1)

1.1)

$$f(x,y) = \sin(x+y) + y^2$$

$$\partial_x f = \cos(x+y) \longrightarrow (\partial_x f)_{(0,0)} = 1$$

$$\partial_y f = \cos(x+y) + 2y \longrightarrow (\partial_y f)_{(0,0)} = 1$$

$$\partial_x^2 f = -\sin(x+y) \longmapsto (\partial_x^2 f)_{(0,0)} = 0$$

$$\partial_x \partial_y f = -\sin(x+y) \longmapsto (\partial_y \partial_x f)_{(0,0)} = 0$$

$$\partial_y^2 f = -\sin(x+y) + 2 \longmapsto (\partial_y^2 f)_{(0,0)} = 2$$

Ergo:

$$\boxed{f(x,y) = \underline{\underline{x + y + y^2}} + \underbrace{\ldots}_{\text{higher orders}}}$$

1.2)

$$f(x,y) = 3xy - x^3 - y^3$$

$$\partial_x f = 3y - 3x^2$$

$$\partial_y f = 3x - y^2$$

$$(\partial_x f, \partial_y f) = (0,0)$$

for

$$\begin{cases} (x,y) = (0,0) \\ \text{and} \\ (x,y) = (1,1) \end{cases}$$

|| EXTREMA or saddle ?
(maxima, minima)

check the 2° derivatives to understand what kind of points they are:

$$\partial_x^2 f = -6x$$

$$\partial_x \partial_y f = 3$$

$$\partial_y^2 f = -6y$$

Therefore it follows that

$(x,y)=(0,0)$

$$\rightarrow H:\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$$\lambda^3-9=0 \Rightarrow \lambda=\pm 3 \begin{cases} >0 \\ <0 \end{cases}$$

This is a saddle point

$(x,y)=(1,1)$

$$H=\begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$$

$$(-6-\lambda)^2-9=36+\lambda^2+12\lambda-9=0$$

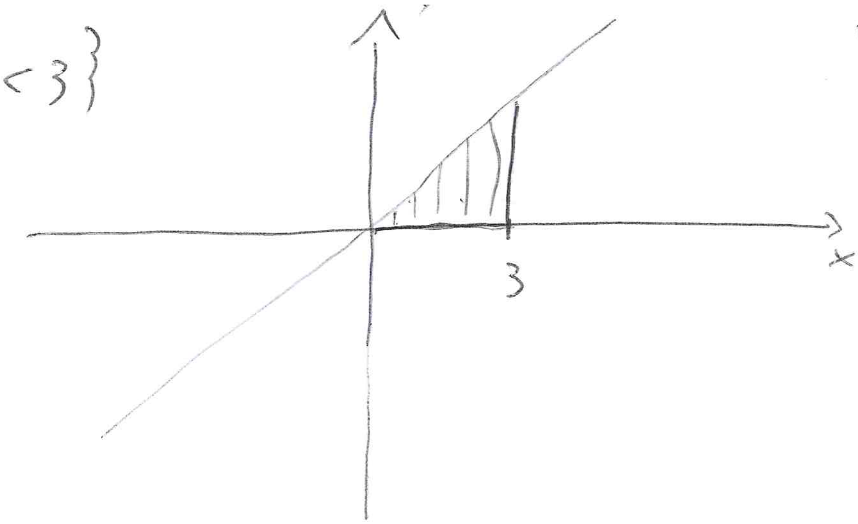
$$\lambda^2+12\lambda+27=0$$

$$\lambda_{1/2}=\frac{-12\pm\sqrt{12^2-27}}{2}$$

$$\lambda_1<0,\ \lambda_2<0 \rightarrow \boxed{(1,1) \text{ is a maximum}}$$

1.3)

$D = \{0 < y < x,\ 0 < x < 3\}$



$$\int_0^3 dx \int_0^x dy \,(3xy - x^3 - y^3) =$$

$$= \int_0^3 dx \left[3x\frac{y^2}{2} - x^3 y - \frac{y^4}{4}\right]_0^x =$$

$$= \int_0^3 dx \left[3x\frac{x^2}{2} - x^4 - \frac{x^4}{4}\right] = \int_0^3 dx \left[\frac{3}{2}x^3 - \frac{5}{4}x^4\right]$$

$$= \left[\frac{3}{2}\frac{x^4}{4} - \frac{5}{4}\frac{x^5}{5}\right]_0^3 = \frac{3}{2}\cdot\frac{3^4}{4} - \frac{5}{4}\frac{3^5}{5} =$$

$$= 3^5\left(\frac{1}{8} - \frac{1}{4}\right) = \boxed{-\frac{3^5}{8}}$$

2)

2.1

$$I = \int_0^{2\pi} d\varphi \int_0^R r\,dr\,\sqrt{R^2-r^2}$$

(This is the result of the transf. $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$)

$w = R^2 - r^2$

$dw = -2r\,dr \Rightarrow r\,dr = -\frac{1}{2}\,dw$

$r = 0 \rightarrow w = R^2$

$r = R \mapsto w = 0$

$$I = 2\pi \int_{-R^2}^{0} -\frac{1}{2}\,dw \cdot w^{1/2} =$$

$$= 2\pi \int_0^{R^2} dw \cdot w^{1/2} = \pi \cdot \left[\frac{w^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^{R^2} =$$

$$= \pi \cdot \left[\frac{w^{3/2}}{3/2}\right]_0^{R^2} = \underline{\underline{\frac{2}{3}\pi R^3}}$$

q.e.d.

$$I = \int_D dx\, dy\; e^{y-2x}$$

$x = u + v$

$y = 2v + 3u$

$$J_{\Phi} = \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \right| = |2-3| = 1$$

Ergo:

$y - 2x = 3u + \cancel{2v} - 2u - \cancel{2v} = u$: $y - 2x < 0 \mapsto u < 0$

$y - 3x = 2v + \cancel{3u} - \cancel{3u} - 3v = -v \to 0 < +v < 1$

Ergo:

$$I = \int_{-\infty}^{0} du \underbrace{\int_{0}^{1} dv}_{1}\; e^{u} = \left[e^{u} \right]_{-\infty}^{0} = \underline{\underline{1}} \;!!!$$

2.3

$$\begin{pmatrix} \cos\varphi \sin\theta & -r\sin\varphi\cos\theta & -r\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{pmatrix}$$

The det is:

$$\cos\varphi\sin\theta\left(-r^2\cos\varphi\sin^2\theta\right) + r\sin\varphi\cos\theta\left(-r\sin\varphi\sin^2\theta - r\cos\varphi\sin\theta\cos\theta\right)$$

$$- r\sin\theta\left(-r\cos\varphi\sin\theta\cos\theta\right) =$$

$$= -r^2\cos^2\varphi\sin\theta - r^2\sin\theta\sin^2\varphi = -r^2\sin\theta \;!!$$

$$\Rightarrow r^2\sin\theta. \quad \theta\in(0,\pi)$$

It follows: $R=3$

$$J = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0 r^2\, dr$$

Άρα:

$$J = 2\pi \cdot (-\cos\theta)\Big|_0^{\pi} \cdot \frac{R^3}{3} = \frac{4}{3}\pi R^3 = \frac{4}{\cancel{3}}\pi \cdot \cancel{27}^{\,9} =$$

$$= 36\pi$$

Ex 3)

$$\partial_x f(x,y) = x^2$$

$$f(x,y) = \frac{x^3}{3} + g(y)$$

$$f(1,y) = \frac{1}{3} + g(y) = \sin(y) + 3$$

$$\rightarrow g(y) = \sin(y) + 3 - \frac{1}{3} = \sin y + \frac{8}{3}.$$

Άρα:

$$f(x,y) = \frac{x^3}{3} + \frac{8}{3} + \sin(y)$$

Check:

$\partial_x f = x^2$! $\quad f(1,y) = \frac{1}{3} + \frac{8}{3} + \sin y = 3 + \sin y$