1.1)

$$\int dx\, x^2 \sin(x^3)$$

$z = x^3$

$dz = 3x^2 dx \rightarrow x^2 dx = \frac{dz}{3}$

$$\int \frac{dz}{3} \sin(z) = -\frac{1}{3}\cos(z) + c = -\frac{1}{3}\cos(x^3) + c$$

check:

$$\frac{d}{dx}\left(-\frac{1}{3}\cos(x^3) + c\right) = -\frac{1}{3} \cdot 3x^2 (-\sin(x^3)) = x^2 \sin(x^3) \quad \checkmark$$

–

$$\int dx\, e^{-x^2} x$$

$z = x^2$

$dz = 2x\, dx \rightarrow x\, dx = \frac{dz}{2}$

$$\int dx\, e^{-x^2} x = \int \frac{dz}{2} e^{-z} = -\frac{1}{2} e^{-z} + c \equiv -\frac{1}{2} e^{-x^2} + c$$

check:

$$\frac{d}{dx}\left(-\frac{1}{2} e^{-x^2} + c\right) = -\frac{1}{2}(-2x) e^{-x^2} = x e^{-x^2} \quad \checkmark$$

–

$$\int dx \frac{x^3}{1+x^4}$$

$z = 1+x^4$

$dz = 4x^3 dx \qquad x^3 dx = \frac{dz}{4}$

$$\int dx \frac{x^3}{1+x^4} = \int \frac{dz}{4} \frac{1}{z} = \frac{1}{4} \ln z + C = \frac{1}{4} \ln(1+x^4) + C$$

Check:

$$\frac{d}{dx}\left(\frac{1}{4}\ln(1+x^4) + C\right) = \frac{1}{4} \cdot 4x^3 \cdot \frac{1}{1+x^4} = \frac{x^3}{1+x^4} \quad \checkmark$$

#

1.2) $\int \ln x \, dx$

$g'(x) = 1 \rightarrow g(x) = x$

$f(x) = \ln x \rightarrow f'(x) = 1/x$

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - x + C$$

Check:

$$\frac{d}{dx}(x \ln x - x) = \ln x + x \cdot \left(\frac{1}{x}\right) - 1 = \ln x \quad \checkmark$$

$$\int dx\, x^2 \ln x$$

$$g'(x) = x^2 \to g(x) = \frac{x^3}{3}$$

$$f(x) = \ln x \mapsto f'(x) = \frac{1}{x}$$

$$\int dx\, x^2 \ln x = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx = \frac{x^3}{3} \ln x - \int \frac{x^2}{3}\, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + c$$

clear:

$$\frac{d}{dx}\left(\frac{x^3}{3} \ln x - \frac{x^3}{9} + c\right) = \frac{3x^2}{3} \ln x + \cancel{\frac{x^3}{3} \cdot \frac{1}{x}} - \cancel{\frac{3x^2}{9}} = x^2 \ln x \quad \checkmark$$

1.3) $I = \int_{-\infty}^{\infty} dx\, x^4 e^{-x^2}$.

We introduce $\alpha$:

$$I(\alpha) = \int_{-\infty}^{\infty} dx\, x^4 e^{-\alpha x^2} \qquad I(\alpha = 1) = I.$$

$$I(\alpha) = -\frac{d}{d\alpha} \int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} = \frac{d^2}{d\alpha^2}\left[\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}\right] = \frac{\partial^2}{\partial \alpha^2} \cdot \left(\sqrt{\frac{\pi}{\alpha}}\right) =$$

$$= \sqrt{\pi} \cdot \frac{d^2}{d\alpha^2}\left(\alpha^{-1/2}\right) = \sqrt{\pi} \cdot \frac{d}{d\alpha}\left(-\frac{1}{2} \alpha^{-3/2}\right) = \sqrt{\pi}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \alpha^{-5/2}$$

$$\boxed{\text{Ergo: } I = I(\alpha = 1) = \frac{3}{4}\sqrt{\pi}.}$$

3'

$$f(x) = \int_0^x dz\, e^{-z^2}$$

$$f(0) = 0$$

$$f'(x) = e^{-x^2} = \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m x^{2m}$$

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \frac{x^{2m+1}}{2m+1} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(2m+1)} x^{2m+1} .$$

2.1) (N.b.: the point $x = 1$ is "regular"... no problem there).

$$I = \int_1^\infty dx \, \frac{x+2}{x^\beta + 1}$$

$f(x) = \dfrac{x+2}{x^\alpha + 1}$ behaves as $\dfrac{1}{x^{\beta+1}} = x^{1-\beta}$ for $x \mapsto \infty$

We can then apply 1-to-1 the reasoning of the handout.

For

$\alpha = 1 - \beta < -1$ it converges, otherwise it diverges.

We then get

$$\boxed{\beta > 2}.$$

2.2)

$$I = \int_0^1 dx \, \frac{\sin x}{x^\beta} \frac{1}{x^{100} + 1}$$

(N.b.: the point $x=1$ is regular for the integrand. All ok there...)

For $x \mapsto 0$ one has

$$f(x) \sim \frac{x}{x^\beta} \cdot 1 = x^{1-\beta} = x^\alpha$$

One has convergence in this case for $\alpha > -1$

$1 - \beta > -1$

$$\boxed{\beta < 2}$$

2.31 

$$I = \int_0^\infty dx\, x^\beta$$

Here we may have problems both for $x \mapsto 0$ and $x \mapsto \infty$ in dependence of the sign of $\beta$.

$$\int_0^\infty dx\, x^\beta = \underbrace{\int_0^1 dx\, x^\beta}_{\text{convergent for } \beta > -1} + \underbrace{\int_1^\infty dx\, x^\beta}_{\text{convergent for } \beta < -1}$$

$\beta > -1 \cup \beta < -1 = \emptyset$ (empty system).

The integral is $\underline{\underline{\text{never}}}$ convergent, but always divergent.

It either diverges in "0" or in "$\infty$".