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Abstracts

On isometric embeddings into the Urysohn universal metric space

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If X is a non-compact Polish space, then X can be embedded in the Urysohn space in such a way that every isometry of X has a unique extension to an isometry of the entire Urysohn space. Some consequences of that will be discussed.

On convergence with respect to a sigma-ideal

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Convergence in measure for sequences of measurable functions on the unit interval is more general than convergence almost everywhere. By the Riesz theorem, f_n is convergent to f in measure on the unit interval if every subsequence of f_n contains a subsequence convergent to f almost everywhere. This equivalence has yielded an abstract notion of convergence for sequences of S-measurable functions with respect to a sigma-ideal J contained in a sigma-algebra S of subsets of a given set Y. In general, J-convergence does not generate a topology but it is possible to introduce J-Cauchy sequences and J-completeness on the space of S-measurable functions on Y. For the unit interval, in the measure case, J-completeness is well known by the Riesz theorem, and in the category case, J-completeness is a nontrivial result due to Wagner-Bojakowska and Wilczynski. We present several operations on abstract measurable spaces which preserve J-completeness. Also, we consider the notions of parametric J-convergence and parametric J-completeness.

On n-reflexive Banach spaces

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In the talk we shall address the following problem posed by Elena Riss in 2000 on the Winter School in Krištanovice (Čech Republic):

Question 1. *Is a separable infinite-dimensional Banach space X reflexive if each net in X has an accumulation point in the weak topology of X?*

By a *net* in a Banach space $(X, \|\cdot\|)$ we understand an ε-net $N \subset X$ for some $\varepsilon > 0$. A subset $N \subset X$ is called an *ε-net* for a subset $B \subset X$ if for every point $x \in B$ there is a point $y \in N$ with $\|x - y\| < \varepsilon$.

It turns out that Question 1 is equivalent to an even more intriguing question concerning ∞-reflexive Banach spaces.

Definition 1. *A Banach space $(X, \|\cdot\|)$ is called* r-reflexive *where $r \in [0, +\infty]$ if for every cover $\mathcal{U}$ of X by weakly open sets there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ that covers an open unit ball $B_1(x) = \{y \in X : \|x - y\| < 1\}$ centered at some point $x \in X$ with $\|x\| \le r$.*

Observe that a Banach space is reflexive if and only if it is 0-reflexive. We shall say that a Banach space X is *ω-reflexive* if it is r-reflexive for some $r \in [0, \infty)$.

It turns out that for infinite-dimensional separable Banach spaces the property appearing in Question 1 is equivalent to the ∞-reflexivity.

Theorem 1. *An infinite-dimensional separable Banach space X is ∞-reflexive (resp. ω-reflexive) if and only if every net in X has an accumulation point (resp. contains a non-trivial convergent sequence) in the weak topology of X.*

So, Question 1 can be reformulated it terms of the r-reflexivity as follows:

Question 2. *Is a (separable) Banach space X reflexive if it is ∞-reflexive? ω-reflexive?*

The first counterexample that comes to mind is the quasireflexive James space J (having codimension 1 in its second dual). We recall that a Banach space X is *quasireflexive* if it has finite codimension in its second dual space X^{**}.

Theorem 2. *The quasireflexive James space J is not ω-reflexive.*

However we do not know if the James space is ∞-reflexive.

Question 3. *Is each quasireflexive Banach space ∞-reflexive? Is the James space ∞-reflexive?*

Our principal result on separable ∞-reflexive Banach spaces asserts that any such a space has Asplund dual. We recall that a Banach space X is *Asplund* if each separable subspace Y of X has separable dual Y^*.

Theorem 3. *Each separable ∞-reflexive Banach space X has Asplund dual X^*.*

Since the Banach space l_1 is not Asplund, Theorem 3 implies the result of [Ba] (asserting that the dual space X^* of a separable ∞-reflexive Banach space X contains no copy of l_1).

The proof of Theorem 3 relies on a characterization of the Asplund property of the dual Banach space in terms of so-called $*$-weak covering properties.

Definition 2. *A Banach space X is defined to satisfy the τ-covering property, where τ is a weaker linear topology on X, if for every bounded subset $B \subset X$ and every sequence $(U_i)_{i=1}^{\infty}$ of τ-open sets in X whose intersection $\bigcap_{i=1}^{\infty} U_i$ is a norm-neighborhood of the origin in X there are points $x_1, \ldots, x_n \in X$ such that $B \subset \bigcup_{i=1}^{n}(x_i + U_i)$.*

If τ is the weak or $*$-weak topology, then we say about the weak or $*$-weak covering properties, briefly, WCP and $*$-WCP.

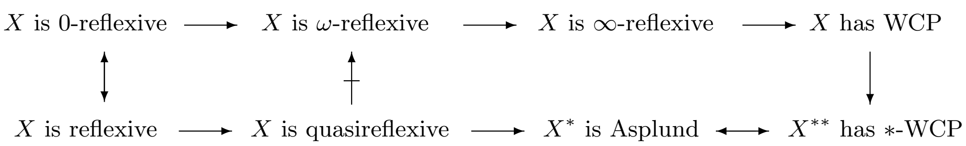
Theorem 3 can be derived from the following theorem that can have an independent value.

Theorem 4. (1) *Each separable ∞-reflexive Banach space has the weak covering property;*

(2) *If a Banach space X has the weak covering property, then the second dual space X^{**} has the $*$-weak covering property;*

(3) *A Banach space X is Asplund if and only if the dual space X^* has the $*$-weak covering property.*

The obtained results fit into the following diagram connecting various reflexivity-like properties and holding for any separable Banach space X:



Now let us discuss some stability properties of r-reflexive spaces and ask some related questions.

Theorem 5. *Let Z be a Banach subspace of a separable Banach space X.*

(1) *If X is an r-reflexive Banach space for some $r \in [0, +\infty]$, then the quotient space X/Z is r-reflexive too.*

(2) *If X is r-reflexive for some $r \in \{0, \omega, \infty\}$, then each Banach space Y isomorphic to X is r-reflexive.*

(3) *If both the spaces Z and X/Z are r-reflexive for some $r \in \{0, \omega, \infty\}$, then X is r-reflexive.*

Question 4. *Is a subspace of a (separable) r-reflexive Banach space r-reflexive (at least for $r \in \{\omega, \infty\}$)?*

Since the r-reflexivity is an isomorphic property for $r \in \{0, \omega, \infty\}$, we may also ask:

Question 5. *Is the r-reflexivity an isomorphic property for arbitrary $r \in (0, +\infty)$?*

Our next question concerns the separability assumption in Theorem 3.

Question 6. *Has each ∞-reflexive Banach space Asplund dual?*

We can give a partial answer for Banach spaces with $\aleph_0$-monolithic dual space. We recall that a topological space X is *monolithic* (resp. $\aleph_0$-*monolithic*) if each (separable) subspace Y of X has network weight $\mathrm{nw}(Y)$ equal to the density $\mathrm{dens}(Y)$ of Y. It is easy to see that each Banach space is monolithic in norm and weak topologies.

We shall say that a Banach space X has ($\aleph_0$-)monolithic dual space, if the dual space X^* is ($\aleph_0$-)monolithic with respect to the $*$-weak topology. It can be shown that a Banach space X has ($\aleph_0$-)monolithic dual if and only if for any (separable) subset $Y \subset X^*$ the annulator $Y^\top = \{x \in X : \forall y^* \in Y\ y^*(x) = 0\}$ has $\mathrm{dens}(X/Y^\top = \mathrm{dens}(Y)$ in X. The latter property was introduced in [BPZ] as the property (1). Since Corson compacta are monolithic, each weakly Lindelöf determined Banach space (=Banach space with Corson dual ball) has monolithic dual.

Proposition 1. *Each ∞-reflexive Banach space with $\aleph_0$-monolithic dual has Asplund dual.*

Question 7. *Has each separable ∞-reflexive Banach space separable dual?*

Unlike the $*$-weak covering property (which is equivalent to the Asplundness of predual), we have very poor information about Banach spaces with the weak covering property. All we know about these spaces is summed up in Theorem 4 and the following

Proposition 2.
(1) *A quotient space of a Banach space with weak covering property has that property too.*
(2) *The product $X \times Y$ of a Banach space X with the weak covering property and a finite-dimensional Banach space Y has the weak covering property.*

Question 8. *Is the weak covering property a Three Space Property? Is it hereditary with respect to subspaces and products?*

Also Questions 2–7 can be posed with the ∞-reflexivity replaced by the weak covering property.

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Absolute Z_∞-spaces: a new dimension class of compacta

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In this talk we shall introduce so called absolute Z_∞-spaces forming a new dimension class of compacta and discuss its relationship with other dimension classes. The concept of an absolute Z_∞-space is related to the notion of a Z_n-set, well studied in infinite-dimensional and geometric topology. By definition, a closed subset A of a topological space X is called a Z_n-*set* where $n \in \omega \cup \{\infty\}$, if every map $f : I^n \to X$ of the n-dimensional cube $I^n = [0,1]^n$ can be uniformly approximated by maps whose images miss the set A. Observe that a subset $A \subset X$ is a Z_0-set if and only if A is closed and nowhere dense in X. It is well-known that every Z_n-set A in an ANR-space X is locally homotopically n-negligible in the sense that for every open set $U \subset X$ the relative homotopy groups $\pi_k(U, U \setminus A)$ vanish for all $k \le n$. Replacing the relative homotopy groups with relative homology groups we arrive to the notion of a homological Z_n-set: a closed subset $A \subset X$ is a *homological Z_n-set* in X if the relative homology groups $H_k(U, U \setminus A)$ vanish for all open sets $U \subset X$ and all $k \le n$. The relationship between Z_n-sets and their homological counterparts is described by

Theorem 1. *A closed subset A of an ANR-space X is a homological Z_n-set if and only if $A \times \{0\}$ is a Z_{n+1}-set in $X \times [-1,1]$.*

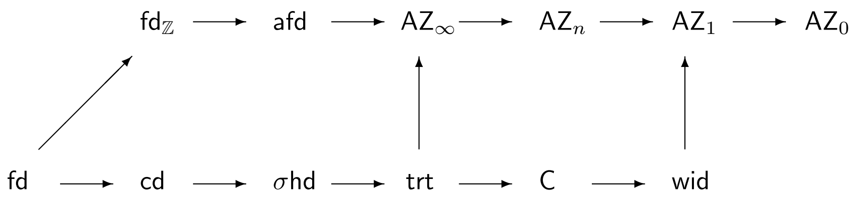
According to an old result of Kroonengerg [6], each closed finite-dimensional subset of the Hilbert cube $Q = [0,1]^\omega$ is a homological Z_∞-set. On the other hand, the Hilbert cube contains closed zero-dimensional subsets (so-called wild Cantor sets) failing to be Z_2-sets, see [9]. Thus homological Z_n-sets behave in more regular and predictable way comparing to their homotopical counterparts.

Having in mind the mention result of Kroonenberg, let us define a compact space X to be an *absolute Z_n-space* if for every embedding $e : X \to Q$ of X into the Hilbert cube Q the image $e(X)$ is a homological Z_n-set in Q. By AZ_n we shall denote the class of all compact metrizable absolute Z_n-spaces. Observe that the class AZ_0 coincides with the class of compact spaces that contain no copy of the Hilbert cube and thus is the largest non-trivial closed hereditary class of metrizable compacta. The class AZ_∞ also is closed with respect to countable unions and taking closed subspaces. Also this class is closed with respect to multiplication by finite-dimensional compacta, more generally by trt-dimensional compacta. The latter class is defined with help of the separation dimension $\mathsf{t}(\cdot)$ introduced by Stainke [8] and its transfinite extension $\mathsf{trt}(\cdot)$ introduced by Arenas, Chatyrko and Puertas [1]. Given a topological space X we write:

- $\mathsf{trt}(X) = -1$ iff $X = \emptyset$;
- $\mathsf{trt}(X) \le \alpha$ where α is an ordinal if each closed subset $B \subset X$ containing more than one point can be separated by a closed subset $C \subset B$ with $\mathsf{trt}(C) < \alpha$.

A topological space X is called trt-*dimensional* if $\mathsf{trt}(X) \le \alpha$ for some ordinal α.

The relationship of the classes AZ_n with other dimension classses are described in the following diagramm in which an arrow $\mathsf{x} \to \mathsf{y}$ means that $\mathsf{x} \subset \mathsf{y}$:



In this diagram

- fd stands for the class of finite-dimensional compacta;
- $\mathsf{fd}_\mathbb{Z}$ is the class of compacta with finite integral cohomological dimension;
- afd is the class of almost finite-dimensional compacta, where a space X is *alsmost finite-dimensional* if $\sup\{\dim(F) : F \underset{cl}{\subset} X \ \dim(F) < \infty\} < \infty$;
- cd is the class of countable-dimensional compacta;
- $\sigma\mathsf{hd}$ is the class of compacta that are countable unions of hereditarily disconnected subspaces;
- trt is the class of trt-dimensional compacta;
- C is the class of compacta with the property C;
- wid is the class of weakly infinite-dimensional compacta.

Now we pose some open problems related to this diagram. The classes cd and $\sigma\mathsf{hd}$ of countable-dimensional and σ-hereditarily disconnected compacta are distinguished by the famous Pol's compactum, and a compact spaces distingushing the classes wid and C was recently constructed by P. Borst [4]. We do not know if the other considered classes also are different.

Question 1. *Is each trt-dimensional compactum σ-hereditarily disconnected? Is each C-compactum trt-dimensional?*

Some immediate questions still are open for the tranfinite dimension trt.

Question 2. *Is the ordinal $\mathsf{trt}(X)$ countable for each trt-dimensional metrizable compactum X? Given a (countable) ordinal α is there a compact (metric) space X with $\mathsf{trt}(X_\alpha) = \alpha$?*

Question 3. *Is $\mathsf{C} \subset \mathsf{AZ}_\infty$? Is $\mathsf{wid} \subset \mathsf{AZ}_2$?*

This question is related to another one:

Question 4. *Let $W \subset Q$ be a closed weakly-infinite dimensional subset (with the property C). Is the complement $Q \setminus W$ homologically trivial?*

Question 5. *What can be said about the classes AZ_n for $n \in \mathbb{N}$? Are they hereditary with respect to taking closed subspaces? Are they pairwise distinct?*

We have defined absolute Z_∞-compacta with help of their embedding into the Hilbert cube. What about embeddings into other spaces resembling the Hilbert cube?

Question 6. *Let A be a compact subset of an absolute retract X whose all points are homological Z_∞-points. Is A a homological Z_∞-set in X if A is an absolute Z_∞-space?*

Compact absolute retracts whose all points are homological Z_∞-points seem to be very close to being Hilbert cubes. By [3] all such spaces fail to be C-spaces and have infinite cohomological dimension with respect to any coefficient group.

Question 7. *Let X be a compact absolute retract whose all points are homological Z_∞-points. Is X strongly infinite-dimensional? Is $X \times [0,1]^2$ homeomorphic to the Hilbert cube? Is X homeomorphic to Q if X has DDP, the Disjoint Disks Property?*

In light of this question we should mention an example of a fake Hilbert cube constructed by Singh [7]. He constructed a compact absolute retract X such that (i) all points of X are homological Z_∞-points, (ii) $X \times [0,1]^2$ and $X \times X$ are homeomorphic to Q but (iii) X contains no proper closed ANR-subspace of X of dimension greater than one.

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Grothendieck property in Sacks model

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For an infinite Boolean algebra $\mathcal{B}$, let $C(K_\mathcal{B})$ be the Banach space of the continuous real-valued functions on its Stone space $K_\mathcal{B}$, with the supremum norm. A Boolean algebra $\mathcal{B}$ is said to have the Grothendieck property whenever each weak-star convergent sequence in $C(K_\mathcal{B})^*$ converges weakly. The purpose of this work is to prove the consistency of the existence of a Boolean algebra with the Grothendieck property and with cardinality less than the continuum cardinal. The negation of this fact follows from $\mathfrak{p} = \mathfrak{c}$.

Schachermayer proved that a necessary (but not sufficient) condition for a Boolean algebra to have the Grothendieck property is that it is not a countable union of a strictly increasing sequence of subalgebras. Just and Koszmider showed that in the model obtained by a product of Sacks forcings, there is a Boolean algebra with cardinality less than $\mathfrak{c}$ and which is not such a union. This motivated us to prove that the Boolean algebra has the Grothendieck property.

Complete Boolean algebras, or even σ-complete Boolean algebras, have the Grothendieck property. There are other properties, such as the subsequential completeness property or the subsequential interpolation property, which are also stronger than the Grothendieck property. However, all of them imply that the Boolean algebra has cardinality at least $\mathfrak{c}$, which is not the case of ours. This illustrates that these properties are quite far from the Grothendieck property.

Ideal convergence and quotient Boolean algebras

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We generalize the Bolzano-Weierstrass theorem (that every bounded sequence of reals admits a convergent subsequence) on ideal convergence. We show examples of ideals with and without Bolzano-Weierstrass property, and give characterizations of BW property in terms of submeasures and extendability to a maximal P-ideal. We show applications to the various orderings of ideals and its Boolean algebras.

Games and metrisability of manifolds

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By a manifold we mean a connected, Hausdorff, locally Euclidean space, while $C_k(X)$ denotes the space of all continuous real-valued functions on X with the compact-open topology. We show that metrisability of a manifold M is equivalent to a number of different conditions involving games on $C_k(M)$. It is also equivalent to the space $C_k(M)$ being Baire and being Volterra, a weakening of the Baire condition.

Descriptive properties of families of automomeomorphisms of the unit interval

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Let $\mathbb{H} \subset C[0,1]$ be the set of all increasing autohomeomorphisms of $[0,1]$. We say that $f \in \mathbb{H}$ is a strictly singular autohomeomorphism, if f has no positive finite derivative at any point, more exactly, f has no positive finite derivative at any point of $(0,1)$ and no one-sided derivative at 0 and 1, right-hand and left-hand, respectively. We show that the family of all strictly singular autohomeomorphisms is Π^1_1–complete, in particular non–Borel. This solves a problem mentioned by Graf, Mauldin and Williams in 1986.

Embedding inverse semigroups into global semigroups of compact groups

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We address the Gutik's problem on characterization of topological semigroups embeddable into global semigroups of topological groups.

By the *global semigroup* $\Gamma(G)$ of a topological group G we understand the hyperspace of non-empty compact subsets of G, endowed with the Vietoris topology and the semigroup operation $A \cdot B = \{a \cdot b : a \in A, b \in B\}$.

Answering the Gutik's question, we prove that a compact topological inverse Clifford semigroup S embeds into a global semigroup $\Gamma(G)$ of a compact topological group G if and only if the semilattice of idempotents of S is zero-dimentional.

We recall that a semigroup S is *inverse* (and *Clifford*) if for any element $x \in S$ there exists a unique inverse element $x^{-1} \in S$, such that $x \cdot x^{-1} \cdot x = x$, $x^{-1} \cdot x \cdot x^{-1} = x^{-1}$ (and $x^{-1} \cdot x = x \cdot x^{-1}$).

Characterization of Spaces with Ideal Convergence Property

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Abstract: We say that a sequence of functions $f_n : X \to \mathbb{R}$ is *I-convergent to a function* $g : X \to \mathbb{R}$ if for every $\varepsilon > 0$ and every $x \in X$ the set $\{n \in \omega : \ |f_n(x) - g(x)| \geq \varepsilon\} \in I$. $\mathcal{IC}(I)$ denotes the class of spaces X where I-convergence implies pointwise convergence on a set from the dual filter $\mathcal{F}(I) = \{B \subseteq \omega | B^c \in I\}$. In Jasinski, Recław, *Ideal Concergence of Continuous Functions,* Topology and its Applications (to appear) we studied $\mathcal{IC}(I)$ in case I was the density ideal or the ideal of bounded subsets of $\omega \times \omega$. Here we attempt to characterize $\mathcal{IC}(I)$ spaces for the entire class of ideals. For example we show that if I is an analytic, atomless P-ideal on ω then $\mathcal{IC}(I) \subseteq s_0$.

Transitive operations and new small subsets of the reals

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For two translation invariant families I and J of subsets of the Cantor space 2^ω such that $I \subseteq J$ we define a family

$$G_t(J, I) = \{A \subseteq 2^\omega \colon (\forall\, B \in I)\; A + B \in J\}.$$

It occurs that many types of small subsets of the reals can be expressed in the form $G_t(J, I)$. Among them there are *everywhere meager* and *everyehere null* sets, new classes of small sets. We define and investigate these classes.

Linearly ordered compacta and projections in Banach spaces

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I will describe an example of a linearly ordered compact K of weight $\aleph_1$, for which the Banach space $C(K)$ does not have a decomposition into a continuous chain of complemented separable subspaces. On the other hand, $C(K)$ can be decomposed into a (discontinuous!) chain of one-complemented separable subspaces. Recall that a subspace is *one-complemented* if there is a norm-one projection onto it. A well-ordered chain $\{E_\xi\}_{\xi<\rho}$ of closed subspaces is *continuous* if E_δ is the closure of $\bigcup_{\xi<\delta} E_\xi$ for every limit ordinal δ.

The above compact space K is a natural continuous order preserving image of a linearly ordered Valdivia compact, therefore $C(K)$ is a closed subspace of a Banach space with a projectional resolution of the identity. This answers two questions due to V. Montesinos and O. Kalenda respectively.

Sandwich-type characterization of completely regular spaces

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All the higher separation axioms in topology, except for complete regularity, are known to have sandwich-type characterizations. We provide a characterization of complete regularity in terms of inserting a continuous real-valued function. The known fact that each continuous real-valued function on a compact subset of a Tychonoff space has a continuous extension to the whole space is obtained as a corollary.

Cardinal invariants for $\mathcal{C}$-cross topologies

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Let X and Y be topological spaces. Consider a family $\mathcal{C}$ of subsets of $X \times Y$ which is closed under finite intersections and such that for each $G \in \mathcal{C}$ the set $G_x = \{y \in Y : (x, y) \in G\}$ has non-empty interior in Y, and the set $G^y = \{x \in X : (x, y) \in G\}$ has non-empty interior in X. Additionally assume, that $\mathcal{C}$ contains a π-base for the product topology on $X \times Y$. The topology on $X \times Y$ generated by $\mathcal{C}$ is called *$\mathcal{C}$-cross topology.*

$\mathcal{C}$-cross topologies are generalizations of the product topology τ, the topology of separate continuity σ or the cross topology γ: See a survey article by M. Henriksen and R. G. Woods, *Separate versus joint continuity: a tale of four topologies* in Top. Appl. 97 (1999), no. 1-2, 175–205. For many cases $\mathcal{C}$-cross topologies fulfill the Kuratowski-Ulam Theorem.

If X is a topological space with the topology λ, then let nwd_λ denotes the ideal of all nowhere dense subset of X, and let $\mathcal{M}_\lambda$ denotes the $\sigma-$ideal of all meager subsets of X. For the plane with various $\mathcal{C}$-cross topologies the following results are obtained:

If $F \in nwd_\tau$, then F is nowhere dense with respect to a $\mathcal{C}$-cross topology, too;

$cof(nwd_\gamma) > 2^\omega$ *and* $cof(\mathcal{M}_\gamma) > 2^\omega$;

$cov(\mathcal{M}_\gamma) = cov(\mathcal{M}_\tau)$;

If X is a not meager subset of the reals, then the square $X \times X$ is not meager with respect to γ. Moreover, $non(\mathcal{M}_\gamma) = non(\mathcal{M}_\tau)$.

Note also, that for the plane $nwd_\tau = nwd_\sigma$.

If X is a metric space and $Y \subseteq X$ is a dense subsets, then $cof(nwd_X) = cof(nwd_Y)$: Here nwd_Z denotes nowhere dense subsets in Z. This last equality does not hold, whenever one considers the plane with the cross topology and the square of the rationals.

Indicatrices of $C^n[0,1]$ functions

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We give necessary and sufficient conditions on a function $f\colon [0,1] \to \{1, 2, \ldots, \mathfrak{c}\}$ under which there exists a $C^n[0,1]$ function $(n = 1, 2 \ldots, \infty)$ $F\colon [0,1] \to [0,1]$ such that for every $y \in [0,1]$, $f(y) = |F^{-1}(y)|$.

First we give such a characterisation for continuous functions F and sketch a construction. Then adding some modifications we give such a characterisation for continuous functions F of $V_n(F) < \infty$ ($V_n(F)$ denotes n-variation). Finally using the theorem of Laczkovich and Preiss (which states that for the above function there exists a homeomorphism $h\colon [0,1] \to [0,1]$ such that $F \circ h \in C^n[0,1]$) we obtain a required characterisation.

Independent Families and Topology

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The talk will be largely inspired by an open problem in infinitary combinatorics concerning functions from ω_1 to ω posed by Holický, Zajiček and Zelený. Although the problem as far as I know remains open, I am going to show that it is somewhat related to σ-independent families. In particular, I am going to reprove one of their results using a slight modification of a theorem on σ-independent families attributed to Tarski.

To obtain this modification, I shall use a tricky technique of some topological flavor, which can be used as well to obtain elegant proofs of classical facts on independent families.

On uniform Eberlein compact spaces

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A compact space is said to be a uniform Eberlein (an Eberlein) compact if it is homeomorphic to a weakly compact subset of a Hilbert (a Banach) space. The main purpose of our research is to examine which purely topological properties distinguish between Eberlein and uniform Eberlein compact spaces. We observe that every metrizable space can be embedded into a uniform Eberlein compact. We show that if X and Y are l-equivalent spaces, and X is a uniform Eberlein compact, then Y also is a uniform Eberlein compact. We don't know if the same statement is true for t-equivalence. We give an example of an Eberlein compact space which can not be represented as a countable union of uniform Eberlein compacts.

Each Abelian group contains subset of arbitrary Prodanov index (with two exceptions)

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The notion of smallness is represented in many parts of mathematics. Here we discuss the smallness in combinatorial sense. For a subset A of a Abelian group G, the Prodanov index of A, denoted by $ind_P^+(A)$, is the cardinal

$ind_P^+(A) = \inf\{\alpha : \text{for every} B \subset G \text{ of size } |B| = \alpha \text{ there are } b, b^{'} \in B \text{ with } b + A \cap b^{'} + A \neq \emptyset\}$.

If the Prodanov index of a subset A is equal to some natural number n it means that there are $n-1$ disjoint shifts of A.

Theorem. *Each infinite Abelian group G contains a subset $A \subset G$ with $ind_P^+(A) = \alpha$ for any cardinal $\alpha > 4$.*

Theorem. *Each infinite Abelian group G which is not isomorphic to $\oplus\mathbb{Z}_2$ contains a subset $A \subset G$ with $ind_P^+(A) = 4$.*

Remark. The group $\oplus\mathbb{Z}_2$ does not contain a subset $A \subset G$ with $ind_P^+(A) = 4$.

Theorem. *Each infinite Abelian group G which is not isomorphic to $\oplus\mathbb{Z}_3$ contains a subset $A \subset G$ with $ind_P^+(A) = 3$.*

Remark. The group $\oplus\mathbb{Z}_3$ does not contain a subset $A \subset G$ with $ind_P^+(A) = 3$.

Applications of pcf theory to topology and measure theory

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We prove that there exists a normal space such that every Baire measure is extendible to a Borel measure but there exists a Baire measure which is not extendible to a regular Borel measure. This gives an answer to a question of Ohta and Tamano and provides a partial answer to a question of Fremlin.

The Near Coherence of Filters Principle does not imply the Filter dichotomy Principle

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We show that there is a forcing extension in which any two ultrafilters on ω are nearly coherent and there is a non-meagre filter that is not nearly ultra. This answers Blass' longstanding question whether the principle of near coherence of filters is strictly weaker than the filter dichotomy principle.

Egorov theorem and Q-ideals

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Last year Marek Balcerzak, Katarzyna Dems and Andrzej Komisarski proved statistical version of Egorov theorem. We extend their result into many other ideals especially analytic P-ideals, which are nicely characterized by Sławomir Solecki. We show ideals for which holds weak and strong version of Egorov theorem. We also characterize such ideals using the notion of Q-ideals.

Topological properties of spaces of measures

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Given a compact space K, we consider the space $P(K)$ of regular probability measures defined on K. Such a space $P(K)$ may be equipped with a natural compact topology (the weak* topology inherited from $C(K)^*$).

We discuss some results and open problems on possible connections between basic topological properties of $P(K)$ and the space K itself. In particular, we mention a ZFC example showing that there is no (*so far...*) natural criterion for separability of $P(K)$.

Universally Kuratowski-Ulam spaces

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We consider a version of the open-open game, which was invented by P. Daniels, K. Kunen and H. Zhou, *On the open-open game*, Fund. Math. 145 (1994), no. 3, 205 - 220. Two players take turns playing with a topological space X. Player I chooses a finite family $\mathcal{A}_0$ of non-empty open subsets of X. Then Player II chooses a finite family $\mathcal{B}_0$ of non-empty open subsets of X such that for each $U \in \mathcal{A}_0$ there exists $V \in \mathcal{B}_0$ with $V \subseteq U$. Similarly at the n-th round Player I chooses a finite family $\mathcal{A}_n$ of non-empty open subset of X. Then Player II chooses a finite family $\mathcal{B}_n$ of non-empty open subsets of X such that for each $U \in \mathcal{A}_n$ there exists $V \in \mathcal{B}_n$ with $V \subseteq U$. If for any natural number k the union $\bigcup\{\mathcal{B}_k \cup \mathcal{B}_{k+1} \cup \ldots\}$ is a dense subset of X, then Player I wins; otherwise Player II wins. The space X is I-*favorable* whenever Player I wins no matter how Player II plays. We say that Player I has a winning strategy.

We have compared notions of uK-U and uK-U* spaces. A space Y is *universally Kuratowski-Ulam* (for short, *uK-U space*), whenever for any topological space X and a meager set $E \subseteq X \times Y$, the set

$$\{x \in X : \{y \in Y : (x, y) \in E\} \text{ is not meager in } Y\}$$

is meager in X, uK-U spaces has been investigated by D. Fremlin, T. Natkaniec and I. Reclaw, *Universally Kuratowski-Ulam spaces*, Fund. Math. 165 (2000), no. 3, 239 - 247. A space Y is *universally Kuratowski-Ulam** (for short, *uK-U** *space*), whenever for a topological space X and a nowhere dense set $E \subseteq X \times Y$ the set

$$\{x \in X : \{y \in Y : (x, y) \in E\} \text{ is not nowhere dense in } Y\}$$

is meager in X, it has been introduced by D. Fremlin, *Universally Kuratowski-Ulam spaces*, a note from: www.essex.ac.uk/maths/staff/fremlin/preprints.htm.

The main results are :

- *Every I-favorable space is universally Kuratowski-Ulam* *,
- *If a compact space* Y *is I-favorable, then the hyperspace* $\exp(Y)$ *with the Vietoris topology is I-favorable, and hence universally Kuratowski-Ulam**,

One may conjecture that there is a compact universally Kuratowski-Ulam space which is not I-favorable.

Questions for Justin Moore

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Uniform spaces through the looking-glass

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A ball structure is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radiuses*. Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},\ B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha),\ B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha)$$

A ball structure $\mathcal{B} = (X, P, B)$ is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
$$B^*(x, \alpha') \subseteq B(x, \alpha),\ B(x, \beta') \subseteq B^*(x, \beta);$$
- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,
$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$
- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$
- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family

$$\{\bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ball structure is said to be a *ballean* (or a *coarse structure*) if it is upper symmetric and upper multiplicative. For motivation to study balleans as the asymptotic counterparts of the uniform topological spaces see [1], [2], [3], [4].

Now we define the mappings which play the part of uniformly continuous mappings on the ballean stage. Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$,there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta).$$

A bijection $f : X_1 \to X_2$ is called an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if f and f^{-1} are $\prec$-mappings. If $X_1 = X_2$ and the identity mapping $id : X_1 \to X_2$ is an asymorphism,we identify $\mathcal{B}_1$ and $\mathcal{B}_2$, and write $\mathcal{B}_1 = \mathcal{B}_2$.

Let $\mathcal{B} = (X, P, B)$ be a ballean. We say that a subset A of X is

- *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $A \subseteq B(x, \alpha)$;
- *large* if there exists $\alpha \in P$ such that $X = B(A, \alpha)$;
- *small* if $X \setminus B(A, \alpha)$ is large for every $\alpha \in P$;
- *thick* if $int(A, \alpha) \neq \varnothing$ for every $\alpha \in P$, where $int(A, \alpha) = \{x \in X : B(x, \alpha) \subseteq A\}$;
- *extralarge* if $int(A, \alpha)$ is large for every $\alpha \in P$;
- *piecewise large* if there exists $\beta \in P$ such that $int(B(A, \beta), \alpha) \neq \varnothing$ for every $\alpha \in P$;
- *pseudodiscrete* if, for every $\alpha \in P$, there exist a bounded subset V of X such that $B(x, \alpha) \cap A = \{a\}$ for every $a \in A \setminus V$.

These observations give a foundation for the following uniform spaces-balleans vocabulary:

dense subset	large subset
nowhere dense subset	small subset
subset with nonempty interior	thick subset
subset with dense interior	extralarge subset
somewhere dense subset	piecewise large subset
discrete subset	pseudodiscrete subset

Using this vocabulary, we get the following cardinal invariants of a ballean:

density($\mathcal{B}$)= $min\{|L|$: *L is a large subset of* $X\}$,

cellularity($\mathcal{B}$)=$sup\{|F|$: *F is a disjoint family of thick subsets of* $X\}$,

spread($\mathcal{B}$)=$sup\{|Y|_{\mathcal{B}}$: *Y is a pseudodiscrete subset of* $X\}$, where $|Y|_{\mathcal{B}} = min\{|Y \setminus V| : V$ is a bounded subset of $X\}$.

Given a cardinal κ, we say that a ballean $\mathcal{B} = (X, P, B)$ is κ-*resolvable* if X can be partitioned to κ large subsets. The *resolvability* of $\mathcal{B}$ is the cardinal

$$res(\mathcal{B}) = sup\{\kappa : \mathcal{B}\ is\ \kappa - resolvable\}.$$

Given a cardinal κ, we say that a ballean $\mathcal{B} = (X, P, B)$ is $\kappa - extraresolvable$ if there exists a family $\mathcal{F}$ of large subsets of X such that $|\mathcal{F}| = \kappa$ and $F \cap F'$ is small whenever F, F' are distinct elements of $\mathcal{F}$. The *extraresolvability* of $\mathcal{B}$ is the cardinal

$$exres(\mathcal{B}) = sup\{\kappa : \mathcal{B}\ is\ \kappa - extraresolvable\}.$$

In the talk I intend to survey the interplays between these invariants and show some of its applications.

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Bolzano Weierstrass property for ideals

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Locally convex topological vector spaces which are reconstructible from their homeomorphism groups

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Let X and Y be open subsets of locally convex topological vector spaces E and F respectively. Denote their homeomorphism groups by $H(X)$ and $H(Y)$, and suppose that φ is an isomorphism between $H(X)$ and $H(Y)$. It is an open question whether the above implies that there is a homeomorphism τ between X and Y such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in H(X)$.

The above is true, however, in the special case that the spaces E and F are normed spaces (1989). The above is also true under the weaker assumption that E and F are normal space, and admit a continuous norm (Leiderman Rubin 2001).

We shall prove here another special case.

Theorem A: Let X and Y be open subsets of locally convex *metrizable* topological vector spaces E and F respectively and φ be an isomorphism between $H(X)$ and $H(Y)$. Then there is a homeomorphism τ between X and Y such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in H(X)$.

Theorem A relies on Theorem B which is stated below. A pair (X, G) in which X is a topological space and G is a subgroup of the group $H(X)$ of all auto-homeomorphisms of X is called a *space-group pair*. A class K of space-group pairs is *faithful* if for every $(X, G), (Y, H) \in K$ and an isomorphism φ between G and H there is a homeomorphism τ between X and Y such that $\varphi(g) = \tau \circ g \circ \tau^{-1}$ for every $g \in G$.

Let (X, G) be a space-group pair and $S \subseteq X$ be open. S is *strongly flexible*, if for every infinite $A \subseteq S$ without accumulation points in X, there is a nonempty open set $V \subseteq X$ such that for every nonempty open set $W \subseteq V$ there is $g \in G$ such that the sets $\{a \in A \mid g(a) \in W\}$ and $\{a \in A \mid \ \}$for some neighborhood U of $a, g{\restriction}U = Id$ are infinite.

Theorem B: Let K be the class of all space-group pairs (X, G) such that
(1) X is regular, first countable and has no isolated points.
(2) For every $x \in X$ and an open neighborhood U of x the set
$\{g(x) \mid g \in G \text{ and } g{\restriction}(X - U) = Id\}$ is somewhere dense.
(3) The family of strongly flexible sets is a cover of X.
Then K is faithful.

Theorem B has applications other than Theorem A.

Maximal abelian self adjoint subalgebras of the Calkin algebra

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The Calkin algebra is the quotient of the algebra of bounded operators on separable Hilbert space modulo the ideal of compact operators; as such it can be viewed as a non-commutative version of the power set of the integers modulo the ideal of finite sets. It is of interest to examine to what extent the large body of set theoretic results about the Cech-Stone compactification of the integers (and its algebra of clopen sets) can be carried over to the context of the Calkin algebra and its state space.

This talk will provide some background on these matters and then look at a specific application to the construction of maximal abelian self adjoint subalgebras (masas) of the Calkin algebra. Canonical examples of masas are obtained, for example, by considering the algebra of diagonal operators with respect to a fixed basis in Hilbert space and then lifting to the Calkin algebra. Multiplication by L^∞ provides similar examples. The structure of other examples remains to be analyzed completely and is likely to depend on set theoretic hyptheses. Some examples of such constructions will be examined.

Fuzzy Set Theory in Encoding Spatial Relations
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Spatial relationships between regions in an image play an important role in scene understanding. Humans are able to quickly ascertain the relationship between two objects, for example B is to the right of A, or B is in front of A, but this has turned out to be a somewhat illusive task for automation. When the objects in a scene are represented by crisp sets, the all-or-nothing definitions of the subsets actually add to the problem of generating such relational descriptions. It is our belief that definitions of spatial relationships based on fuzzy set theory, coupled with a fuzzy segmentation will yield realistic results.

Towards a structure theory for T_5 compact spaces
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We describe a context in which the class of compact ccc T_5 spaces looks quite close to the class of compact metric spaces.

The combinatorics of the Baire group
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We study subgroups of $\mathbb{Z}^{\mathbb{N}}$ which possess group theoretic properties of boundedness type, analogous to properties introduced by Menger (1924), Hurewicz (1925), Rothberger (1938), and Scheepers (1996). (The studied properties were introduced independently by Kocinac and Okunev).

We obtain purely combinatorial characterizations of these properties, and combine them with other techniques to solve several questions of Babinkostova, Kocinac, and Scheepers.

An informal thesis emerging from our study is that the Baire group is a “universal” group with respect to boundedness properties of groups.

This paper is available online at http://arxiv.org/abs/math.GN/0508146

Examples of function spaces which are non-separable topological Hilbert manifolds
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Using Toruńczyk's characterization theorem [1], we show that the space $C_B(X,Y)$ of bounded continuous mappings from X into Y is a topological manifold modelled on the Hilbert space of weight 2^ω, with respect to the topology of uniform convergence, under the following three assumptions:

(1) X is a noncompact, separable and metrizable space,

(2) Y is a complete metric space which is an ANRU (ANR in uniform sense, a notion introduced by Nguyen To Nhu [2]),

(3) the components of Y have diameters bounded away from zero.

Compact polyhedra satisfy assumptions (2) and (3) for Y. The assumption (2) and (3) can be replaced by "Y is a connected complete Riemannian manifold”, where the metric is determined by the geodesics.

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Complete nonmeasurabilitty in regular families of small sets

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MSC: Primary 03E75; Secondary 03E35, 28A05, 28A99
Keywords: Lebesgue measure, Baire property, measurable set, algebraic sum

The main motivation of this paper is the following theorem which is known in the literature as Four Poles Theorem.

Theorem 1 (Brzuchowski, Cichon, Grzegorek, Ryll-Nardzewski). *Let X be any Polish space and let I be a σ–ideal with Borel base. Assume that $A \subseteq I$ is point-finite family i.e. for each $x \in X |\{A \in A : x \in A\}| < \omega$ and $\bigcup A = X$. Then there exists a subfamily $A_0 \subseteq A$ such that $\bigcup A_0$ is I–nonmeasurable i.e. for each $B \in B(X)$ $B \triangle \bigcup A_0 \notin I$.*

In various cases it is possible to obtain more than nonmeasurability of the union of a subfamily A_0. Namely, the intersection of this union with any measurable set that is not in I is nonmeasurable (recall, the measurability is understood here in the sense of belonging to the σ-algebra generated by the family of Borel sets and I). Such strong conclusion can be obtained for the ideal of first Baire category sets under the assumption that A is a partition (see [3]).

In this paper we show how to obtain complete nonmeasurability of the union of subfamily of A assuming that the family A is in some sense regular. We avoid to use any additional set-theoretic assumptions.

Definition 1. *Let (X, I) be a Polish ideal space. Let $A \subseteq X$. We say that A is completely I–nonmeasurable if*

$$\forall B \in B(X) \;\; A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset.$$

Let us notice that A is completely $[X]^{\leq\omega}$–nonmeasurable iff A is a Bernstein set. A set A is completely $\mathbb{L}$–nonmeasurable if A has full outer measure and its inner measure is zero.

Now, we we deal with c.c.c. ideals. Let us recall that for a set $A \subseteq X$ by $[A]_I$ we denote the Borel envelope of A, i.e. the minimal (mod I) Borel set containing A.

Theorem 2. *Let X be an uncountable Polish space. Let $I \subseteq P(X)$ be a c.c.c σ-ideal with Borel base. Assume that we have a family $F \subseteq I$ satisfying the following conditions*

(1) *F is point-finite.*
(2) $(\forall B \in B+(X))(B \subseteq [\bigcup F]_I \rightarrow |\{F \in F : F \cap B \neq \emptyset\}| = 2^\omega)$.

Then there exists a subfamily $F' \subseteq F$ such that $\bigcup F'$ is completely I–nonmeasurable in $[\bigcup F]_I$.

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