_\nu p_1^\nu + m_N)] \\ &= \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2}{\cosh^2\delta}(4p_1\cdot p_2 + 4m_n^2) = \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2}{\cosh^2\delta}2(m_{\tilde{G}}^2 - 2m_N^2 + 2m_N^2) \\ &= \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2}{\cosh^2\delta}2m_{\tilde{G}}^2\;,\end{aligned} \tag{6.73}$$

where we have used that

$$\sum_s u^\alpha(\boldsymbol{p_1},s)\bar{u}^\beta(\boldsymbol{p_1},s) = (\gamma_\mu p_1^\mu + m_N)^{\alpha\beta}\;,$$
$$\sum_r v^\alpha(\boldsymbol{p_2},r)\bar{v}^\beta(\boldsymbol{p_2},r) = (-\gamma_\mu p_2^\mu + m_N)^{\alpha\beta}\;.$$

In the third line, we applied $2p_1\cdot p_2 = m_{\tilde{G}}^2 - 2m_N^2$. Thus, the final result for the decay widths reads

$$\Gamma_{\tilde{G}\to\bar{N}N} = \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2}{4\pi\cosh^2\delta}\,p_{f,\tilde{G}NN}\;. \tag{6.74}$$

The calculation of the $\tilde{G}\to\bar{N}N_*$ decay width is similar. The decay is described by the last term in Eq. (6.70) and corresponds to the Feynman diagram in Fig. 6.3. Using Eq. (1.35) and performing similar steps as in the calculation of $\Gamma_{\tilde{G}\to\bar{N}N}$, we obtain the following expression for the decay width (in the rest frame of the glueball):

$$\Gamma_{\tilde{G}\to\bar{N}N_*} = \frac{p_{f,\tilde{G}N_*N}}{8\pi m_{\tilde{G}}^2}\overline{|\mathcal{M}_{\tilde{G}\to\bar{N}N_*}|^2}\;, \tag{6.75}$$

where

$$p_{f,\tilde{G}N_*N} = \frac{1}{2m_{\tilde{G}}}\sqrt{(m_{\tilde{G}}^2 - m_{N_*}^2 - m_N^2)^2 - 4m_{N_*}^2 m_N^2} \ . \tag{6.76}$$

Using the Feynman rules at the end of Sec. 1.2.2 and the corresponding Feynman diagram we find

$$\overline{|\mathcal{M}_{\tilde{G}\to\bar{N}N}|^2} = 2(g_{\tilde{G}\Phi}^{N_f=2})^2 \tanh^2\delta\left[m_{\tilde{G}}^2 - (m_N + m_{N_*})^2\right] \ . \tag{6.77}$$

Thus, the decay width reads

$$\begin{aligned}\Gamma_{\tilde{G}\to\bar{N}N_*+\text{ h.c.}} &= 2\Gamma_{\tilde{G}\to\bar{N}N_*}\\ &= \frac{(g_{\tilde{G}\Phi}^{N_f=2})^2\tanh^2\delta}{2\pi m_{\tilde{G}}^2}[m_{\tilde{G}}^2 - (m_N + m_{N_*})^2]\ p_{f,\tilde{G}N_*N} \ .\end{aligned} \tag{6.78}$$

With this result and Eq. (6.74), we compute the branching ratio of the two pseudoscalar decay processes:

$$\begin{aligned}\frac{\Gamma_{\tilde{G}\to\bar{N}N}}{\Gamma_{\tilde{G}\to\bar{N}_*N+\text{ h.c.}}} &= \frac{m_{\tilde{G}}^2 p_{f,\tilde{G}NN}}{2\sinh^2\delta\left[M_{\tilde{G}}^2 - (m_N + m_{N_*})^2\right] p_{f,\tilde{G}N_*N}}\\ &\simeq 0.85\ \frac{p_{f,\tilde{G}NN}}{p_{f,\tilde{G}N_*N}}\\ &\simeq 1.96 \ .\end{aligned} \tag{6.79}$$

The numerical values are obtained by using Eq. (2.159) with $m_0 = (460 \pm 136)$ MeV [12] as well as taking the pseudoscalar glueball mass $M_{\tilde{G}} = 2.6$ GeV [91]. Although the phase space of $\tilde{G} \to \bar{N}N$ is much larger (neglecting phase space the ratio would be 0.85), the decay width is only slightly larger than for $\tilde{G} \to \bar{N}_*N$. Thus, we predict a sizably strong decay $\tilde{G} \to \bar{N}_*N$ (although the numerical value depends on the model parameter m_0). For this reason, it seems promising to search for the pseudoscalar glueball in the process

$$p + \bar{p} \to \tilde{G} \to p + \bar{p}(1535) + \text{ h.c.} \ . \tag{6.80}$$

at the future PANDA experiment [21].

6.3.2. Three-flavor case

In the $N_f = 3$ case, we follow the same procedure as for the $N_f = 2$ case. Replacing the determinant term in Eq. (6.44) by the pseudoscalar glueball field $\tilde{G}$, we obtain the following chirally as well as parity and charge-conjugation invariant Lagrangian:

$$\begin{aligned}\mathcal{L}_{\tilde{G}B}^{N_f=3} &= i\,\frac{\tilde{g}_1 + \tilde{g}_2}{2}\,\tilde{G}\ \text{Tr}(\bar{B}_{M_*}B_N - \bar{B}_N B_{M_*} - \bar{B}_{N_*}B_M + \bar{B}_M B_{N_*})\\ &- i\,\frac{\tilde{g}_1 - \tilde{g}_2}{2}\,\tilde{G}\ \text{Tr}(\bar{B}_N\gamma_5 B_M + \bar{B}_M\gamma_5 B_N + \bar{B}_{N_*}\gamma_5 B_{M_*} + \bar{B}_{M_*}\gamma_5 B_{N_*}) \ .\end{aligned} \tag{6.81}$$

The pseudoscalar glueball $\tilde{G}$ couples to the baryonic field combinations $\bar{B}_{M_*}B_N$, $\bar{B}_M B_{N_*}$, $\bar{B}_N\gamma_5 B_M$, and $\bar{B}_{M_*}\gamma_5 B_{N_*}$. If we take the pseudoscalar glueball mass to be 2.6 GeV [91], for kinematic reasons only the following two decays should be sizable:

$$\tilde{G} \to N(1440)N \ , \qquad \tilde{G} \to N(1535)N \ . \tag{6.82}$$

While the latter is in agreement with the $N_f = 2$ study (as it should), the first decay emerges when mixing among the baryon fields is considered.

Chapter 7.

Conclusions and outlook

At a fundamental level the strong interaction is described by the quantum field theory called quantum chromodynamics (QCD). This theory is non-perturbative and not analytically solvable in the low-energy regime. A possible way to study QCD at such energies makes use of low-energy effective Lagrangians, which obey (some of) the same symmetries as the underlying QCD. In such an approach, the degrees of freedom are no longer quarks and gluons as in QCD, but the physically observable particles: mesons and baryons (i.e., hadrons).

We investigated the so-called extended linear sigma model (eLSM), which is a chiral effective approach to describe hadron masses and interactions in the low-energy regime. In order to reproduce the known features of the underlying theory, quantum chromodynamics, it is based on a linear realization of the chiral $U(N_f)_L \times U(N_f)_R$ symmetry (as well as dilatation invariance), but also includes the anomalous, explicit, as well as spontaneous breaking of chiral symmetry. The eLSM was first constructed in Ref. [1] including (pseudo)scalar and (axial-)vector mesons, as well as glueballs [2] for the case $N_f = 2$. Later on, it has been extended to the cases $N_f = 3$ and $N_f = 4$, see Refs. [3, 4, 5, 6, 7, 8] and Ref. [9] respectively. For $N_f = 2$, in the baryonic sector of the eLSM, the nucleon and its chiral partner have been investigated in Refs. [12, 13, 14, 15] featuring a mirror assigment [10, 11] which allows for a chirally invariant mass term.

As the enlargement from $N_f = 2$ to $N_f = 3$ has already been performed in the mesonic sector, in this work we have generalized the baryonic sector of the eLSM to $N_f = 3$. This extension was not as straightforward as in the mesonic sector. For $N_f = 2$, mesons are represented by a 2×2 matrix, which for $N_f = 3$, has to be extended to a 3×3 matrix. In contrast, in the baryonic sector for $N_f = 2$, the nucleons are described by a spinor isodoublet $\Psi_N = (p, n)^T$ (where p and n represent the proton and the neutron, respectively), which for $N_f = 3$ has to be extended to a 3×3 matrix containing the $J^P = \frac{1}{2}^+$ baryon octet, see Eq. (3.55):

$$\begin{pmatrix} \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\ \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix} .$$

Also the inclusion of the chiral-partner multiplet with $J^P = \frac{1}{2}^-$ generalizes not as straightforwardly from the $N_f = 2$ discussion to the case $N_f = 3$.

To address these issues, we first constructed baryon fields in a chiral quark-diquark picture, i.e., we combined a quark with a diquark, where the latter is a (pseudo)scalar state with antisymmetric flavor and color structure. For $N_f = 3$, the diquark transforms just as an antiquark. Thus, the construction of baryons resembles the construction of mesons as quark-antiquark states. In this way, the 3×3 matrix structure arises naturally. In combination with the requirement of chirally invariant mass terms in the Lagrangian (mirror assignment), we are finally lead to four baryon multiplets. Two of them are called N_1 and N_2. They behave under chiral transformations as follows:

$$N_{1R} \to U_R N_{1R} U_R^\dagger \, , \quad N_{1L} \to U_L N_{1L} U_R^\dagger \, , \qquad N_{2R} \to U_R N_{2R} U_L^\dagger \, , \quad N_{2L} \to U_L N_{2L} U_L^\dagger \, .$$

The remaining two fields are called M_1 and M_2. They show chiral transformations which from the left-hand side are mirrored when compared to the previous fields:

$$M_{1R} \to U_L M_{1R} U_R^\dagger \, , \quad M_{1L} \to U_R M_{1L} U_R^\dagger, \qquad M_{2R} \to U_L M_{2R} U_L^\dagger \, , \quad M_{2L} \to U_R M_{2L} U_L^\dagger \, .$$

In turn, this allows to construct a baryonic eLSM Lagrangian describing three-flavor baryon octets and their interactions with mesons. Besides the invariance under parity and charge-conjugation transformations, the baryonic sector (initially) has a chiral $U(3)_L \times U(3)_R$ symmetry. (Later on, we included a further term incorporating the $U(1)_A$ anomaly.)
The Lagrangian contains twelve parameters, ten of which describe the interactions with (pseudo)scalar and (axial-)vector mesons. The remaining two parameters, $m_{0,1}$ and $m_{0,2}$, resemble mass parameters and formally break the dilatation invariance. However, we assume that they arise from (dilatation-invariant) interactions with a glueball and/or a four-quark state, after spontaneous symmetry breaking induces a non-vanishing vacuum expectation value of the glueball and/or the four-quark state. The resulting values of the two mass parameters are a measure for contributions of the baryon masses arising from other sources than the chiral condensate, e.g. a gluon and/or a four-quark condensate.

Since the full $N_f = 3$ analysis of the model is difficult, we investigated its reduced version with $N_f = 2$ flavors. In this case, the model contains four nucleonic doublets, which mix to form physical fields assigned to the nucleon $N(939)$ and the resonances $N(1440)$, $N(1535)$, and $N(1650)$. We determined the parameters of the model by fitting the masses (5% uncertainty assumed), decay widths, and the axial coupling constants of the aforementioned states to experimentally known data [81], as well as axial coupling constants resulting from lattice-QCD calculations [68]. Numerically minimizing the corresponding χ^2 functional, we have found three acceptable sets of parameter values corresponding to almost equally deep minima. In two of these cases, the values of $m_{0,1}$ and $m_{0,2}$ are small and thus indicate that all masses are mainly generated by spontaneous chiral symmetry breaking. In contrast, in the third minimum the values are almost as large as the vacuum mass of the nucleon, which shows that the masses contain contributions from other condensates.

For all three solutions, the model describes most of the masses, decay widths, and axial coupling constants in good agreement with data, see Tab. 5.2. However, in each minimum, the decay width of $N(1535) \to N\eta$ comes out too small.

Furthermore, the model allows to identify the pairs

$$N(939) \text{ and } N(1535) \quad \text{as well as} \quad N(1440) \text{ and } N(1635) \tag{7.1}$$

as chiral partners. This is the case for all three minima, as shown by a study of the limiting process in which chiral symmetry is restored.

Considering the problem with the $N(1535) \to N\eta$ decay width, it is interesting to note that the experimental value is larger than expected at a first sight. Namely, it is almost as large as the one of $N(1535) \to N\pi$, which appears to be surprising, because flavor symmetry predicts $\Gamma_{N(1535)\to N\eta} \simeq 0.17\ \Gamma_{N(1535)\to N\pi}$, see Eq. (6.1). Obviously, the predictions of flavor symmetry contradict experimental data. (The ratio would be even smaller when phase space regarded.) We elaborate this point in detail by constructing a Lagrangian based only on $U(3)_V$ flavor symmetry (i.e., without full chiral symmetry and without terms parametrizing the axial anomaly). This allows to consider the decays of the whole baryon octet $\{N(1535),\ \Lambda(1670),\ \Sigma(1620),\ \Xi(?)\}$ into ground-state baryons $\{N(939),\ \Lambda(1116),\ \Sigma(1193),\ \Xi(1338)\}$ and a pseudoscalar meson. We find that decays involving a pion and a kaon are described in good agreement with experiment. However, as expected, the model predicts too small values for the decay width of $N(1535) \to N\eta$, but also the decay $\Lambda(1670) \to \Lambda(1116)\eta$ is underestimated. In general, we expect that the coupling of all baryons of the octet $\{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$ to a ground-state baryon and an η (and η') meson cannot be correctly described by flavor symmetry. On the other hand, if we study the decays of the octet $\{N(1650),\ \Lambda(1800),\ \Sigma(1750),\ \Xi(?)\}$ in the same way, the decays into an η meson (in particular $N(1650) \to N\eta$ and $\Lambda(1800) \to \Lambda\eta$) are reproduced with reasonable accuracy.

This illustrates why the eLSM, as previously considered, could not reproduce the $N(1535) \to N\eta$ decay width. Since it is based on a chiral symmetry, it also obeys flavor symmetry and thus underestimates this decay in accordance with the previous discussion.
We conclude that $N(1535)$ [and $\Lambda(1670)$, $\Sigma(1620)$, $\Xi(?)$] couple more strongly to the η meson than follows from flavor-symmetry considerations. We propose that the effects of the axial $U(1)_A$ anomaly can explain this increase.

Indeed, the implementation of the mirror assignment in the eLSM allows to construct an anomalous term without destroying the chiral $SU(N_f)_L \times SU(N_f)_R$ symmetry. As already mentioned, the mirror assignment allows to construct a chirally invariant mass term. In a very similar way, we have constructed a further pseudoscalar term. It couples the nucleon and its chiral partner, and is symmetric under $SU(2)_L \times SU(2)_R$ transformations, but breaks the symmetry under $U(1)_A$ transformations. It generates a further coupling term to η and η'.

In the two-flavor case, we have shown that (assuming $N(1535)$ as chiral partner of the nucleon) the anomaly term enhances the probability of the decay $N(1535) \to N\eta$, if we choose the coupling constant properly.
In the three-flavor case with four baryon octets, we found that, after fixing the $N(1535) \to N\eta$ decay width, similar terms parametrizing the $U(1)_A$ anomaly enhance also the decay $\Lambda(1670) \to \Lambda\eta$. This shows that the increase caused by the anomaly term is in agreement with experimental data. We have also seen that it predicts an enhanced $N(1535)N(939)\eta'$ coupling through the inclusion of the anomaly, which is in agreement with the analysis of scattering data.

Moreover, the anomaly term includes some interactions between the baryons, which allowed us to identify the octet $\{N(1535), \Lambda(1670), \Sigma(1620), \Xi(?)\}$ as chiral partners of the ground-state baryons (in agreement with the results from the two-flavor eLSM studies above).

To conclude, the anomaly causes an increase in the η couplings which is in accordance with experiment. As a consequence, we propose that it is crucial to add such a term to the above introduced eLSM.

Note, the mathematical structure of the coupling of a pseudoscalar glueball to baryons is very similar to the one of the anomaly term. Thus, we investigated these interactions in the two- as well as three-flavor case. We found that the pseudoscalar glueball couples strongly to the nucleon and its chiral partner $N(1535)$ and possibly to $N(1440)N$. Thus, we assume that the study of the process $p + \bar{p} \to p + \bar{p}(1535)+$ h.c. [which can be performed at the future PANDA experiment [21]] is a promising and interesting channel to search for the pseudoscalar glueball.

Finally, in order to decide which scenario is preferable, in the future it will be important to study the $N_f = 3$ case (where more experimental data exist).
As a first attempt, we used the Lagrangian in Eq. (4.1) or (4.75), where all parameters are already determined by the two-flavor study, see Tab. 5.1. In this way, one can estimate the hyperon masses and decay widths. The obtained results do not meet the experimental predictions, see Tabs. 7.1 and 7.2, as expected. In accordance to what we have said above, because there is no anomaly term implemented in the original $N_f = 2$ fit.

Thus, as an important outlook of this work, we suggest to perform again an independent fit in the $N_f = 3$ case including the discussed anomalous terms (which increase the number of parameters by two). In this context, it will also be interesting to further investigate the necessity of the large-N_c effective four-point interaction terms [last two lines in Eq. (4.1)]

quantity	minimum 1	minimum 2	minimum 3	experiment
$m_{\Lambda(1116)}$ [MeV]	1099.31	1098.91	1063.54	1115.68
$m_{\Lambda(1600)}$ [MeV]	1611	1613	1570	1600
$m_{\Lambda(1670)}$ [MeV]	1545	1567	1504	1670
$m_{\Lambda(1800)}$ [MeV]	1641	1604	1610	1800
$m_{\Sigma(1193)}$ [MeV]	1065.44	1063.70	1062.70	1189.37
$m_{\Sigma(1660)}$ [MeV]	1534	1538	1534	1660
$m_{\Sigma(1620)}(m_{\Sigma(1560)})$ [MeV]	1447	1463	1435	1620 (1560)
$m_{\Sigma(1750)}(m_{\Sigma(1620)})$ [MeV]	1536	1507	1551	1750 (1620)
$m_{\Xi(1318)}$ [MeV]	938.90	1539.35	938.90	1314.86
$m_{\Xi(1690)}$ [MeV]	1430	1243	1559	1690
$m_{\Xi(?)}$ [MeV]	1561	1494	1429	?
$m_{\Xi(?)}$ [MeV]	1658	1757	1663	?

Table 7.1. Hyperon masses estimated by the eLSM (without anomaly term in the baryonic sector) using the parameter values from the $N_f = 2$ fit.

which had to be included to prevent a pairwise mass degeneration. Moreover, the effects of the mass difference between the scalar and pseudoscalar diquark can be investigated (which is experimentally evident but not regarded by the effective eLSM approach).

In conclusion, the description of baryons in a chiral framework is a challenging topic. Here, we have developed extensions to the three-flavor case by implementing four $J = 1/2$ baryonic multiplets as well as by including novel terms parametrizing the axial anomaly in the baryonic sector. Yet, there is a lot to do in the future: besides the full description of the three-flavor case mentioned above, one should also include additional resonances, such as the ones with quantum numbers $J = 3/2$, and study various meson-nucleon and nucleon-nucleon scattering processes involving strangeness. Eventually, one should apply the developed chiral model at nonzero density in order to describe nuclear matter and neutron stars. In particular, recent discoveries of gravitational waves emitted by colliding neutron stars allow to get detailed information of the properties of such dense-matter objects in the next decades. In this way, new and demanding constraints on the nuclear-matter equation of state and hence on the model describing baryons, such as the chiral one presented in this thesis, will be obtained.

quantity	minimum 1	minimum 2	minimum 3	experiment
$\Gamma_{\Lambda(1600)\to\Sigma\pi}$ [MeV]	36.1406	35.6605	52.2479	$5-150$
$\Gamma_{\Lambda(1600)\to N\bar{K}}$ [MeV]	0.0300399	0.154541	556.082	$7.5-75$
$\Gamma_{\Lambda(1670)\to\Sigma\pi}$ [MeV]	249.673	282.52	217.155	$6.25-27.5$
$\Gamma_{\Lambda(1670)\to\Lambda\eta}$ [MeV]	4.33818	6.08877	3.8697	$2.5-12.5$
$\Gamma_{\Lambda(1670)\to N\bar{K}}$ [MeV]	0.467557	98.0468	690.322	$5-15$
$\Gamma_{\Lambda(1800)\to\Sigma\pi}$ [MeV]	273.254	305.129	62.4845	seen
$\Gamma_{\Lambda(1800)\to\Lambda\eta}$ [MeV]	26.7181	15.2243	10.0952	$2-44$
$\Gamma_{\Lambda(1800)\to N\bar{K}}$ [MeV]	180.034	30.9138	155.777	$50-160$
$\Gamma_{\Sigma(1660)\to\Sigma\pi}$ [MeV]	96.8161	113.742	123.073	seen
$\Gamma_{\Sigma(1660)\to\Lambda\pi}$ [MeV]	26.0344	24.0284	27.4744	seen
$\Gamma_{\Sigma(1660)\to N\bar{K}}$ [MeV]	0.11939	0.0257815	0.0614872	$4-60$
$\Gamma_{\Sigma(1620)\to\Sigma\pi}$ [MeV]	146.779	901.018	1341.46	-
$\Gamma_{\Sigma(1620)\to\Lambda\pi}$ [MeV]	88.2844	113.046	83.498	-
$\Gamma_{\Sigma(1620)\to N\bar{K}}$ [MeV]	188.626	477.353	503.434	-
$\Gamma_{\Sigma(1750)\to\Sigma\pi}$ [MeV]	1359.91	32.5598	68.9303	< 12.8
$\Gamma_{\Sigma(1750)\to\Lambda\pi}$ [MeV]	80.6213	81.4006	32.8549	seen
$\Gamma_{\Sigma(1750)\to\Sigma\eta}$ [MeV]	24.616	23.7114	7.68779	$9-88$
$\Gamma_{\Sigma(1750)\to N\bar{K}}$ [MeV]	564.279	109.491	4.203	$6-64$

Table 7.2. Hyperon decay widths estimated by the eLSM (without anomaly term in the baryonic sector) using the parameter values from the $N_f = 2$ fit.

Appendix A.

Derivation of Noether's theorem

We consider a classical field theory and assume the spacetime variables to transform according to

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu \ ,$$

while the fields transform as

$$\Phi_a(x) \to \phi'_a(x') = \phi_a(x) + \delta\phi_a(x) \ .$$

Since we want to study continuous (connected) symmetry transformations it is sufficient to restrict to infinitesimal transformations. This is possible because in the case of continuous connected groups every transformation can be expressed as an (infinite) sequence of infinitesimal transformations. Regarding the transformation of the fields we have to bring to our minds that the variation $\delta\phi_a(x)$ includes not only the change of the "form" of the field but also a part arising from the change of the spacetime variable,

$$\begin{aligned}\delta\phi_a(x) &\equiv \phi'_a(x') - \phi_a(x) \\ &= \phi'_a(x') - \phi_a(x') + \phi_a(x') - \phi_a(x) \\ &= \Delta\phi_a(x') + [\partial_\mu\phi_a(x)]\delta x^\mu + \mathcal{O}(\delta x^2) \\ &= \Delta\phi_a(x) + [\partial_\mu\phi_a(x)]\delta x^\mu + \mathcal{O}(\delta x^2, \delta\phi_a\delta x^\mu) \ , \end{aligned} \tag{A.1}$$

where $\Delta\phi_a(x') \equiv \phi'_a(x') - \phi_a(x')$ is the variation of the field at a fixed spacetime point x'. A useful property of this difference, which is often called total variation of the field, is the commutativity with spacetime derivatives. (A fact that we will use in later calculations.) Furthermore, in Eq. (A.1), we have applied the Taylor expansion $\phi_a(x') = \phi_a(x) + [\partial_\mu\phi_a(x)]\delta x^\mu + \mathcal{O}(\delta x^2)$. Hence, $\Delta\phi_a(x') = \Delta\phi_a(x) + \mathcal{O}(\delta\phi_a\delta x^\mu)$ shows that the total variation at the transformed spacetime point is equal to the variation at the spacetime point x^μ up to first order.

We denote a theory to be invariant under a certain transformation if the equations of motion are unaffected by the transformation, which are the Euler-Lagrange equations (1.28), obtained by demanding $\delta S = 0$.
However, the Lagrangian leading to certain equations of motions is not unique. The Lagrangian is defined only up to a surface term. The physics (i.e., the equations of motion)

that follow from the Euler-Lagrange equations do not change if we add a divergence term to the given Lagrangian density,

$$\tilde{\mathcal{L}} = \mathcal{L} + \partial_\mu \Theta^\mu \ , \tag{A.2}$$

where $\Theta^\mu = \Theta^\mu(\phi_a, x)$ is an arbitrary function. The invariance of the Euler-Lagrange equation can be seen easily by considering

$$\delta\tilde{S} = \int_\Omega \mathrm{d}^4x \ \tilde{\mathcal{L}} = \delta S + \delta \int_\Omega \mathrm{d}^4x \ \partial_\mu \Theta^\mu = \delta S + \oint_{\Sigma(\Omega)} \mathrm{d}\sigma_\mu \ \delta\Theta^\mu = \delta S \ , \tag{A.3}$$

where we used the commutativity of the variation with the spacetime derivative under the integral. The surface integral vanishes since the fields are held constant on the surface. Consequently, to prove a theory to be symmetric, in general it is too restrictive to claim the Lagrangian to be invariant, since it is allowed to change up to a surface term. Instead, we have to require that the action functional does not change upon such transformations. Thus, a symmetry implies

$$0 \stackrel{!}{=} \delta S \equiv S[\phi'_a(x')] - S[\phi_a(x)] \ . \tag{A.4}$$

Rewriting this condition leads to a proof of Noether's theorem which predicts a conserved quantity if the condition is fulfilled. We represent the action functionals by spacetime integrals over the Lagrangian density and add a zero,

$$\begin{aligned} 0 = \delta S = &\int_{\Omega'} \mathrm{d}^4x' \Big[\mathcal{L}(\phi'_a(x'), \partial'_\mu \phi'_a(x')) - \mathcal{L}(\phi_a(x'), \partial'_\mu \phi_a(x')) \Big] + \\ &+ \int_{\Omega'} \mathrm{d}^4x' \mathcal{L}(\phi_a(x'), \partial'_\mu \phi_a(x')) - \int_\Omega \mathrm{d}^4x \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)) \ . \end{aligned} \tag{A.5}$$

In the first line, the spacetime variable is unchanged and only the fields differ. We Taylor-expand the first Lagrangian in Eq. (A.5) around $\phi_a(x')$ up to first order in $\Delta\phi_a(x') = \phi'_a(x') - \phi_a(x')$,

$$\begin{aligned} \mathcal{L}(\phi'_a(x'), \partial'_\mu \phi'_a(x')) \simeq \mathcal{L}(\phi_a(x'), \partial'_\mu \phi_a(x')) + \frac{\partial \mathcal{L}(\phi_a(x'), \partial'_\mu \phi_a(x'))}{\partial \phi_a(x')} \Delta\phi_a(x') \\ + \frac{\partial \mathcal{L}(\phi_a(x'), \partial'_\mu \phi_a(x'))}{\partial(\partial'_\mu \phi_a(x'))} \Delta\big(\partial'_\mu \phi_a(x')\big) \ , \end{aligned}$$

so that the first line of Eq. (A.5) can be expressed as

$$
\begin{aligned}
&\int_{\Omega'} \mathrm{d}^4x' \Big[\mathcal{L}(\phi'_a(x'), \partial'_\mu\phi'_a(x')) - \mathcal{L}(\phi_a(x'), \partial'_\mu\phi_a(x'))\Big] = \\
&\quad\simeq \int_\Omega \mathrm{d}^4x \Big[\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial\phi_a(x)}\Delta\phi_a(x) + \frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial(\partial_\mu\phi_a(x))}\partial_\mu\Delta\phi_a(x)\Big] \\
&\quad= \int_\Omega \mathrm{d}^4x \Big[\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial\phi_a(x)}\Delta\phi_a(x) + \partial_\mu\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial(\partial_\mu\phi_a(x))}\Delta\phi_a(x) \\
&\qquad - \Big(\partial_\mu\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial(\partial_\mu\phi_a(x))}\Big)\Delta\phi_a(x)\Big] \\
&\quad= \int_\Omega \mathrm{d}^4x \Big[\Big(\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial\phi_a(x)} - \partial_\mu\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial(\partial_\mu\phi_a(x))}\Big)\Delta\phi_a(x) \\
&\qquad + \partial_\mu\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial(\partial_\mu\phi_a(x))}\Delta\phi_a(x)\Big] \\
&\quad= \int_\Omega \mathrm{d}^4x\ \partial_\mu\frac{\partial\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))}{\partial(\partial_\mu\phi_a(x))}\Delta\phi_a(x)
\end{aligned} \tag{A.6}
$$

where we have used that the total variation $\Delta\phi_a(x)$ commutes with the 4-gradient and that $\phi_a(x)$ fulfills the classical equation of motion (1.28).[1]
The variable substitution of $x'^\mu \to x^\mu$ can also be applied to the first integral in the second line of Eq. (A.5). The Jacobi determinant is given by

$$
\begin{aligned}
\Big|\frac{\partial x'^\mu}{\partial x^\nu}\Big| &= \det(g^\mu{}_\nu + \partial_\nu\delta x^\mu) = \exp[\ln\det(g^\mu{}_\nu + \partial_\nu\delta x^\mu)] \\
&= \exp[\mathrm{Tr}\ln(g^\mu{}_\nu + \partial_\nu\delta x^\mu)] \simeq \exp(\partial_\nu\delta x^\nu) \simeq 1 + \partial_\nu\delta x^\nu\ ,
\end{aligned} \tag{A.7}
$$

where have we used that $\ln\det A = \mathrm{Tr}\ln A\ \forall\ A$ being a symmetric positive definite matrix. We have Taylor-expanded the logarithm as well as the exponential function and neglected terms of second order in δx^μ. The Taylor expansion of the Lagrangian in the second line of Eq. (A.5) around x^μ is given by

$$
\mathcal{L}(\phi_a(x'), \partial'_\mu\phi_a(x')) \simeq \mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x)) + \partial_\nu\big[\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))\big]\delta x^\nu + \mathcal{O}(\delta x^2)\ . \tag{A.8}
$$

Then, with Eqs. (A.7) and (A.8), the first integral in the second line of Eq. (A.5) finally reads

$$
\begin{aligned}
&\int_{\Omega'} \mathrm{d}^4x' \mathcal{L}(\phi_a(x'), \partial'_\mu\phi_a(x')) = \\
&\quad\simeq \int_\Omega \mathrm{d}^4x\ \Big\{\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x)) + \partial_\nu\big[\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))\big]\delta x^\nu + \mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))\partial_\nu\delta x^\nu\Big\} \\
&\quad= \int_\Omega \mathrm{d}^4x\ \Big[\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x)) + \partial_\nu\mathcal{L}(\phi_a(x), \partial_\mu\phi_a(x))\delta x^\nu\Big]\ .
\end{aligned} \tag{A.9}
$$

[1]Note that we directly replaced the spacetime variable x'^μ by x^μ, where the corrections that accompany this substitution are of second order in infinitely small quantities and negligible. This can be understood without the explicit computation of expansions because the integrand itself is already infinitesimal.

We insert the rewritten integrals (A.6) and (A.9) into the condition (A.5) and obtain

$$\begin{aligned} 0 = \delta S &= \int_\Omega \mathrm{d}^4x \, \left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta\phi_a + \partial_\nu \mathcal{L} \delta x^\nu\right] \\ &= \int_\Omega \mathrm{d}^4x \, \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a - \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a + g^\mu{}_\nu \mathcal{L}\Big) \delta x^\nu\Big] \\ &= \int_\Omega \mathrm{d}^4x \, \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a - \theta^\mu{}_\nu \delta x^\nu\Big) \, , \end{aligned} \tag{A.10}$$

where we dropped the arguments, rewrote the total variation with $\Delta\phi_a = \delta\phi_a + \partial_\nu \phi_a \delta x^\nu$ [Eq. (A.1)] and introduced the energy-momentum tensor,

$$\theta^\mu{}_\nu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a + g^\mu{}_\nu \mathcal{L} \, . \tag{A.11}$$

Since the spacetime volume was chosen arbitrarily we recognize that the integrand itself has to vanish, in order to fulfill the condition. The result is a continuity equation $\partial_\mu \mathcal{J}^\mu = 0$ for the conserved so-called Noether current density

$$\mathcal{J}^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a - \theta^\mu{}_\nu \delta x^\nu \, , \tag{A.12}$$

which constructively proves Noether's Theorem, because we only have to evaluate the volume integral of the continuity equation,

$$0 = \int_V \mathrm{d}^3\boldsymbol{r} \, \partial_\mu \mathcal{J}^\mu(x) = \int_V \mathrm{d}^3\boldsymbol{r} \, \partial_0 \mathcal{J}^0(x) - \int_V \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{J}}(x) = \int_V \mathrm{d}^3\boldsymbol{r} \, \partial_0 \mathcal{J}^0(x) \, ,$$

to obtain the conserved quantity which is called Noether charge,

$$Q = \int_V \mathrm{d}^3\boldsymbol{r} \, \mathcal{J}^0(x) \, . \tag{A.13}$$

To conclude, every symmetry of the action functional upon a continuous transformation of the fields and/or the spacetime variables implies a conserved current density (1.58) and therefore a conservation law of the Noether charge (1.60).

Appendix B.

Gauge Fixing and Ghost Fields in QCD

The second quantisation of the QCD Lagrangian (1.99) poses some problems. They arise from the fact that the gluon fields have the additional degree of freedom of gauge transformations. In the functional (i.e., path) integral formulation, one would naively consider the generating functional [compare Ref. [36] for more details) for gluon fields to be

$$Z_{\text{gluons,naive}}[\vec{J}_\mu] = \mathcal{N} \int \mathcal{D}\vec{A}\ \exp\Big\{ i \int \mathrm{d}^4x\ \Big[\mathcal{L}_{\text{gluons}}(A_a^\mu(x), \partial_\mu A_a^\mu(x)) + J_\mu^a(x) A_a^\mu(x) \Big] \Big\}\ ,$$

we would integrate over redundant information. Note that we shortened the expression by introducing the abbreviation $\mathcal{D}\vec{A} = \prod_a \mathcal{D}A_a^\mu(x)$ as a product of path integral measures. In order to eliminate this additional degree of freedom, we have to restrict the path-integral of the generating functional to a set of gluon gauge fields that contains only one A_a^μ of each set of gauge-equivalent field configurations. We can do this by means of a trick, due to Faddeev and Popov. We introduce a "gauge-fixing" condition, called $F_x[A_a^\mu] = 0$. This can be any arbitrary gauge condition, such as for example the Lorenz gauge $F_x[A_a^\mu] = \partial_\mu A_a^\mu(x) = 0$. Now, we want to constrain the functional integral to cover only the configurations that fulfill the gauge-fixing condition. Therefore, we have to insert a functional delta distribution, $\delta[F_x[A_a^\mu]]$. To do so legally, we insert a 1 under the integral of the naively considered generating functional in the form,

$$1 = \mathcal{N}' \int \mathcal{D}\vec{\theta}\ \delta[F_x[A_{a,\theta}^\mu]] \det\Big(\frac{\delta F_x[A_{a,\theta}^\mu]}{\delta\theta_b(y)} \Big)\ ,$$

where we defined $\mathcal{D}\vec{\theta} = \prod_a \mathcal{D}\theta_a(x)$ and $\mathcal{N}'$ is a normalization constant. The quantities θ_a are the parameters of the color gauge transformation and $A_{a,\theta}^\mu$ denotes the gauge-transformed field. The generating functional (initially for vanishing sources $\vec{J}_\mu = 0$) reads then,

$$
\begin{aligned}
&Z_{\text{gluons}}[\vec{J}_\mu = 0] = \\
&= \mathcal{N}\mathcal{N}' \int \mathcal{D}\vec{A} \int \mathcal{D}\vec{\theta} \; \det\Big(\frac{\delta F[A^\mu_{a,\theta}]}{\delta\theta_b}\Big)\delta[F[A^\mu_{a,\theta}]] \; \exp\Big\{i \int \mathrm{d}^4x \; \mathcal{L}_{\text{gluons}}(A^\mu_a, \partial_\mu A^\mu_a)\Big\} = \\
&= \mathcal{N}\mathcal{N}'\Big(\int \mathcal{D}\vec{\theta}\Big) \int \mathcal{D}\vec{A}_\theta \; \det\Big(\frac{\delta F[A^\mu_{a,\theta}]}{\delta\theta_b}\Big)\delta[F[A^\mu_{a,\theta}]] \; \exp\Big\{i \int \mathrm{d}^4x \; \mathcal{L}_{\text{gluons}}(A^\mu_{a,\theta}, \partial_\mu A^\mu_{a,\theta})\Big\} = \\
&= \mathcal{N}'' \int \mathcal{D}\vec{A} \; \det\Big(\frac{\delta F[A^\mu_a]}{\delta\theta_b}\Big)\delta[F[A^\mu_a]] \; \exp\Big\{i \int \mathrm{d}^4x \; \mathcal{L}_{\text{gluons}}(A^\mu_a, \partial_\mu A^\mu_a)\Big\} \; ,
\end{aligned}
$$

where we used that the action in the exponent as well as the integration measure is gauge invariant. The invariance of the latter can be understood by considering the fact that we integrate over all gauge field configurations and therefore it cannot make a difference if they are gauge transformed or not. Since $A^\mu_{a,\theta}$ reduced now to a simple integration variable, we renamed it to A^μ_a in the last line. Furthermore we dropped the explicit spacetime dependences in order to simplify the notation and absorbed the normalization constants $\mathcal{N}$ and $\mathcal{N}'$ and the gauge-parameter integral into a new normalization constant $\mathcal{N}''$.
Although it appears that we have nothing done but including a factor of 1, as desired the functional integral over A^μ_a is now restricted by the delta function to physically unequal field configurations.
The determinant of the functional derivative of the gauge-fixing condition with respect to the parameters of the gauge transformation (which is often called Faddeev-Popov determinant) can be rewritten as a functional integral over Grassmannian fields[1]. Therefore we introduce artificial fields whose particle excitations are called ghosts or more precise Faddeev-Popov ghosts. The name arises due to the unnaturalness of being bosons with spin zero but at the same time being anticommuting Grassmannian fields, meaning that they are scalar particles that obey Fermi-Dirac statistics. This violation of the spin-statistics theorem is acceptable because ghosts are not physical particles and they are introduced merely to represent the Faddeev-Popov determinant in a convenient way.
With the abbreviation

$$
M_{ab}(x,y) = \frac{\delta F_x[A^\mu_a]}{\delta\theta_b(y)} \; ,
$$

the Faddeev-Popov determinant can then be written as

$$
\begin{aligned}
\det(iM_{ab}(x,y)) &= \int \mathcal{D}\vec{\bar{\eta}}\mathcal{D}\vec{\eta} \; \exp\Big(-i \int \mathrm{d}^4x\mathrm{d}^4y \; \bar{\eta}_a(x) M_{ab}(x,y)\eta_b(y)\Big) = \\
&\equiv \int \mathcal{D}\vec{\bar{\eta}}\mathcal{D}\vec{\eta} \; \exp\Big(-i \int \mathrm{d}^4x\mathrm{d}^4y \; \mathcal{L}_{\text{ghosts}}\Big) \; ,
\end{aligned}
$$

where $\bar{\eta}_a$ and η_a represent the antighost and ghost fields. Furthermore we introduced the ghost Lagrangian $\mathcal{L}_{\text{ghosts}}$ in order to shorten the notation. In the case of QCD which is a

[1]Every determinant can be written as an integral over Grassmannian variables.

non-abelian gauge theory this ghost term will include interactions between ghosts and the gluon gauge fields. This can be seen by explicitly choosing a gauge-fixing condition and expanding $M_{ab}(x,y)$. For example if we use Lorenz gauge, $F_x[A^\mu_{U,a}] = \partial_x \cdot A_{U,a}(x) = 0$, we finally obtain [e.g. [36], Chap. 5.7]

$$M_{ab}(x,y) = \Big\{\delta_{ab}\Box_x - g f_{abc}[\partial_x \cdot A^c(x) + A^c(x)\cdot\partial_x]\Big\}\delta^{(4)}(x-y) \ ,$$

and

$$\begin{aligned}&\det(iM) = \\ &= \int \mathcal{D}\bar\eta_a \mathcal{D}\eta_a \ \exp\Big[-i\int \mathrm{d}^4x\mathrm{d}^4y \ \bar\eta_a(x)\delta_{ab}\Box_x\delta^{(4)}(X-Y)\eta_b(y) - i\int \mathrm{d}^4 \ g f^{abc}(\partial\bar\eta_a)\eta_b A^\mu_c\Big] \ ,\end{aligned}$$

where the last term describes the ghost-gluon vertex, which also has to be taken into account in QCD calculations.

Going back to arbitrary gauge-fixing conditions again, the generating functional now reads

$$\begin{aligned}&Z_{\text{gluons}}[\vec{J}_\mu] = \\ &= \mathcal{N}''' \int \mathcal{D}\vec{A}\mathcal{D}\vec{\bar\eta}\mathcal{D}\vec\eta \ \delta[F[A^\mu_a]] \ \exp\Big\{-i\int \mathrm{d}^4x\mathrm{d}^4y \ \mathcal{L}_{\text{ghosts}} + i\int \mathrm{d}^4x \ \big(\mathcal{L}_{\text{gluons}} + J^a_\mu A^\mu_a\big)\Big\} \ ,\end{aligned}$$

where we already included the source term and absorbed $-i$ factors from the replacement $\det M \to \det(iM)$ into $\mathcal{N}'''$.
Finally we can eliminate the functional delta distribution by performing the replacement

$$F_x[A^\mu_a] \to F_x[A^\mu_a] - C_a(x) \ ,$$

where $C_a(x)$ can be any scalar function with color index, and "averaging" over gauge-fixing conditions differing in C_a by including a Gaussian weight factor of

$$\int \mathcal{D}\vec{C} \exp\Big\{-i\frac{1}{2\xi}\int \mathrm{d}^4x \ C_a C^a\Big\}$$

and its inverse which will be absorbed into the normalization constant that we again denote with $\mathcal{N}$. The parameter ξ is a gauge parameter and can be chosen arbitrarily. Finally the generating functional for gluon fields reads

$$\begin{aligned}&Z_{\text{gluons}}[\vec{J}_\mu] = \\ &= \mathcal{N}\int \mathcal{D}\vec{C} \exp\Big\{-i\frac{1}{2\xi}\int \mathrm{d}^4x \ C_a C^a\Big\}\int \mathcal{D}\vec{A}\mathcal{D}\vec{\bar\eta}\mathcal{D}\vec\eta \ \delta[F[A^\mu_a]] \ \times \\ &\qquad \times \exp\Big\{-i\int \mathrm{d}^4x\mathrm{d}^4y \ \mathcal{L}_{\text{ghosts}} + i\int \mathrm{d}^4x \ \big(\mathcal{L}_{\text{gluons}} + J^a_\mu A^\mu_a\big)\Big\} \\ &= \mathcal{N}\int \mathcal{D}\vec{A}\mathcal{D}\vec{\bar\eta}\mathcal{D}\vec\eta \ \exp\Big\{-i\int \mathrm{d}^4x\mathrm{d}^4y \ \mathcal{L}_{\text{ghosts}} + i\int \mathrm{d}^4x \ \Big(\mathcal{L}_{\text{gluons}} - \frac{F[A^\mu_a]F[A^\mu_a]}{2\xi} + J^a_\mu A^\mu_a\Big)\Big\} \ ,\end{aligned}$$

where we used the delta distribution to perform the integral over C_a. Effectively we have added a new gauge-fixing term

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\xi} F_x[A_a^\mu] F_x[A_a^\mu]$$

to the Lagrangian, which (in connection with the ghost field) solves the problem of the additional degree of freedom of gauge transformations.
In conclusion, it is straightforward to include also fermions and to obtain the "quantised" QCD Lagrangian,

$$\mathcal{L}_{\text{QCD}} = \bar{q}(x)\big(i\gamma^\mu D_\mu - m\big)q(x) - \frac{1}{2}\,\text{Tr}(\mathcal{G}_{\mu\nu}\mathcal{G}^{\mu\nu}) - \frac{1}{2\xi} F_x^2[A_a^\mu] + \bar{\eta}_a(x) M_{ab}(x,y) \eta_b(y) \ , \tag{B.1}$$

where the gauge-fixing condition $F_x[A_a^\mu]$ still has to be chosen.

Appendix C.

Decay widths of $N^* \to NP$ in the two-flavor eLSM in the mirror assignment

We consider the following Lagrangian describing the decay of the chiral partner of the nucleon N^* into the nucleon N and a pseudoscalar meson P:

$$\begin{aligned}\mathcal{L}_{NPN^*} =& - i\mathcal{A}\ \bar{N}PN^* + \mathcal{B}\ \bar{N}\gamma_\mu(\partial^\mu P)N^* = \\ =& - i\bar{N}P(\mathcal{A} - \mathcal{B}\not{q})N^* \ ,\end{aligned} \tag{C.1}$$

where we replaced the covariant partial derivative of P with $-iq^\mu$, where q^μ is the momentum of P, pointing into the vertex. Furthermore, we used the Feynman-slash notation $\not{q} = \gamma_\mu q^\mu$. If we consider the model (2.140), $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} := \frac{(\hat{g}_1 - \hat{g}_2)}{4\cosh\theta} Z \ , \quad \mathcal{B} := -\frac{\hat{c}_1 + \hat{c}_2}{4\cosh\theta} Zw \ . \tag{C.2}$$

With Eq. (1.51) the associated tree-level decay width reads in terms of the amplitude reads

$$\Gamma_{N^* \to NP} = \lambda_P \frac{p_f}{8\pi m_{N^*}^2} \overline{|i\mathcal{M}_{fi}|^2} \ , \tag{C.3}$$

where p_f is the absolute momentum of the particles in the final state,

$$p_f = \frac{1}{2m_{N^*}} \sqrt{(m_{N^*}^2 - m_N^2 - m_P^2)^2 - 4m_N^2 m_P^2} \ , \tag{C.4}$$

compare Eq. (1.48) and the factor λ_P is added by hand:

- For $P = \pi$, it pays attention to the three possible isospin states of the pion, i.e.,

$$\lambda_\pi = 3 \ . \tag{C.5}$$

- For $P = \eta$, it takes into account that $\eta = \eta_N \cos\theta_P + \eta_S \sin\theta_P$ with the non-strange $\eta_N \equiv (\bar{u}u + \bar{d}d)/\sqrt{2}$ and the strange contribution $\eta_S \equiv \bar{s}s$. They mix with an angle of θ_P . Since it is assumed that the amplitude of the decay $N^* \to N\eta_S$ is massively suppressed, to good approximation it holds $\Gamma_{N^* \to N\eta} = \cos^2 \Phi_P \Gamma_{N^* \to N\eta_N}$. That is,

$$\lambda_\eta = \cos^2 \theta_P \ . \tag{C.6}$$

In this work, we use the pseudoscalar mixing angle $\theta_P = -44.6°$, which was obtained in the studies of Ref. [3].

The amplitude of this decay follows from the Feynman rules as summarized in Sec. [1.2.2]:

$$i\mathcal{M}_{N^* \to NP} = \bar{u}_\alpha(\boldsymbol{p}, r)[-i(\mathcal{A} - \mathcal{B}\not{q})]^{\alpha\beta} u_\beta(\boldsymbol{k}, s) \tag{C.7}$$

with the Dirac spinors $u_\alpha(\boldsymbol{p}, r)$ and $u_\beta(\boldsymbol{k}, s)$ of positive energy, where $\boldsymbol{p}$ is the out going three-momentum of the nucleon and $\boldsymbol{k}$ is the in-going three-momentum of the (decaying) chiral partner. The parameters r and k are the spins of the nucleon and its chiral partner, respectively. The Greek indices α and β are Dirac spinor indices, which are understood to be summed over. We calculate the squared amplitude, which is averaged over initial and summed over final spins:

$$\begin{aligned}
&\overline{|i\mathcal{M}_{N^* \to NP}|^2} = \\
=&\frac{1}{2}\sum_{r,s} \bar{u}_\beta(\boldsymbol{k}, s)[i(\mathcal{A} - \mathcal{B}\not{q})]^{\beta\alpha} u_\alpha(\boldsymbol{p}, r)\bar{u}_\gamma(\boldsymbol{p}, r)[-i(\mathcal{A} - \mathcal{B}\not{q})]^{\gamma\delta} u_\delta(\boldsymbol{k}, s) = \\
=&\frac{1}{2}\sum_{s} \bar{u}_\beta(\boldsymbol{k}, s)[i(\mathcal{A} - \mathcal{B}\not{q})]^{\beta\alpha} (\not{p} + m_N)^{\alpha\gamma}[-i(\mathcal{A} - \mathcal{B}\not{q})]^{\gamma\delta} u_\delta(\boldsymbol{k}, s) = \\
=&\frac{1}{2}[i(\mathcal{A} - \mathcal{B}\not{q})]^{\beta\alpha} (\not{p} + m_N)^{\alpha\gamma}[-i(\mathcal{A} - \mathcal{B}\not{q})]^{\gamma\delta} (\not{k} + m_{N^*})^{\delta\beta} = \\
=&\frac{1}{2}\operatorname{Tr}\Big\{[i(\mathcal{A} - \mathcal{B}\not{q})](\not{p} + m_N)[-i(\mathcal{A} - \mathcal{B}\not{q})](\not{k} + m_{N^*})\Big\} = \\
=&\frac{\mathcal{A}^2}{2}\operatorname{Tr}\big[(\not{p} + m_N)(\not{k} + m_{N^*})\big] + \frac{\mathcal{B}^2}{2}\operatorname{Tr}\big[\not{q}(\not{p} + m_N)\not{q}(\not{k} + m_{N^*})\big] \\
&-\frac{\mathcal{AB}}{2}\operatorname{Tr}\big[(\not{p} + m_N)\not{q}(\not{k} + m_{N^*})\big] - \frac{\mathcal{BA}}{2}\operatorname{Tr}\big[\not{q}(\not{p} + m_N)(\not{k} + m_{N^*})\big] = \\
=&\frac{\mathcal{A}^2}{2}\operatorname{Tr}\big(\not{p}\not{k} + m_N m_{N^*}\big) + \frac{\mathcal{B}^2}{2}\operatorname{Tr}\big(\not{q}m_N\not{q}m_{N^*} + \not{q}\not{p}\not{q}\not{k}\big) \\
&-\frac{\mathcal{AB}}{2}\operatorname{Tr}\big(\not{p}\not{q}m_{N^*} + m_N\not{q}\not{k}\big) - \frac{\mathcal{BA}}{2}\operatorname{Tr}\big(\not{q}\not{p}m_{N^*} + \not{q}m_N\not{k}\big) =
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\mathcal{A}^2}{2} p_\mu k_\mu \operatorname{Tr}\left(\gamma^\mu\gamma^\nu\right) + \frac{\mathcal{A}^2}{2} m_N m_{N^*} \operatorname{Tr}\left(\mathbb{1}_4\right) + \frac{\mathcal{B}^2}{2} m_N m_{N^*} q_\mu q_\nu \operatorname{Tr}\left(\gamma^\mu\gamma^\nu\right) \\
&+ \frac{\mathcal{B}^2}{2} q_\mu p_\nu q_\sigma k_\tau \operatorname{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau\right) - \mathcal{A}\mathcal{B} m_{N^*} p_\mu q_\nu \operatorname{Tr}\left(\gamma^\mu\gamma^\nu\right) - \mathcal{A}\mathcal{B} m_N q_\mu k_\nu \operatorname{Tr}\left(\gamma^\mu\gamma^\nu\right) = \\
=&4\frac{\mathcal{A}^2}{2}(p^\mu k^\mu + m_N m_{N^*}) + 4\frac{\mathcal{B}^2}{2}(m_N m_{N^*} q_\mu q^\mu + q_\mu p^\mu q_\nu k^\nu - q_\mu p_\nu q^\mu k^\nu + q_\mu p_\nu q^\nu k^\mu) \\
&- 4\mathcal{A}\mathcal{B}(m_{N^*} p_\mu q^\mu + m_N q_\mu k^\mu) = \\
=&4\frac{\mathcal{A}^2}{2}(E_N m_{N^*} + m_N m_{N^*}) + 4\frac{\mathcal{B}^2}{2}(m_N m_{N^*} m_P^2 + q_\mu p^\mu E_P m_{N^*} - m_P^2 E_N m_{N^*} \\
&+ E_P m_{N^*} p_\mu q^\mu) - 4\mathcal{A}\mathcal{B}(m_{N^*} p_\mu q^\mu + m_f E_P m_N) = \\
=&4 m_{N^*} m_N \frac{\mathcal{A}^2}{2}(\frac{E_N}{m_N} + 1) + 4\frac{\mathcal{B}^2}{2}(m_{N^*} m_N m_P^2 + 2\frac{m_{N^*}^2 - m_N^2 - m_P^2}{2} E_P m_{N^*} - m_P^2 E_N m_{N^*}) \\
&- 4\mathcal{A}\mathcal{B}(m_{N^*}\frac{m_{N^*}^2 - m_N^2 - m_P^2}{2} + m_N E_P m_{N^*}) = \\
=&4 m_{N^*} m_N \frac{\mathcal{A}^2}{2}(\frac{E_N}{m_N} + 1) + 4 m_{N^*} m_N \frac{\mathcal{B}^2}{2}[m_P^2 + (m_{N^*}^2 - m_N^2 - m_P^2)\frac{E_P}{m_N} - m_\eta^2 \frac{E_N}{m_N}] \\
&- 4 m_{N^*} m_N \mathcal{A}\mathcal{B}(\frac{m_{N^*}^2 - m_N^2 - m_P^2}{2m_N} + E_P) \;, \qquad \text{(C.8)}
\end{aligned}
$$

where from the fifth to the sixth line, we took into account that traces of an odd number of Dirac matrices vanish, and from the seventh to the eighth line we used the relation $\operatorname{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ and $\operatorname{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\tau) = 4(g^{\mu\nu}g^{\sigma\tau} - g^{\mu\sigma}g^{\nu\tau} + g^{\mu\tau}g^{\nu\sigma})$. Furthermore, using the decay-width expression (1.51), implies that we are in the rest frame of the initial particle, i.e., $k^\mu = (m_{N^*}, 0)^T$. Finally, we applied $q_\mu q^\mu = m_P^2$ and

$$
\begin{aligned}
k^2 = (p+q)^2 &= p_\mu p^\mu + 2p_\mu q^\mu + q_\mu q^\mu \\
\Leftrightarrow \qquad m_{N^*}^2 &= m_N^2 + 2p_\mu q^\mu + m_P^2 \\
\Leftrightarrow \qquad p_\mu q^\nu &= \frac{m_{N^*}^2 - m_N^2 - m_P^2}{2} \;. \qquad \text{(C.9)}
\end{aligned}
$$

Furthermore, we used the relativistic energy-momentum relation for N and P:

$$
E_N = \sqrt{p_f^2 + m_N^2} \;, \quad E_P = \sqrt{p_f^2 + m_P^2} \;. \qquad \text{(C.10)}
$$

Finally, the averaged squared amplitude reads

$$
\begin{aligned}
\overline{|i\mathcal{M}_{fi}|^2} = 4 m_{N^*} m_N \Big\{ \;& \frac{\mathcal{A}^2}{2}\left(\frac{E_N}{m_N} + 1\right) + \frac{\mathcal{B}^2}{2}\left[\left(m_{N^*}^2 - m_N^2 - m_P^2\right)\frac{E_P}{m_N} + m_\eta^2\left(1 - \frac{E_N}{m_N}\right)\right] \\
&- \mathcal{A}\mathcal{B}\left(\frac{m_{N^*}^2 - m_N^2 - m_P^2}{2m_N}\right) + E_P)\Big\} \;. \qquad \text{(C.11)}
\end{aligned}
$$

With Eq. (C.3) and $\mathcal{A}$ and $\mathcal{B}$ as given in Eq. (C.2), the decay width of $N^* \to NP$ is calculated to be

$$\Gamma_{N^*\to NP} = \lambda_\eta \frac{p_f}{2\pi}\frac{m_N}{m_{N^*}}\Big\{ \quad \frac{Z^2}{2}\frac{(\hat{g}_1-\hat{g}_2)^2}{16\cosh^2\theta}\left(\frac{E_N}{m_N}+1\right) + \frac{(\hat{c}_1+\hat{c}_2)^2 Z^2 w2}{32\cosh^2\theta}\left[(m_{N^*}^2 - m_N^2 - m_P^2)\frac{E_P}{m_N} + m_\eta^2\left(1-\frac{E_N}{m_N}\right)\right] + \frac{Z^2 w(\hat{c}_1+\hat{c}_2)}{4\cosh\theta}\frac{\hat{g}_1-\hat{g}_2}{4\cosh\theta}\left(\frac{m_{N^*}^2 - m_N^2 - m_P^2}{2m_N}\right) + E_P)\Big\} \ . \quad \text{(C.12)}$$

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Curriculum Vitae **Lisa Olbrich**

born on October 24, 1989 in Aschaffenburg, unmarried

Doctorate (*doctor philosophiae naturalis*, Dr. phil. nat.)

10/2014 – today Doctoral candidate at the Institute of Theoretical Physics
Johann Wolfgang Goethe – University Frankfurt am Main

- Extension and optimization of a model using explorative data analysis
- Comparison of theory and experiment with statistical methods
- Publications in renowned journals
- Regular presentations in the working group and technical seminars addressing colleges as well as international experts

Education

10/2012 – 09/2014 Master of Science, Institute of Theoretical Physics "with honors" (1.0)
Johann Wolfgang Goethe – University Frankfurt am Main

10/2009 – 09/2012 Bachelor of Science, Institute of Theoretical Physics "very good" (1.3)
Johann Wolfgang Goethe – University Frankfurt am Main

09/2000 – 06/2009 General matriculation standard (1.9)
Advanced courses: mathematics and physics (both "very good")
Hanns–Seidel–Gymnasium Hösbach, Bavaria

Further studies *soft-skill courses (Helmholtz Graduate School, HGS-HIRe):*

3 days – 06/2017 Leadership qualities
Leadership and Career Development

4 days – 06/2016 Teamwork and project management
Leading Teams in a Research Environment

4 days – 11/2015 Research
Making an Impact as an Effective Researcher

Publications

Phys. Rev. D L. Olbrich, M. Zétényi, F. Giacosa, and D. H. Rischke, *Influence of the axial anomaly on the decay $N(1535) \to N\eta$*, PRD **97** (2018) 014007,

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