Semi-rigorous statistical inference: fitting multiplicities in Au-Au collisions at $\sqrt{s_{NN}} = 62.4$ and 200 GeV

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Sources


Data:

Multiplicity: Au-Au at $\sqrt{s_{NN}} = 200$ GeV

$|\eta| < 0.26$
Multiplicity: Au-Au at $\sqrt{S_{NN}} = 62.4$ GeV

$| \eta | < 0.26$
Negative binomial distribution

\[
P(n; p, k) = \frac{k(k + 1)(k + 2)\ldots(k + n - 1)}{n!}(1 - p)^n p^k
\]

0 ≤ p ≤ 1,  \(k\) is a positive real number

\(n = 0, 1, 2, \ldots\) - the number of charged particles in an event

\[(k, p) \rightarrow \left( k, \bar{n} = \frac{k(1 - p)}{p} \right)\]

\(\bar{n}\) - expectation value of \(n\)
The least-squares test statistic

$$\chi^2_{LS}(\vec{n}; \bar{n}, k) = \sum_{i=1}^{m} \frac{(n_i - \nu_i(\bar{n}, k))^2}{\text{err}_i^2}$$

- $\vec{n} = (n_1, n_2, \ldots, n_m)$ - vector of data (entries)
- $\text{err}_i$ - uncertainty of the $i$th measurement
- $\nu_i = N \cdot P(i - 1; \bar{n}, k)$ - expected number of entries
- $N = \sum n_i$ - total number of events

The least-squares test statistic

$$\chi^2_{min}(\vec{n}) = \chi^2_{LS}(\vec{n}; \hat{n}, \hat{k})$$

- $\hat{n}, \hat{k}$ - estimators of parameters $\bar{n}$ and $k$
Goodness-of-fit: \( p \)-value

The probability of obtaining the value of the test statistic equal to or greater than the value just obtained for the present data set (i.e. \( \chi^2_{min} \)), when repeating the whole experiment many times (repeating measurement of \( \overrightarrow{n} \)):

\[
p = P(\chi^2 \geq \chi^2_{min}) = \int_{\chi^2_{min}}^{\infty} g(t)dt,
\]

\( g(t) \) - probability density function of \( \chi^2_{min} \), NOT KNOWN USUALLY

\( \chi^2_{min}(\overrightarrow{n}) \) - statistic because a function of multidimensional random variable \( \overrightarrow{n} \)
Assume the significance level $\alpha$ in advance.

($\alpha = 0.1\%$, here)

If $p < \alpha$, a hypothesis should be rejected ("bad fit").

If $p \geq \alpha$, a hypothesis can not be rejected ("good fit").
Errors on PHENIX multiplicity measurements

\[ \text{err}_i^2 = \sigma_{i,\text{stat}}^2 + \sigma_{i,\text{syst}}^2 \]

\[ \sigma_{i,\text{stat}} = \sqrt{n_i} , \quad \sigma_{i,\text{syst}} = 3 \cdot \sigma_{i,\text{stat}} = 3 \cdot \sqrt{n_i} \]

\[ \chi^2_{PHEN} (\vec{n}; \bar{n}, k) = \frac{1}{10} \cdot \sum_{i=1}^m \frac{(n_i - \nu_i(\bar{n}, k))^2}{n_i} = \frac{1}{10} \cdot \chi^2_N (\vec{n}; \bar{n}, k) \]

\( \chi^2_N \) - Neyman’s \( \chi^2 \) test statistic, asymptotically \( \chi^2 \) distributed!

Jerzy Spława-Neyman (1894-1981)

\[ \Rightarrow \text{PHENIX } \chi^2 \text{ function is NOT } \chi^2 \text{ distributed!} \]
the distribution $g(t)$ of a function $t(z)$ of a random variable $z$ with the known p.d.f. $f(z)$:

$$g(t) = f(z(t)) \left| \frac{dz}{dt} \right|$$

$$g(t; n_{dof}) = 10 f(10t; n_{dof})$$

$p$-value of PHENIX test statistic:

$$p = \int_{10 \cdot \chi^2_{PHEN, min}}^{\infty} f(t; n_{dof}) dt$$
$\chi^2$ (chi-square) distribution

$0 \leq t \leq +\infty,$

$n = 1, 2, \ldots$ - the number of degrees of freedom

$$f(t; n) = \frac{1}{2^{n/2} \Gamma(n/2)} t^{n/2-1} \cdot e^{-t/2}$$

$E[t] = n, \quad V[t] = 2n$
Distributions: Au-Au at $\sqrt{s_{NN}} = 200$ GeV

Figure: Uncorrected multiplicity distributions for bins with $n_i > 5$ (left) and $n_i > 60$ (right).
Distributions: Au-Au at $\sqrt{s_{NN}} = 200$ GeV

Figure: Uncorrected multiplicity distributions for bins with $n_i > 5$ (left) and $n_i > 60$ (right).
Results: Au-Au at $\sqrt{s_{NN}} = 200$ GeV

<table>
<thead>
<tr>
<th>Centr.</th>
<th>$N$</th>
<th>$\hat{k}$</th>
<th>$\hat{n}$</th>
<th>$\chi^2_{PHEN}/n_{dof}$</th>
<th>$p$-value</th>
<th>$%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-5</td>
<td>652579</td>
<td>289.0± 2.9</td>
<td>61.86± 0.01</td>
<td>0.57</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5-10</td>
<td>657571</td>
<td>168.1± 1.2</td>
<td>53.91± 0.01</td>
<td>0.61</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10-15</td>
<td>658258</td>
<td>116.4± 0.7</td>
<td>46.50± 0.01</td>
<td>0.53</td>
<td>0</td>
<td>0</td>
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<tr>
<td>15-20</td>
<td>659302</td>
<td>86.9± 0.5</td>
<td>39.72± 0.01</td>
<td>0.43</td>
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<tr>
<td>20-25</td>
<td>658461</td>
<td>69.1± 0.4</td>
<td>33.56± 0.01</td>
<td>0.34</td>
<td>0</td>
<td>0</td>
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<tr>
<td>25-30</td>
<td>659337</td>
<td>57.9± 0.3</td>
<td>28.0± 0.01</td>
<td>0.28</td>
<td>6.7·10^{-8}</td>
<td></td>
</tr>
<tr>
<td>30-35</td>
<td>659021</td>
<td>48.3± 0.3</td>
<td>23.02± 0.01</td>
<td>0.16</td>
<td>0.76</td>
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<tr>
<td>35-40</td>
<td>660937</td>
<td>41.3± 0.2</td>
<td>18.64± 0.01</td>
<td>0.19</td>
<td>0.12</td>
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<tr>
<td>40-45</td>
<td>661422</td>
<td>34.6± 0.2</td>
<td>14.84± 0.01</td>
<td>0.21</td>
<td>0.015</td>
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<tr>
<td>45-50</td>
<td>661577</td>
<td>27.9± 0.2</td>
<td>11.56± 0.005</td>
<td>0.23</td>
<td>0.011</td>
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<tr>
<td>50-55</td>
<td>661877</td>
<td>21.9± 0.1</td>
<td>8.81± 0.004</td>
<td>0.30</td>
<td>7.8·10^{-5}</td>
<td></td>
</tr>
</tbody>
</table>
Distributions: Au-Au at $\sqrt{s_{NN}} = 62.4$ GeV

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<th>$p$-value</th>
<th>$%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-5</td>
<td>607075</td>
<td>227.9±2.5</td>
<td>44.67±0.01</td>
<td>0.19</td>
<td>5.6·10$^{-3}$</td>
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<tr>
<td>5-10</td>
<td>752263</td>
<td>143.9±1.1</td>
<td>37.96±0.01</td>
<td>0.12</td>
<td>14.4</td>
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<tr>
<td>10-15</td>
<td>752739</td>
<td>116.2±0.9</td>
<td>31.53±0.01</td>
<td>0.13</td>
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<td>15-20</td>
<td>752492</td>
<td>88.5±0.6</td>
<td>26.07±0.01</td>
<td>0.11</td>
<td>30.9</td>
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<tr>
<td>20-25</td>
<td>752182</td>
<td>69.2±0.5</td>
<td>21.35±0.01</td>
<td>0.22</td>
<td>2.4·10$^{-3}$</td>
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</tr>
<tr>
<td>25-30</td>
<td>752095</td>
<td>53.6±0.4</td>
<td>17.30±0.01</td>
<td>0.23</td>
<td>1.8·10$^{-3}$</td>
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</tr>
<tr>
<td>30-35</td>
<td>751324</td>
<td>40.3±0.3</td>
<td>13.84±0.005</td>
<td>0.26</td>
<td>4.3·10$^{-4}$</td>
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<td>35-40</td>
<td>751639</td>
<td>31.8±0.2</td>
<td>10.89±0.004</td>
<td>0.15</td>
<td>3.5</td>
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<tr>
<td>40-45</td>
<td>750852</td>
<td>25.2±0.2</td>
<td>8.42±0.004</td>
<td>0.22</td>
<td>0.062</td>
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<tr>
<td>45-50</td>
<td>751348</td>
<td>22.0±0.2</td>
<td>6.41±0.003</td>
<td>343</td>
<td>0</td>
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</tbody>
</table>
Conclusions

1. Caution is necessary, when one infers about quality of a fit when the distribution of a test statistic is not known. Then inference from the condition $\chi^2/n_{dof} \sim 1$ could be confused.

2. Adding statistical and systematic errors in quadrature could change properties of the LS test statistic entirely.

3. As far as PHENIX Au-Au data are concerned, only for 6 from 21 cases of collision energy and centrality the NBD hypothesis can not be rejected.