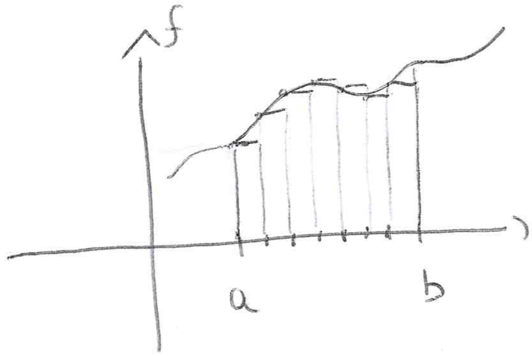
Integrals

$f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$ (a,b) segment $\subseteq D$.



Let us "divide" the segment in N small pieces of length δx / $N\delta x = b-a$.

(For simplicity let us take them all with equal length δx).

Let us also consider the points

$$x_m = a + \delta x \cdot m \qquad m = 0, 1, \dots N$$

We then construct the sum

$$\sum_{m=0}^{N=(b-a)/\delta x} f(x_m)\,\delta x$$

The definite integral $\int_a^b f(x)\,dx$ is defined as the following limit:

$$\int_a^b f(x)\,dx = \lim_{\delta x \mapsto 0^+} \sum_{m=0}^{N=(b-a)/\delta x} f(x)\,\delta x$$

Intuitively: $\delta x \mapsto dx$ "infinitesimal"

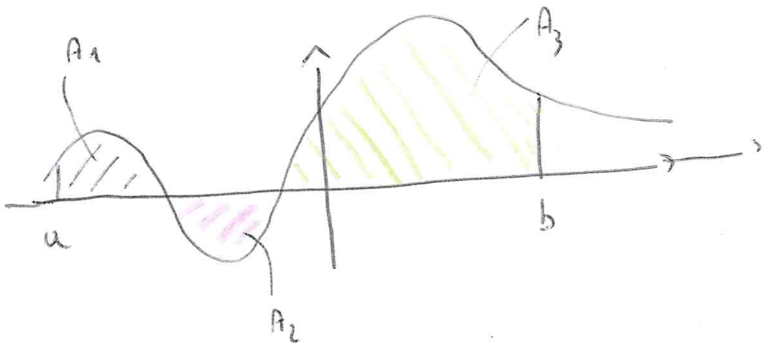
$$\sum_{n=0}^{N} \mapsto \int_a^b \qquad \left(\int = \text{long } S \text{ (Latin)}, \text{ stands for sum} \right)$$

it is clear that if $f(x) > 0$

$$I = \int_a^b f(x)\,dx$$

is the area between the x-axis and the function $f(x)$.

However, in general the sign is not always positive:



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 \,;$$

out of the definition it follows that:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx \quad \text{and} \quad \int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f\,dx + \beta \int_a^b g(x)\,dx.$$

Moreover, one defines that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

In this way the integral is well defined regardless if the lower limit is effectively smaller than the upper limit.

The indefinite integral is defined as

$$\int f(x)\,dx = F(x) + C$$

where $F(x) \,/\, \frac{dF(x)}{dx} = f(x)$.

<u>Property 1</u>

$$F(x) = \int_{x_0}^{x} f(z)\,dz \text{ is a primitive function of } f(x).$$

$$\frac{dF(x)}{dx} = \lim_{h \mapsto 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \mapsto 0} \frac{\int_{x_0}^{x+h} f(z)\,dz - \int_{x_0}^{x} f(z)\,dz}{h} =$$

$$= \lim_{h \mapsto 0} \frac{\cancel{\int_{x_0}^{x} f(z)\,dz} + \int_{x}^{x+h} f(z)\,dz - \cancel{\int_{x_0}^{x} f(z)\,dz}}{h} = \lim_{h \mapsto 0} \frac{f(x)\cdot(x+h-x)}{h}$$

$$= \lim_{h \mapsto 0} \frac{f(x)\cdot h}{h} = f(x) \qquad \text{q.e.d.}$$

Note: $F(x)$ is such that $F(x_0) = 0$ 'per construction'.

The function

$$G(x) = \int_{x_1}^{x} f(z)\,dz$$

is <u>also</u> a primitive of $f(x)$. $F(x)$ and $G(x)$ differ only by a constant

$$F(x) - G(x) = \int_{x_0}^{x} f(z)\,dz + \int_{x}^{x_1} f(z)\,dz = \int_{x_0}^{x_1} f(z)\,dz = \text{constant}.$$

q.e.d.

Property 2:

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b$$

This important property can be proven as follows:

$$\int_a^b f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx = -\int_{x_0}^a f(x)\,dx + \int_{x_0}^b f(x)\,dx = F(b) - F(a)$$

[Note, if we had used $G(x) = \int_{x_1}^x f(x)\,dx$ the result does not change:

$$\int_a^b f(x)\,dx = G(b) - G(a)$$]

Thus, the integral is the 'inverse operation' of the derivative:

$$\int \frac{d\,f(x)}{dx}\,dx = f(x) + c\,;$$

In this sense, both the integration and the derivation can be interpreted as operators on the space of real functions:

$$\begin{cases} \dfrac{d}{dx}: & f(x) \longmapsto f'(x) \\ \int: & f(x) \longmapsto \int f(x)\,dx \end{cases}$$

<u>Elementary integrals:</u>

Being $\int f(x)\,dx = F(x)$ with $\frac{dF}{dx} = f$ we can get the integral of elementary functions by using the properties of derivatives.

For instance:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c; \text{ in fact: } \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{n+1}{n+1}\,x^{n+1-1} = x^n.$$

$$\int \frac{1}{x}\,dx = \ln|x| + c;$$

$$\int e^x\,dx = e^x + c;$$

$$\int \sin x\,dx = -\cos x + c; \quad \int \cos x\,dx = \sin x + c;$$

$$\int \frac{1}{\cos^2 x}\,dx = \tan x + c;$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c;$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + c;$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \ln\left(x + \sqrt{1+x^2}\right) + c;$$

Peculiarity of integrals

$\int f(x)\, g(x)\, dx$ is NOT expressible as a combination of the integrals $\int f(x)dx$ and $\int g(x)dx$. (This is different from derivatives where this was possible: product rule...).

This fact has an important consequence: not all the integrals "can be solved", i.e. not all the integrals can be expressed as a combination of elementary functions.

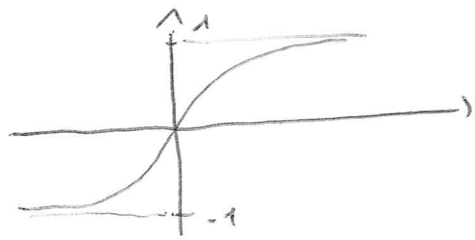
For instance, the integral

$$\int e^{-x^2} dx$$

is of this type. It cannot be 'solved'.

Integrals can then be used to define new functions. In the previous case one defines

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$



It occurs in a variety calculations in statistics and evaluation of errors.

Another notable examples is:

$$\operatorname{li}(x) = \int_0^x \frac{dz}{\ln z}$$

Of course, through $\int$ it is possible to generate an "∞" of new functions.

Integration per substitution:

$\int f(x)dx$

$x = g(z)$

$dx = \frac{dg(z)}{dz} dz$

$\int f(x)dx = \int f(g(z))\frac{dg(z)}{dz} dz = F(z).$

In order to get the result as function of x substitute the inverse function $z = z(x)$ / $x = g(z(x))$.

Examples:

- $\int x e^{-x^2} dx$; $x = \sqrt{z}$; $dx = \frac{1}{2\sqrt{z}} dz$

$\int x e^{-x^2} dx = \int \sqrt{z} \cdot e^{-z} \frac{1}{2\sqrt{z}} dz = \int \frac{e^{-z}}{2} dz = -\frac{e^{-z}}{2} + c = -\frac{e^{-x^2}}{2} + c.$

(Note: WRITING $z = x^2$ IS also OK...) X

- $\int \frac{f'(x)\,dx}{1+f^2(x)}$; $(x = f^{-1}(z)$ / $f(f^{-1}(z)) = z$; but this is not the easier way)

$z = f(x)$ $\quad dz = f'(x)dx$

$\int \frac{f'(x)dx}{1+f^2(x)} = \int \frac{dz}{1+z^2} = \arctan z + c = \arctan f(x) + c$

X: in fact: $z = x^2$; $dz = 2x\,dx$, $x\,dz = \frac{dz}{2}$

$\int x\,dx\,e^{-x^2} = \int \frac{dz}{2} e^{-z} = -\frac{e^{-z}}{2}.$

Integration per parts:

$$\frac{d}{dx}\left(f(x)\, g(x) \right) = f'(x)\, g(x) + f(x)\, g'(x)$$

Integrate:

$$\underbrace{\int \frac{d}{dx}\left(f(x)\, g(x) \right) dx}_{f(x)\, g(x)} = \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

Ergo:

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

Example:

$$\int x \ln x \, dx$$

$g'(x) = x$; $g(x) = x^2/2$

$f(x) = \ln x \mapsto f'(x) = \frac{1}{x}$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{1}{x} \frac{x^2}{2}\, dx = \frac{x^2}{2} \ln x - \frac{1}{2} x^2 + C$$

check:

$$\frac{d}{dx}\left(\frac{x^2}{2} \ln x - \frac{1}{2} x^2 \right) = x \ln x + \frac{x^2}{2} \frac{1}{x} - \frac{1}{2} 2x = x \ln x \ \checkmark$$

Derivation of parameters

$$\int_a^b x e^{-x} dx$$

(I take a definite integral in this case ... I explain later why).

One could solve this integral by parts, but we show here a different way.

$$\int_a^b x e^{-x} dx = \left(\int_a^b x e^{-\alpha x} dx \right)_{\alpha=1} =$$

$\left(\text{Notice that } g(\alpha) = \int_a^b x e^{-\alpha x} dx \right.$

$\left. \text{is a well defined function of } \alpha! \right)$

$$= \left[-\frac{d}{d\alpha} \int_a^b e^{-\alpha x} dx \right]_{\alpha=1} = \left[-\frac{d}{d\alpha} \left(-\frac{1}{\alpha} e^{-\alpha x} \right)_a^b \right]_{\alpha=1}$$

$$= \left[\frac{d}{d\alpha} \left(\frac{1}{\alpha} e^{-\alpha b} - \frac{1}{\alpha} e^{-\alpha a} \right) \right]_{\alpha=1} = \left[-\frac{1}{\alpha^2} e^{-\alpha b} - \frac{b}{\alpha} e^{-\alpha b} + \frac{1}{\alpha^2} e^{-\alpha a} + \frac{a}{\alpha} e^{-\alpha a} \right]_{\alpha}$$

$$= e^{-a} - e^{-b} + a e^{-a} - b e^{-b}.$$

Note that we could have done it also for an indefinite integral using a "partial derivative":

$$\int_a^b x\, e^{-x}\, dx = \left[-\frac{\partial}{\partial \alpha} \underbrace{\int e^{-\alpha x}\, dx}_{} \right]_{\alpha = 1}$$

this is a function of two variables; $g = g(x, \alpha)$.

Derive only w.r.t. α! (Just as if x where a constant parameter).

$$= \left[-\frac{\partial}{\partial \alpha} \left(-\frac{1}{\alpha} e^{-\alpha x} \right) \right]_{\alpha = 1} = \left[-\frac{1}{\alpha^2} e^{-\alpha x} - \frac{x}{\alpha} e^{-\alpha x} \right]_{\alpha = 1}$$

$$= -e^{-x} - x e^{-x} = -(1+x)\, e^{-x}.$$ (+ constant)

Check:

$$\frac{d}{dx} \left(-(1+x) e^{-x} \right) = -e^{-x} + (1+x) e^{-x} = x e^{-x} \quad \checkmark$$