Differential equations

obviously, diff. eq. in more dimensions can be very complicated.

They are expressed via "partial derivatives" ...

$$F\left(\frac{\partial^n f}{\partial x^n}, \frac{\partial^n f}{\partial x^{n-1}\partial y}, \dots \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = 0$$

"Field theory" and "Quantum field theory" deal with such equations.

Here we can simply discuss some basic examples.

Very Simple differential eq. of 1° order with partial derivatives

$$\frac{\partial f(x,y)}{\partial x} = y$$

Solution:

$$f(x,y) = yx + \underbrace{g(y)}_{\text{generic function...}}$$

There is not only an initial condition, but an initial function...

For instance

$$\boxed{f(1,y) = y^2}$$ "initial condition"

$$\Rightarrow g(y) = y^2 - y$$

$$\boxed{f(x,y) = xy + y^2 - y}$$

Let us now consider a system of 2 diff. eqs:

$$\begin{cases} \partial_x f = x^2 \\ \partial_y f = y^3 \end{cases}$$

$$f = \frac{x^3}{3} + g(y)$$

$$\partial_y f = \frac{\partial g}{\partial y} = y^3 \rightarrow g(y) = \frac{y^4}{4} + c$$

$$\boxed{f(x,y) = \frac{x^3}{3} + \frac{y^4}{4} + c}$$

In this case one initial condition

(such as $f(0,0) = 0 \rightarrow c = 0$) is enough).

NOTE: Indeed, not always there are solutions out of such a system:

$$\begin{cases} \partial_x f = xy \\ \partial_y f = x \end{cases}$$

out of the first eq:

$$f(x,y) = \frac{x^2}{2} y + g(y)$$

$$\partial_y f = \frac{x^2}{2} + g'(y) = x$$

However, $g'(y)$ is a function of y and not of x...

If you try to force it:

$$g'(y) = x - \frac{x^2}{2}$$

$$g(y) = \left(x - \frac{x^2}{2}\right) y$$

You then spoil the 1° eq: $f(x,y) = \frac{x^2}{2} y + \left(x - \frac{x^2}{2}\right) y$

$$\boxed{\partial_x f = xy + (1-x)y}$$

Next step: 3 important equations with partial derivatives up to 2° order.

Laplace equation, wave equation, and diffusion equation.

Note that:

- 2° order are "natural"... they occur in most physical problems.
(Newton eq. is of 2° order, but also the eq. for electric and magnetic fields are such... this indeed goes back to the fact that the $\mathcal{L}$ (Lagrangian) contains only terms with 1° order).

- We study the "simple form of the sols." and also we do not take into account "sources"...

Operator: $\Delta = \partial_x^2 + \partial_y^2$

Let us study

$$\boxed{\Delta f(x,y) = 0}$$

A simple solution can be found by including only terms up to 1° order and a mix-term:

$$f(x,y) = \alpha + \beta x + \gamma y + \delta x y$$

In fact.

$$\begin{cases} \partial_x^2 f(x,y) = 0 \\ \partial_y^2 f(x,y) = 0 \end{cases}$$

separately!

However this is only a "limited" form of the possible solution.

In fact, we can have $f(x,y)$ such that

$$\partial_x^2 f = -\partial_y^2 f \neq 0$$

$$\Downarrow$$

$$(\partial_x^2 + \partial_y^2) f = 0$$

A trick, often used to solve partial diff. eq., is to make a "Separationansatz":

$$\boxed{f(x,y) = U(x)W(y)}$$

Then:

$$0 = \left(\partial_x^2 + \partial_y^2\right) f(x,y) = \left(\partial_x^2 + \partial_y^2\right) U(x)W(y)$$

$$= \left[\partial_x^2 U(x)\right] W(y) + U(x)\left[\partial_y^2 W(y)\right] = 0$$

$$\left[\partial_x^2 U(x)\right] W(y) = -U(x)\left[\partial_y^2 W(y)\right]$$

$$\frac{\partial_x^2 U(x)}{U(x)} = -\frac{\partial_y^2 W(y)}{W(y)}$$

There is only one possibility!

$$\frac{\partial_x^2 U(x)}{U(x)} = -\frac{\partial_y^2 W(y)}{W(y)} = K = \text{const.}$$

For simplicity, let us consider $K > 0$!

$$\frac{\partial_x^2 U(x)}{U(x)} = K$$

$$\boxed{\frac{d_x^2 U}{dx^2} - K U(x) = 0}$$

$$U(x) = \left(\alpha e^{\sqrt{K}x} + \beta e^{-\sqrt{K}x}\right).$$

For the other eq. we get:

$$\frac{d^2 W}{dy^2} + K W(y) = 0$$

$$W(y) = \gamma \sin(\sqrt{K}\, y) + \delta \cos(\sqrt{K}\, y)$$

Ergo the full solution (with separation):

$$\boxed{f(x,y) = \left(\alpha e^{\sqrt{K}x} + \beta e^{-\sqrt{K}x}\right)\left(\gamma \sin(\sqrt{K}\, y) + \delta \cos(\sqrt{K}\, y)\right)}$$

Diffusion equation :

$$\boxed{\left[\kappa \partial_t - \partial_x^2\right] \phi(t, x) = 0}$$

Write down the "separation postulate"

$$\phi(t, x) = U(x)\, W(t)$$

$$\Downarrow$$

Find a solution of this form.

Wave eq.:

$$\phi(t, x)$$

$$\boxed{\left(\frac{1}{c^2}\partial_t^2 - \partial_x^2\right)\phi(t, x) = 0}$$

$$\Box = \frac{1}{c^2}\partial_t^2 - \partial_x^2 \qquad \text{"Quabla"}$$

We could also work with the "separation ansatz" and we would find a "special solution".
However, it is possible to find the most general solution by studying

$$f(x - ct)$$

(real function which depend only on $(x - ct)$ ...)
close to $x - ct = 0$ we can express it through a Taylor series

$$f(x-ct) = \sum_{n=0}^{\infty} c_n (x-ct)^n .$$

WAY 1: Let us show that

$$f(x-ct) = (x-ct)^{\alpha}$$

is a solution of the wave eq. (independently on α).

$$\partial_t f(x-ct) = -c\,\alpha\,(x-ct)^{\alpha-1}$$

$$\partial_t^2 f(x-ct) = (-c)(-c)\,\alpha(\alpha-1)\,(x-ct)^{\alpha-2}$$

$$= c^2\,\alpha(\alpha-1)(x-ct)^{\alpha-2}$$

$$\partial_x f(x-ct) = \alpha\,(x-ct)^{\alpha-1} \quad ;$$

$$\partial_x^2 f(x-ct) = \alpha(\alpha-1)\,(x-ct)^{\alpha-2} \quad ;$$

$$\frac{1}{c^2}\partial_t^2 f(x-ct) = \alpha(\alpha-1)\,(x-ct)^{\alpha-2} = \partial_x^2 f(x-ct)$$

q.e.d.

Being α general we have shown that $f(x-ct)$ (which, at least close to $x-ct \simeq 0$, can be expressed in a general form as a power series) is a solution.

Way 2: $f(x-ct)$

$f(\xi)$ whereas $\xi = x - ct$

$$\partial_t f(\xi) = \frac{d f(\xi)}{d\xi} \frac{\partial \xi}{\partial t}$$

$$\begin{cases} \partial_t^2 f(\xi) = \dfrac{d^2 f(\xi)}{d\xi^2}\left(\dfrac{\partial \xi}{\partial t}\right)^2 = c^2 \dfrac{d^2 f(\xi)}{d\xi^2} \\ \partial_x^2 f(\xi) = \dfrac{d^2 f(\xi)}{d\xi^2}\left(\dfrac{\partial \xi}{\partial x}\right)^2 = \dfrac{d^2 f(\xi)}{d\xi^2} \end{cases}$$

Ergo

$$\left(\frac{1}{c^2}\partial_t^2 - \partial_x^2\right) f(x-ct) = 0$$ is a sol.

It is important to understand that $f(\xi)$ is really generic:

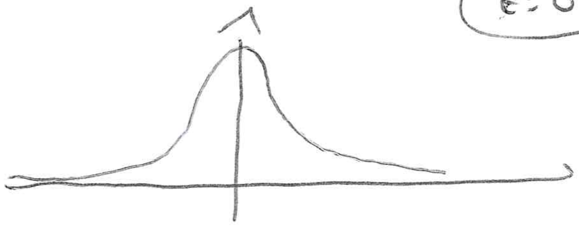
$\sin(x-ct)$, $\tanh(x-ct)$, ...

These are all solutions!!!!

Note that, if $f(x)$ is a function (such as a gaussian ...)

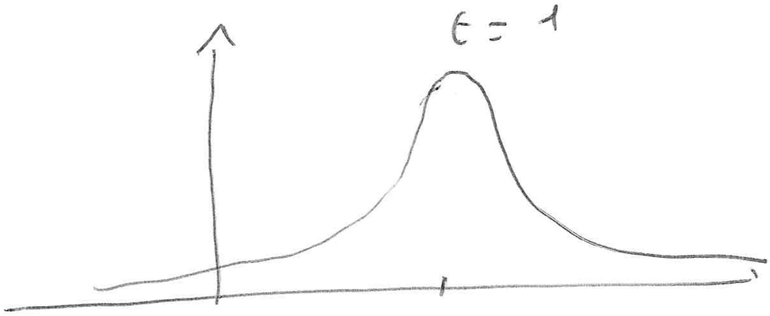
$t = 0$

$$f(x) = e^{-\alpha x^2}$$



Then

$$f(x - ct) = e^{-\alpha (x - ct)^2}$$



The "wave packet" moves from the left to the right.

However, there is another possibility: wave moving to the left...

$$g(x+ct)$$

is also a solution.

Therefore:

$$\phi(t,x) = f(x-ct) + g(x+ct)$$

is the most general sol. of $\Box\phi(t,x)=0$

$f(x)$ and $g(x)$ are generic function.

A unique solution is obtained by considering initial solution:

$$\begin{cases} \phi(0,x) = \alpha(x) \\ \left[\partial_t \phi(t,x)\right]_{t=0} = \varphi(x) \end{cases}$$

$\alpha(x)$ and $\varphi(x)$ are given functions...

The solution is then unique.

Example:

$$\begin{cases} \phi(t=0,x) = e^{-\lambda x^2} \\ \partial_t \phi(t=0,x) = 0 \end{cases}$$

$$\boxed{\phi(t=0,x) = f(x) + g(x) = e^{-\lambda x^2}}$$

$$\partial_t \phi(t,x) = -c\frac{df(\xi)}{d\xi} + c\frac{dg(\eta)}{d\eta} \qquad \begin{aligned} \xi &= x - ct \\ \eta &= x + ct \end{aligned}$$

$$\Big[\partial_t \phi(t,x)\Big]_{t=0} = -c\frac{df(x)}{dx} + c\frac{dg(x)}{dx}$$

Ergo here: $\quad f' - g' = 0$

$$f + g = e^{-\lambda x^2}$$

$$f' - g' = 0$$

$$f(x) - g(x) = K$$

$$f + g = 0 \Rightarrow f = -g$$

$f = g + K$

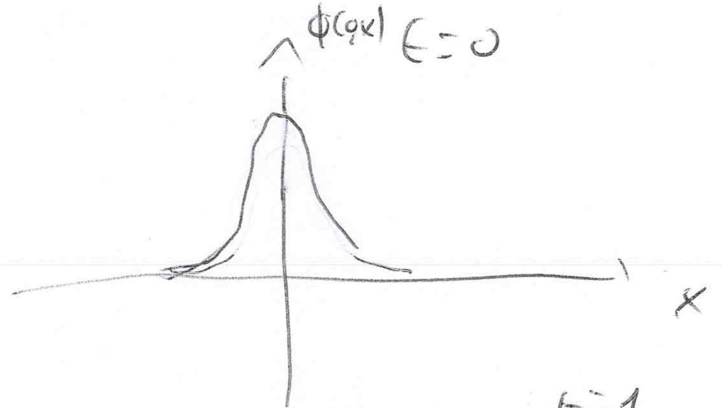
Put it back:

$2f + K = e^{-\lambda x^2}$

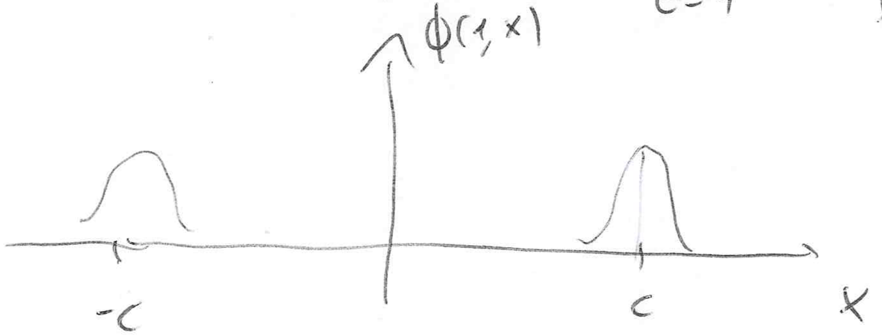
$f(x) = \frac{1}{2} e^{-\lambda x^2} - \frac{K}{2} \Rightarrow g(x) = \frac{1}{2} e^{-\lambda x^2} + \frac{K}{2}$

Ergo, the final solution is:

$$\boxed{\phi(t,x) = \frac{1}{2} e^{-\lambda (x-ct)^2} + \frac{1}{2} e^{-\lambda (x+ct)^2}}$$



Half goes right, half goes left!!



In general for

$$\begin{cases} \phi(t=0,x) = \chi(x) \\ \left[\partial_t \phi(t,x)\right]_{t=0} = \psi(x) \end{cases}$$

$$\Downarrow$$

$$\phi(t,x) = \frac{\chi(x-ct) + \chi(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} dx' \, \psi(x')$$

is the most general solution fulfilling the "initial conditions".