# Trace

$$\mathrm{Tr}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = a_{11} + a_{22} = \sum_{i=1}^{N=2} a_{ii}$$

In general for a $N \times N$ matrix:

$$\mathrm{Tr} A = \sum_{i=1}^{N} a_{ii}$$

Properties:

$$\mathrm{Tr}[A+B] = \mathrm{tr}[A] + \mathrm{tr}[B]$$

$$\mathrm{Tr}[c \cdot A] = c\,\mathrm{Tr}[A]$$

$$\mathrm{Tr}[A] = \mathrm{Tr}[A^t]$$

$$\mathrm{Tr}[A \cdot B] = \mathrm{Tr}[B \cdot A]$$

(Achtung: $\mathrm{Tr}[AB] \neq \mathrm{Tr}[A] \cdot \mathrm{Tr}[B]$

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB] = \mathrm{Tr}[BCA]$$

$$\mathrm{Tr}[1_N] = N.$$

# Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det A = a_{11}a_{22} - a_{12}a_{21} = \sum_{i,j=1}^{2} a_{1i}\, a_{2j}\, \varepsilon_{ij}$$

whereas $\varepsilon_{ij}$ / $\varepsilon_{12} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{21} = -1$

In fact

$$\varepsilon_{12}\, a_{11} a_{22} + \varepsilon_{21}\, a_{12} a_{21} = a_{11}a_{22} - a_{12}a_{21} = \det A.$$

In general, for $A$ being an $N \times N$ matrix:

$$\boxed{\det A = \sum_{i_1, \ldots, i_N = 1}^{N} \varepsilon_{i_1 i_2 \ldots i_N}\, a_{1 i_1} a_{2 i_2} \cdots a_{N i_N}}$$

$$\varepsilon_{i_1 \ldots i_N} = \begin{cases} 0 & \text{if two indices are equals} \\ 1 & \text{even permutation} \\ -1 & \text{odd} \quad \text{" " " "} \end{cases}$$

Properties of the determinant :

$\det(c \cdot A) = c^N \det A$

$\det(A^t) = \det A$

Achtung :

$(\det(A+B) \neq \det A + \det B)$

$\det(A \cdot B) = \det A \cdot \det B$

$\det 1_N = 1$

# Inverse matrix



$A$ is a $N \times N$ matrix.

$A^{-1}$ is the inverse matrix of $A$ if

$$A \cdot A^{-1} = A^{-1} \cdot A = 1_N$$

$$\boxed{A^{-1} \text{ exist only if } \det A \neq 0 \,!}$$

For instance for a $2 \times 2$ matrix:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- The inversion of matrix is an operation which has to be performed when solving complicated problems.

- The set of $N \times N$ matrices is not a group under the matrix operation because not all the elements have the inverse.

- The set of $N \times N$ matrices such that $\det A \neq 0$ form a group (a nonabelian one) under the matrix multiplication.

Important consequence:

$$\boxed{\det A^{-1} = \frac{1}{\det A}}$$

Infact, being $AA^{-1} = 1$ we have

$$\det(AA^{-1}) = 1$$

$$\det A \cdot \det A^{-1} = 1$$

$$\det A^{-1} = \frac{1}{\det A}$$

We also see here that only for $\det A \neq 0$ the matrix $A^{-1}$ is defined.

## Commutator:

$A_1, A_2$ : $N \times N$ matrices

$$[\cdot,\cdot] : (N\times N)\times(N\times N) \longmapsto N\times N$$

$$[A_1, A_2] = A_1 A_2 - A_2 A_1$$

The commutator is a basic quantity which plays a crucial role when discussing operators.

obviously:

$$[A_2, A_1] = -[A_1, A_2]$$

$$\mathrm{Tr}\left([A_1,A_2]\right) = \mathrm{Tr}\left(A_1A_2 - A_2A_1\right) = \mathrm{Tr}(A_1A_2) - \mathrm{Tr}(A_2A_1) = 0$$

$$[A, A] = 0$$

$[A, A^t] \neq 0$ $\quad ( [A, A^t] = A \cdot A^t - A^t \cdot A \neq 0$.. But if $A^t = A \to [A, A^t] =$

$$[A_1,[A_2,A_3]] + [A_3,[A_1,A_2]] + [A_2,[A_3,A_1]] =$$

x

$$[A_1, A_2 A_3] = A_2 [A_1, A_3] + [A_1, A_2] A_3$$

$$[A_1 A_2, A_3] = A_1 [A_2, A_3] + [A_1, A_3] A_2$$

...

...

# EXPONENTIAL MATRIX FORM

Consider a $N \times N$ matrix $A$. What does

$$e^{A}$$

mean?

At first sight it makes no sense. We have, for instance, something like

$$e^{\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}}$$

However, let us remind which is the Taylor expansion of the exponential function:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We can then make the expression $e^{A}$ meaningful by "defining"

$$e^{A} = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

In fact, we know how to calculate $A^2, A^3, \dots$

Important:

$$e^{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \neq \begin{pmatrix} e^{a_{11}} & e^{a_{12}} \\ e^{a_{21}} & e^{a_{22}} \end{pmatrix} !!!$$

Never!!!!

Indeed, we can make sense of each expression of the form

$$\text{“} f(A) \text{”}$$

whereas $f(x) = \sum_{n=0}^{\infty} c_n x^n$ by defining

$$f(A) = \sum_{n=0}^{\infty} c_n \underline{A}^n$$

$$c_n = \frac{1}{n!} \left( \frac{d^n f(x)}{dx^n} \right)_{x=0}.$$

But... let us not forget the following:

"Matrices are <u>not</u> numbers"

$$(A_1 + A_2)^2 = A_1^2 + A_2^2 + A_1 \cdot A_2 + A_2 \cdot A_1$$

Only if $[A_1, A_2] = 0$ we can write

$$(A_1 + A_2)^2 = A_1^2 + A_2^2 + 2A_1A_2$$

What about

$$e^{A_1} \cdot e^{A_2} \, ?$$

This is a rather complicated object...

if $[A_1, A_2] = 0$: $\quad e^{A_1} \cdot e^{A_2} = e^{A_1 + A_2}$

but in general we have:

$$e^{A_1} \cdot e^{A_2} = e^{A_1 + A_2 + \frac{1}{2}[A_1, A_2] + \frac{1}{12}[A_1, [A_1, A_2]] + \dots}$$

Backer - Campbell - Hausdorff formula

Again; matrices are not numbers...

# IMPORTANT PROPERTY

$$B = e^{A}$$

$$\boxed{\det B = e^{\mathrm{Tr}\, A}}$$

Let us show it in this way: we introduce a "small parameter $\varepsilon$ and we consider only terms up to $\varepsilon$ ... we neglect $\varepsilon^2$ and higher.

$$B = e^{\varepsilon A} = 1 + \varepsilon A + \dots = \begin{pmatrix} 1+\varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & 1+\varepsilon a_{22} \end{pmatrix}$$

$$\det B = (1+\varepsilon a_{11})(1+\varepsilon a_{22}) - \varepsilon^2 (a_{12}\, a_{21})$$

$$= 1 + \varepsilon (a_{11} + a_{22}) + \dots$$

$$= 1 + \varepsilon \,\mathrm{Tr}\, A = e^{\varepsilon \,\mathrm{Tr}\, A}$$

As an exercise → prove it up to order $\varepsilon^2$ included.