Derivability

def:

$f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$; $x_0 \in D$

$f(x)$ is derivable in x_0 if the limit $\lim_{h \mapsto 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists.

One writes:

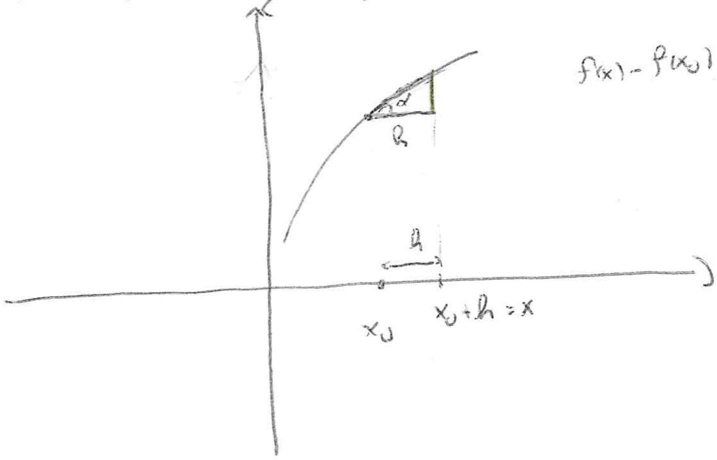
$$\lim_{h \mapsto 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) = \left(\frac{df}{dx}\right)_{x=x_0}$$

Note that, writing $x_0 + h = x$

$$f'(x_0) = \lim_{x \mapsto x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Geometrical meaning: $f'(x_0) = \tan \alpha$ (tangent to the function in x_0)

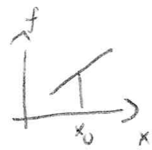
$f(x) - f(x_0) \simeq \tan\alpha \cdot (x - x_0)$ for x very close to x_0



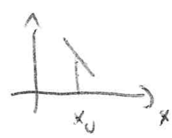
For x very close to x_0:

$f(x) \simeq f'(x_0) \cdot (x - x_0)$;

$\to f'(x_0) > 0$

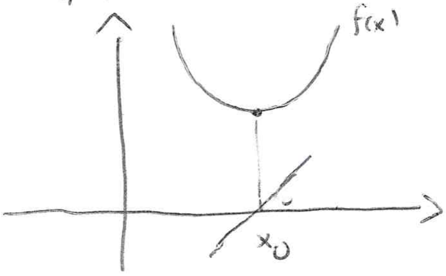


$\to f'(x_0) < 0$

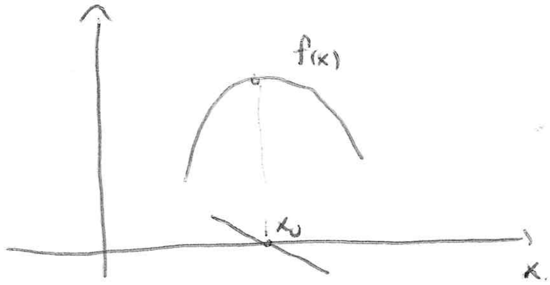


When $f'(x_0) = 0$ ↦ the function is 'flat' (horizontal)

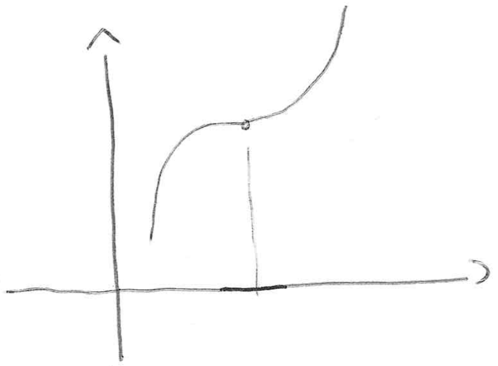
Examples:



A sufficient condition is $f''(x_0) > 0$



A sufficient condition is $f''(x_0) < 0$



A necessary condition is $f''(x_0) = 0$

You have a flex if (sufficient condition) $\begin{cases} f''(x_0) = 0 \\ f'''(x_0) \neq 0 \end{cases}$

In general, the study of derivatives is important in a countless nr of examples in mathematics, physics, engineering, ...

Example 1: derivative of x^2. $f(x) = x^2$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^2 - x_0^2}{h} =$$

$$= \lim_{h \to 0} \frac{x_0^2 + h^2 + 2x_0 h - x_0^2}{h} = \lim_{h \to 0} \frac{h^2 + 2x_0 h}{h} =$$

$$= \lim_{h \to 0} \left(h + 2x_0 \right) = 2x_0 .$$

In general, we write: $\boxed{f'(x) = 2x.}$

$\left(\text{Generalization: } f(x) = x^n \mapsto f'(x) = n\, x^{n-1} \right).$

Example 2: $f(x) = \sin x$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} =$$

$$= \lim_{h \to 0} \sin x \cdot \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} + \lim_{h \to 0} \cos x \cdot \underbrace{\frac{\sin h}{h}}_{1} = \cos x.$$

$$\frac{d}{dx}(\sin x) = \cos x .$$

Similarly, one can evaluate the derivatives of all elementary functions.

Table with derivatives

$f(x)$	$f'(x)$
x^{λ}	$\lambda x^{\lambda-1}$ $(\lambda \in \mathbb{R})$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1+\tan^2 x = \frac{1}{\cos^2 x}$
e^x	e^x
$\arctan x$	$\frac{1}{1+x^2}$
$\arcsin x$	$-\frac{1}{\sqrt{1-x^2}}$

$f(x)$	$f'(x)$
a^x	$\ln(a)\cdot a^x$
$\ln(\lvert x\rvert)$	$1/x$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$

$$\begin{cases} \cosh x = \frac{1}{2}(e^x + e^{-x}) \\ \sinh x = \frac{1}{2}(e^x - e^{-x}) \end{cases}$$

Basic properties:

Product: $[f(x)\cdot g(x)]' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

Ratio: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\cdot g(x) - f(x) g'(x)}{g^2}$

Chain: $f(x) = g(h(x)) \rightarrow f'(x) = g'(h(x))\cdot h'(x)$

example: $f(x) = e^{x^2} \rightarrow f'(x) = 2x\cdot e^{x^2}$.

The product of derivative is important: it assumes that - given the list of elementary functions - the derivative of a combination of elementary functions is also expressable as a (different) combination of elementary functions.

This is not so for the integration.

Theorem: $f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$; $x_0 \in D$; $f(x)$ derivable in x_0 $\Rightarrow$ $f(x)$ is continuous in x_0.

Per hypothesis the limit

$$\lim_{x \mapsto x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists and is well defined. Then, for $x \simeq x_0$ $\quad f(x) - f(x_0) = f'(x_0) \cdot (x - x_0)$.

This means that:

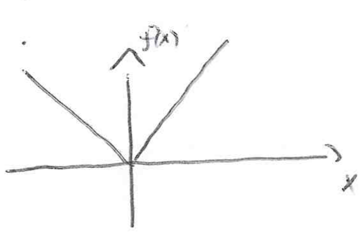
$$\lim_{x \mapsto x_0} \left(f(x) - f(x_0) \right) = \lim_{x \mapsto 0} f'(x_0)(x - x_0) = 0 \Rightarrow \underline{\underline{\lim_{x \mapsto x_0} f(x) = f(x_0)}}.$$

The last term is the definition of continuity in x_0. q.e.d.

$$\boxed{f(x) \text{ derivable in } x_0 \rightarrow f(x) \text{ continuous in } x_0.}$$

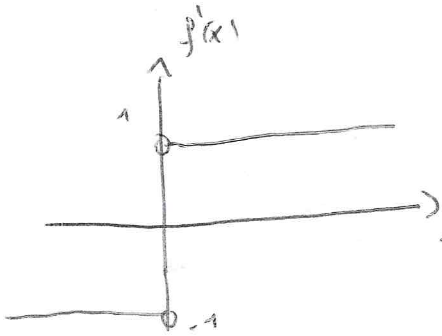
ACHTUNG: the opposite is <u>not</u> true!

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



$$\lim_{x \mapsto 0} f(x) = f(0) = 0.$$

<u>But</u>: $\displaystyle \lim_{h \mapsto 0} \frac{f(h) - f(0)}{h}$ does <u>not</u> exist!



(This is, btw, the sign-function)

In fact, one has:

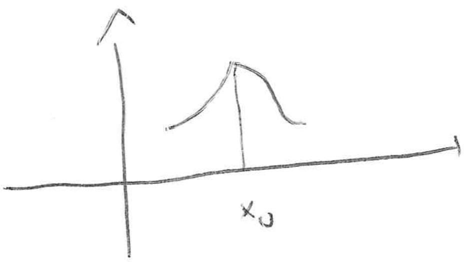
$$\lim_{h \mapsto 0^+} \frac{f(h)}{h} = \lim_{h \mapsto 0^+} \frac{h}{h} = 1$$

and

$$\lim_{h \mapsto 0^-} \frac{f(h)}{h} = \lim_{h \mapsto 0^-} \frac{h}{h} = -1 .$$

Thus, the limit is not determined and therefor $f(x)$ is not derivable.

In general, wherever you have something like



$f(x)$ is not derivable in x_0.

Summarizing: (DOES **NOT** imply that)

f continuous in x_0 $\cancel{\Longrightarrow}$ f is derivable in x_0.

Taylor series around the point $x_0 = 0$ (also called McLaurin)

$f(x): D \subseteq \mathbb{R} \mapsto \mathbb{R}$; $x_0 = 0 \in D$.

We write the function $f(x)$ as a sum of polynomes:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

What are the numbers $a_0, a_1, a_2, \dots$?

$$f(0) = a_0 + a_1 \cdot 0 + \dots = a_0 \qquad \Rightarrow \quad a_0 = f(0).$$

$$\begin{cases} f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots \\ f'(0) = a_1 \end{cases} \qquad \Rightarrow \quad a_1 = f'(0).$$

$$\begin{cases} f''(x) = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \dots \\ f''(0) = 2a_2 \end{cases} \qquad \Rightarrow \quad a_2 = \frac{1}{2} f''(0).$$

$$\begin{cases} f'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4 x + \dots \\ f'''(0) = 3 \cdot 2\, a_3 \end{cases} \qquad \Rightarrow \quad a_3 = \frac{1}{3 \cdot 2} f'''(0).$$

...

In general one gets: $\boxed{a_n = \frac{1}{n!} f^{(n)}(0)}$

Thus:

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} f^{(m)}(0)\, x^m$$

Some considerations are necessary:

- $f(x) = e^x = \sum_{m=0}^{\infty} \frac{1}{m!} x^m$

  This is valid for each $x \in \mathbb{R}$ (which is the domain of the function).

- $f(x) = \frac{1}{1-x}$; $D = (-\infty, 1) \cup (1, \infty)$

  $f(x) = \sum_{m=0}^{\infty} x^m$ (i.e. : $f^{(m)}(0) = m!$, therefore each coeff. $a_m = 1$ in this case!)

  However, this equivalence is valid only if $x = (-1, 1)$. This is namely the region of convergence of the series. (for $x=2$: $\sum_{n=0}^{\infty} 2^n = \infty \ldots$)

In general; $f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)\, x^m}{m!}$ is valid for $x \in D$, where D is the domain of $f(x)$ and x is part of D for which the summation is finite.

The precise definitions of "convergence" goes beyond the present discussion, but the intuitive meaning should be clear.

The Taylor-expansion is a good way to approximate a certain function f(x).

$$f(x) = \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{N} a_m x^m + O_q(x^{m+1}) \approx \sum_{m=0}^{N} a_m x^m .$$

That is, in the vicinity of $x_0 = 0$ we can approximate the function f(x) up to a given order.

<u>Example.</u>

$f(x) = \sin x$

$f(0) = \sin(0) = 0 = a_0;$

$f'(x) = +\cos x; \quad f'(0) = 1 = a_1;$

$f''(x) = -\sin x; \quad f''(0) = 0 = a_2;$

$f'''(x) = -\cos x; \quad f'''(0) = -1; \rightarrow a_3 = -\frac{1}{3!}$

$$\boxed{\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} X^{2m+1}}$$

(In this case $X = \mathbb{R}$).

When approximating $\sin x$ with $f(x, N) = \sum_{m=0}^{N} \frac{(-1)^m}{(2m+1)!} X^{2m+1}$ the approximation gets better and better - also for large x - when taking N large enough.

"Important" expansions:

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} = 1 - \frac{1}{2!} x^2 + \frac{x^4}{4!} + \dots$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1} = x - \frac{1}{3!} x^3 + \dots$$

$$\cosh x = \sum_{m=0}^{\infty} \frac{1}{(2m)!} x^{2m} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\sinh x = \sum_{m=0}^{\infty} \frac{1}{(2m+1)} x^{2m} = 1 + \frac{1}{3!} x^3 + \dots$$

$$\ln(1+x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{m} x^m = 1 + x - \frac{1}{2} x^2 + \dots$$

Comment on Taylor expansion and <u>perturbation</u> theory:

$$f(x) = e^{\lambda x} = 1 + \lambda x + \frac{\lambda^2 x^2}{2} + O_3(x^3)$$

λ = small parameter (such as coupling constant in perturbation theory)

The smaller is λ, the better is the approximation, even if we stop at the second order.

In Physic very often we can only calculate the "approximations" of the form $1 + \lambda x + \frac{\lambda^2 x^2}{2} + \dots$ but not the "full" result $e^{\lambda x}$.

Generic x_0

obviously, there is nothing special in having chosen $x_0 = 0$ as a starting point of our expansion. One can repeat all the steps leading to:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad a_n = \frac{1}{n!} \left(\frac{d^n f(x)}{dx^n} \right)_{x = x_0} = \frac{1}{n!} f^{(n)}(x_0).$$

L'Hopital's rule

consider $f(x), g(x) \; / \; \lim_{x \mapsto x_0} f(x) = \lim_{x \mapsto x_0} g(x) = 0.$

$$\lim_{x \mapsto x_0} \frac{f(x)}{g(x)} = \left[\frac{0}{0} \right].$$

Now, by Taylor-expanding both function around x_0:

$$f(x) = a_1 (x - x_0) + a_2 (x - x_0)^2 + \dots \qquad f'(x) = a_1 + 2 a_2 (x - x_0) + \dots$$

$$g(x) = b_1 (x - x_0) + b_2 (x - x_0)^2 + \dots \qquad g'(x) = b_1 + 2 b_2 (x - x_0) + \dots$$

Let us assume that $a_1 \neq 0 \vee b_1 \neq 0$ (we exclude $a_1 = b_1 = 0$), then:

$$\lim_{x \mapsto x_0} \frac{f(x)}{g(x)} = \lim_{x \mapsto x_0} \frac{a_1 (x - x_0) + a_2 (x - x_0)^2}{b_1 (x - x_0) + b_2 (x - x_0)^2} = \frac{a_1}{b_1} = \lim_{x \mapsto x_0} \frac{f'(x)}{g'(x)}$$

N.b.:

$$\lim_{x \mapsto x_0} \frac{f(x)}{g(x)} = \frac{a_1}{b_1} \equiv \begin{cases} 0 & \text{if } a_1 = 0 \\ \pm\infty & \text{if } b_1 = 0 \\ \frac{a_1}{b_1} \in \mathbb{R} \setminus \{0\} & \text{if } a_1 \neq 0, b_1 \neq 0. \end{cases}$$

Thus:

$$\lim_{x \mapsto x_0} \frac{f(x)}{g(x)} = \lim_{x \mapsto x_0} \frac{f'(x)}{g'(x)}$$

In the case in which $f'(x_0) = g'(x_0) = 0$ one can go further and evaluate the 2-nd derivative:

$$\lim_{x \mapsto x_0} \frac{f(x)}{g(x)} = \lim_{x \mapsto x_0} \frac{f'(x)}{g'(x)} = \lim_{x \mapsto x_0} \frac{f''(x)}{g''(x)}$$

and so on and so forth.

Example:

$$\lim_{x \mapsto 0} \frac{\sin x}{x} = \lim_{x \mapsto 0} \frac{\cos x}{1} = \cos 0 = 1.$$

It is a very simple and useful trick to calculate limits.

It can be easily generalized to indeterminate form of the type $\left[\frac{\infty}{\infty}\right]$ and also for the case $x_0 = \pm\infty$.

A peculiar example:

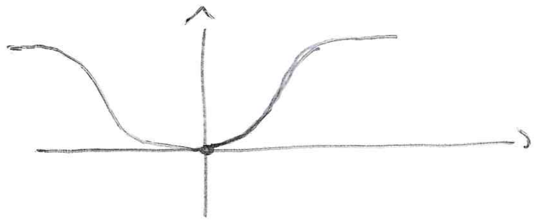
$$f(x) = e^{-\frac{1}{x^2}}$$

strictly speaking, $x=0$ is not a point of the domain of $f(x)$.

However, one can easily see that

$\lim_{x\mapsto 0} f(x) = e^{-\infty} = 0$ → the function can be easily continued in $x=0$.

$\lim_{x\mapsto \pm\infty} f(x) = 0$



$f'(x) = 2x^{-3} e^{-1/x^2}$

$f'(0) = 0$.

if we try to expand the function we find that $f^{(n)}(0) = 0$.

The Taylor expansion is zero: This has clearly to do with the fact that $x=0$ was not originally a point of D.

Moreover, this kind of singularity is very peculiar:

if the exact result of a calculation is

$$e^{-\frac{1}{\lambda^2 x^2}}$$

whereas λ is a small parameter. In this case we cannot expand the result as a series in λ... this is an example of a "non-perturbative" result.