1.1)

$M = \{ B \,/\, \det B \neq 0 \}$.

$(M, \cdot)$ is a group.

Infact, the operation is internal:

A, B such that $\det A \neq 0$, $\det B \neq 0$.

$\det(AB) = \det A \det B \neq 0 \longrightarrow AB \in M$.

Then:

- $A(BC) = (AB)C$ "obvious".
- $1_N = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in M$: Yes! $\det 1_N = 1 \neq 0$
- $B \in M \to B^{-1} \in M$: Yes. In fact:

$$\det B^{-1} = \frac{1}{\det B} \neq 0.$$

The group is not abelian because in general

$$AB \neq BA.$$

1.2)

$M = \{ B \,/\, B^t B = B B^t = 1$ and $\det B = +1 \}$.

$(M, \cdot)$ is a group.

Internal operation:

$B_1, B_2 \in M \qquad B_1 B_2$ is such.

$$(B_1 B_2)^t (B_1 B_2) = B_2^t \underbrace{B_1^t B_1}_{1} B_2 = B_2^t B_2 = 1_N$$

and

$$\det(B_1 B_2) = \det B_1 \det B_2 = (+1)(+1) = +1.$$

Then:

- $B_1 (B_2 B_3) = (B_1 B_2) B_3$ ✓
- $1_N \in M$: Yes. $1_N^t 1_N = 1_N 1_N^t = 1_N$ and $\det 1_N = 1$.
- $B \in M \to B^{-1} \in M$. In fact.

$$(B^{-1})^t (B^{-1}) = (B B^t)^{-1} = (1_N)^{-1} = 1_N$$

$$\det B^{-1} = \frac{1}{\det B} = +1!$$

$(M, \cdot)$ is a group. This is a subgroup of $O(N)$ and is called $SO($

1.3) $M = \{ B \,/\, B^t B = B B^t = 1_N$ and $\det B = -1 \}$.

$(M, \cdot)$ is not a group.

In fact the operation is not internal.

if $B_1, B_2 \in M$ then $B_1 B_2$ doesn't.

$\det(B_1 B_2) = \det B_1 \det B_2 = (-1)(-1) = +1$!

Moreover, also the identity 1_N does not belong to M.

2)

$$B = e^{\varepsilon A} = 1 + \varepsilon A + \frac{\varepsilon^2}{2} A^2 + \underbrace{\mathcal{O}(\varepsilon^3)}_{\text{neglect } \varepsilon^3 \text{ and higher} \ldots}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A^2 = AA = \begin{pmatrix} a_{11}^2 + a_{12} a_{21} & a_{11} a_{12} + a_{12} a_{22} \\ a_{21} a_{11} + a_{22} a_{21} & a_{21} a_{12} + a_{22}^2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 + \varepsilon a_{11} + \frac{\varepsilon^2}{2}\left(a_{11}^2 + a_{12} a_{21}\right) & \varepsilon a_{12} + \frac{\varepsilon^2}{2}\left(a_{11} a_{12} + a_{12} a_{22}\right. \\ \varepsilon a_{21} + \frac{\varepsilon^2}{2}\left(a_{21} a_{11} + a_{22} a_{21}\right) & 1 + \varepsilon a_{22} + \frac{\varepsilon^2}{2}\left(a_{21} a_{12} + a_{22}^2\right) \end{pmatrix}$$

$$\det B = 1 + \varepsilon(a_{11} + a_{22}) + \varepsilon^2 a_{11} a_{22} + \frac{\varepsilon^2}{2}\left(a_{11}^2 + a_{12} a_{21} + a_{21} a_{12} + a_{2}\ldots\right.$$
$$- \varepsilon^2 a_{12} a_{21} + \mathcal{O}(\varepsilon^3)$$

$$= 1 + \varepsilon(a_{11} + a_{22}) + \frac{\varepsilon^2}{2}\Big(a_{11}^2 + \cancel{a_{12} a_{21}} + a_{22}^2 + \cancel{a_{21} a_{12}}$$
$$\cancel{- 2 a_{12} a_{21}} + 2 a_{22} a_{21} \Big)$$

$$= 1 + \epsilon(a_{11} + a_{22}) + \frac{\epsilon^2}{2}(a_{11} + a_{22})^2 + \mathcal{O}(\epsilon^3)$$

$$= 1 + \epsilon \mathrm{Tr}[B] + \frac{\epsilon^2}{2}(\mathrm{Tr}(B))^2 + \mathcal{O}(\epsilon^3)$$

$$= e^{\epsilon \mathrm{Tr}[B]} \quad \text{(up to order } \epsilon^3\text{)}$$

ex. 3

$c = \cos\theta$

$s = \sin\theta$

3.1)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Ergo:

$$f(x(u,v), y(u,v)) = \frac{1}{2} a^2 (cu - sv)^2 + \frac{1}{2} b^2 (su + cv)^2$$

$$+ \alpha (cu - sv)(su + cv) =$$

$$= \frac{1}{2} a^2 c^2 u^2 + \frac{1}{2} a^2 s^2 v^2 - a^2 cs(uv)$$

$$+ \frac{1}{2} b^2 s^2 u^2 + \frac{1}{2} b^2 c^2 v^2 + b^2 sc(uv)$$

$$+ \alpha c s u^2 + \alpha c^2 (uv) - \alpha s^2 (uv) - \alpha s c v^2 =$$

$$\boxed{\begin{aligned} &= \frac{1}{2} u^2 \left(a^2 c^2 + b^2 s^2 + 2\alpha cs\right) + \frac{1}{2} v^2 \left(a^2 s^2 + b^2 c^2 - 2\alpha sc\right) \\ &+ uv\left(-a^2 cs + b^2 sc + \alpha c^2 - \alpha s^2\right) = f(x(u,v), y(u,v)) \end{aligned}}$$

3.2)

$$f(x(u,v), y(u,v)) = \frac{1}{2}\tilde{a}^2 u^2 + \frac{1}{2}\tilde{b}^2 v^2$$

if:

$$-a^2 cs + b^2 sc + \alpha(c^2 - s^2) = 0 .$$

$$\alpha(c^2 - s^2) = (a^2 - b^2)\, sc$$

$$\alpha \cos(2\theta) = \frac{(a^2 - b^2)}{2} \cdot \sin(2\theta)$$

$$\boxed{\tan(2\theta) = \frac{2\alpha}{a^2 - b^2}}$$

$$\boxed{\theta = \frac{1}{2}\arctan\left(\frac{2\alpha}{a^2 - b^2}\right)}$$

$$\begin{cases} \tilde{a}^2 = a^2 c^2 + b^2 s^2 + 2\alpha sc \\ \tilde{b}^2 = a^2 s^2 + b^2 c^2 - 2\alpha sc \end{cases}$$

3.3

$$\tan(2\vartheta) = \infty \rightarrow \quad 2\vartheta = \frac{\pi}{2} \mapsto \vartheta = \frac{\pi}{4}$$

Ergo:

$$\boxed{a = b}$$

Ex 4)

$$B^t A B = D = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$$

$$B^t A B D^{-1} = D D^{-1} = 1_N$$ whereas $D^{-1} = \mathrm{diag}(\lambda_1^{-1}, \dots, \lambda_N^{-1})$

Multiply from the left with B:

$$\underbrace{B B^t}_{1} A B D^{-1} = B$$

$$A B D^{-1} = B$$

... and then from the right by B^t:

$$A B D^{-1} B^t = B B^t = 1_N$$

Ergo:

$$\boxed{A^{-1} = B D^{-1} B^t}$$