Aufgabe 1.1

$$f(x,y) = \frac{xy^2}{x^5+y^5}$$

For $y = x$ I get $F(x) = f(x,x) = \frac{x^3}{x^5+x^5} = \frac{1}{2x^2}$

$$\lim_{x \mapsto 0} F(x) = \infty$$

If, on the contrary we chose $y = x^2$ we get

$$G(x) = f(x,x^2) = \frac{x^5}{x^5+x^{10}} \simeq \frac{x^5}{x^5} \simeq 1$$

(in fact for x small $x^{10} << x^5$: $\lim_{x \mapsto 0} \frac{x^{10}}{x^5} = 0$ )

Formally:

$$\lim_{x \mapsto 0} G(x) = \lim_{x \mapsto 0} \frac{x^5}{x^5+x^{10}} = \lim_{x \mapsto 0} \frac{1}{1+x^5} = 1$$

$F(x)$ and $G(x)$ have different limits for $x \mapsto 0$:

$\lim_{(x,y) \mapsto (0,0)} f(x,y)$ does <u>not</u> exist.

2) $f(x,y) = \ln(x^2 - y)$

$x^2 - y > 0$

Ergo:

$f(x,y): D \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$

whereas $D = \{(x,y) : y < x^2\}$

3) $f = \ln(x^2 - y)$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 - y} \qquad \frac{\partial f}{\partial y} = -\frac{1}{x^2 - y}$$

$$\frac{\partial^2 f}{\partial x^2} = -2\frac{x+y}{(x^2-y)^2} \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{2x}{(x^2-y)^2} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{(x^2-y)^2}$$

4) $f = x^\alpha y^\beta$

$$\partial_x f = \frac{\partial f}{\partial x} = \alpha x^{\alpha-1} y^\beta$$

$$\partial_y \partial_x f = \alpha \beta x^{\alpha-1} y^{\beta-1}$$

$$\begin{cases} \partial_y f = \beta x^\alpha y^{\beta-1} \\ \partial_x \partial_y f = \beta \alpha x^{\alpha-1} y^{\beta-1} \end{cases}$$

Ergo :

$$\partial_y \partial_x f = \partial_x \partial_y f \qquad \forall\, \alpha, \beta$$

Ex. 2)

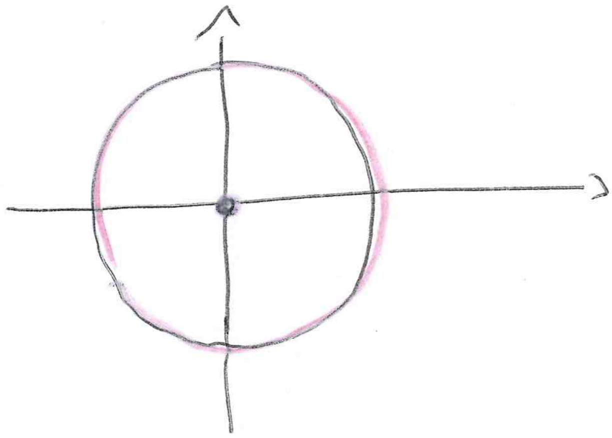
$$f(x,y) = (x^2+y^2-1)^2$$

$$\vec{\nabla} f = (\partial_x f, \partial_y f) = \left( 4x(x^2+y^2-1) \,,\, 4y(x^2+y^2-1) \right)$$

$$\vec{\nabla} f = 0$$

for

$$\begin{cases} x = y = 0 \\ \quad \vee \text{ (or)} \\ x^2+y^2 = 1 \end{cases}$$



Points where $\vec{\nabla} f = 0$.

$x = y = 0$

$\vec{\nabla} f = 0 = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$

$$\begin{cases} \vec{T}_1 = (1,0,0) \\ \vec{T}_2 = (0,1,0) \end{cases}$$

Similarly, these are also the tg-vectors for the points $(1,0)$ and $(1/\sqrt{2}, 1/\sqrt{2})$ for which $\vec{\nabla} f = (0,0)$.

The situation is different for $(x, y) = (1, 1)$ 

$$\left(\partial_x f\right)_{(1,1)} = 4$$

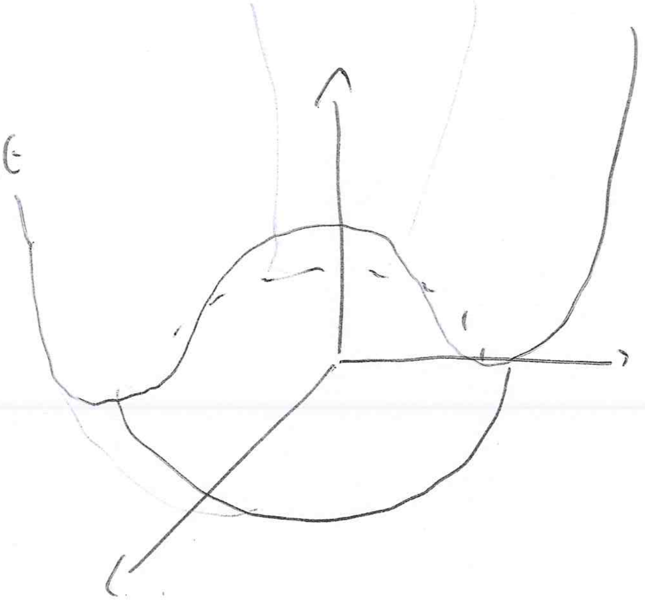
$$\left(\partial_y f\right)_{(1,1)} = 4$$

Ergo:

$$\vec{T}_1 = \frac{1}{\sqrt{1+16}} (1, 0, 4)$$

$$\vec{T}_2 = \frac{1}{\sqrt{17}} (0, 1, 4)$$

3) Mexican hat



Ex. 3

$$f(x,y) = x^2 + y^2 + \alpha x y$$

1) $$\partial_x f = 2x + \alpha y$$

$$\partial_y f = 2y + \alpha x$$

Ergo $\vec{\nabla} f = (0,0)$ for $x = y = 0$.

2) $$\partial_x^2 f = 2$$

$$\partial_y^2 f = 2$$

$$\partial_x \partial_y f = \alpha = \partial_y \partial_x f$$

Ergo the Hesse matrix is

$$H = \begin{pmatrix} 2 & \alpha \\ \alpha & 2 \end{pmatrix}.$$

3) The eigenvalues are found for

$$(2-\lambda)(2-\lambda) - \alpha^2 = 0.$$

$$\lambda^2 - 4\lambda + 4 - \alpha^2 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4(4-\alpha^2)}}{2}$$

For instance, for $\alpha = 0$ $\lambda_1 = \lambda_2 = 2$.

Note that we can rewrite it as:

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 16 + 4\alpha^2}}{2} = \frac{4 \pm 2|\alpha|}{2} = 2 \pm |\alpha|$$

$\lambda_1 = 2 + |\alpha| > 0 \quad \forall \alpha.$

$\lambda_2 = 2 - |\alpha|$

We realise that:

$$\begin{cases} |\alpha| < 2 \rightarrow \lambda_2 > 0 \rightarrow (0,0) \text{ is a MINIMUM} \\ |\alpha| > 2 \rightarrow \lambda_2 < 0 \rightarrow (0,0) \text{ is a saddle point.} \end{cases}$$

($\alpha = \pm 2$ is a boundary case...)