# Similarities in Multiparticle Production Processes in $pp$ Collisions as Imprints of Nonextensive Statistics

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**Abstract:** The transverse momentum $p_T$ spectra of both hadrons and jets produced in $pp$ collisions at beam energies from hundreds GeV to a few TeV exhibit power-law behavior of $1/p_T^n$ at high $p_T$, with similar power indices $n$. The respective nonextensivity parameters for transverse momenta distributions and the global nonextensivity parameter obtained from multiplicities are compared. In particular, data on kaon to pion (charged particles) multiplicity ratio are analyzed, assuming that the reaction occurs in some nonextensive environment. The values of the corresponding nonextensivity parameters were found to be similar, strongly indicating the existence of a common mechanism behind all these observables.

**Keywords:** $pp$ collisions; multiparticle production; nonextensive statistics

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For some time now it is known that transverse momentum spectra of different kinds measured in multiparticle production processes can be described by a quasi power-law formula (Throughout the whole article a natural units (Boltzmann constant and speed of light) are used by setting its values to unity. Because data, which we discuss, are presented at midrapidity (i.e., for $\eta \sim 0$) and for large transverse momenta, $p_T \gg m$, the energy $E$, transverse energy $E_T$ and transverse mass $m_T = \sqrt{m^2 + p_T^2}$ is roughly equal to its transverse momentum $p_T$, which we shall use in what follows.)

$$h(p_T) = C \left(1 + \frac{p_T}{nT}\right)^{-n}. \tag{1}$$

For large values of transverse momenta, $p_T \gg nT$, Equation (1) becomes scale free (independent of $T$) power distribution, $1/p_T^n$. This was first proposed in [1,2] as the simplest formula extrapolating exponential behavior observed for low $p_T$ to power behavior at large $p_T$. At present it is known as the QCD-inspired Hagedorn formula [3,4]. This distribution is usually interpreted in terms of the statistical model of particle production, employing the Tsallis non-extensive statistics [5–7], with $n = 1/(q-1)$. In this case q is known as a parameter of nonextensivity.

Distribution (1) and its numerous modifications, nowadays, have been successfully used for a description of multiparticle production processes in a wide range of incident energy (from a few GeV up to a few of TeV) and in broad range of transverse momenta (see, for example, reviews [8–10]). The most recent applications of this approach come from the STAR and PHENIX Collaborations at RHIC [11,12] and from CMS [13,14], ALICE [15,16] and ATLAS [17] Collaborations at LHC (see also a recent compilations [18,19]). In particular, it turned out that it successfully describes a transverse momenta of charged particles measured by the LHC experiments, the flux of which changes by over 14 orders of magnitude [20–23].

It is now empirically well-documented that transverse spectra of both hadrons and jets exhibit a power-law behavior of $1/p_T^n$ at hight $p_T$. This observation (usually interpreted in terms of non-extensive Tsallis statistics) meets difficulty. Whereas in [21,22] it has been advocated that the power indices $n$ for hadrons are systematically greater than those for jets, in [24,25] the values of

corresponding power indices were found to be similar, strongly indicating the existence of a common mechanism behind all of these processes.

The experimental distributions of transverse momenta

$$f(p_T) = \frac{d^2\sigma}{p_T dp_T d\eta}$$

(2)

for hadrons are usually fitted directly by Tsallis distribution (1). For jets we observe deviation from the power-law at tail of distribution which is caused by kinematical constrains [25].

Recently, QCD-based formula was proposed in the form [20–23]

$$h''(p_T) = C'' \left(1 + \frac{p_T^2}{p_{T0}^2}\right)^{-n/2} \alpha_S^2(p_T)g''(p_T),$$

(3)

where a running coupling constant

$$\alpha_S(p_T) = \frac{12\pi}{27 \ln[10 + (p_T/0.25)^2]}$$

(4)

and a dumping factor

$$g''(p_T) = \frac{(1 - 2p_T/\sqrt{s})^{13}}{(1 - p_T/\sqrt{s})^{1/2}}$$

(5)

which nicely accommodate data on the tail of distribution. Experimental data interpreted in terms of QCD-based formula (3) leads to conclusion that the power indices for jets ($n \simeq$ 4–5) are systematically smaller than those for hadrons ($n \simeq$ 6–10). It should be noticed that the parameter $n$ appearing in equation (5) is not exactly the slope parameter of transverse spectra. In fact, the differences in power indices (if really exist) are much smaller. For example, at center of mass energy of two colliding protons, $\sqrt{s}$ = 7 TeV we have $\Delta n = n_{hadron} - n_{jet}$ = 6.6–5.4 = 1.2. A part of this difference ($\Delta n \simeq$ 0.5) comes from the factor $\alpha_S^2(p_T)$. The other part of differences comes from the fitting procedure using quasi-power distribution with damping factor $g''(p_T)$. Such inkling comes from the fact that whereas for hadrons $n$ decrease with energy, the one for jets is constant or even increase with energy (for lower energies we are closer to kinematical limit and fluxes of jets for $p_T/\sqrt{s} = const$ are higher).

In Ref. [25], jets observed at midrapidity and in a narrow jet cone defined by $R = \sqrt{\Delta\eta^2 + \Delta\phi^2}$ (where $\Delta\phi$ and $\Delta\eta$ are, respectively, the azimuthal angle and the pseudorapidity of hadrons relative to that of the jet) have been examined (where data on $p + \bar{p}$ and $p + p$ interactions, covering a wide energy range from 0.54 TeV up to 7 TeV, obtained by CMS [26], CDF II [27], D0 [28] and UA2 [29] experiments was used for the analysis).

Transverse momentum spectra of jets, spanning over 12 orders of magnitude in the observed cross sections, can be successfully described by the conditional Tsallis distribution

$$h'(p_T) = h(p_T|U) = C' \left(1 + \frac{p_T}{nT}\right)^{-n} g'(p_T),$$

(6)

where

$$g'(p_T) = \left(\frac{U - p_T}{U}\right)^{\nu-1} \left(1 - \frac{p_T}{nT + U}\right)^{2-\nu-n}$$

(7)

and $U$ denotes the available energy for particle production with $\nu$ degrees of freedom. For large energy ($U \gg p_T$) and large number of degrees of freedom ($\nu \gg 1$), the conditional probability distribution (6) reduces to the single particle distribution given by Equation (1).

Spectra exhibit a power-law bahavior of $1/p_T^n$ for which $n \approx$ 7–8 ($n \approx$ 8 at $\sqrt{s}$ =8 00 GeV and decrease with energy, $n = 8(\sqrt{s}/800)^{-0.064}$. Of course many other parameterizations are possible in the limited interval of energy (0.5 TeV–7 TeV). For example $n = 4 + 10(\sqrt{s})^{-0.137}$ for jets and

$n = 4 + 15.6(\sqrt{s})^{-0.2}$ for charged particles. However, all parameterizations lead to a conclusion that $n_{jet} \simeq n_{hadron}$ at TeV energies (1.2 TeV–1.4 TeV).

Comparison of power indices $n$ for jets with those for charged particles produced in minimum bias collisions is shown in Figure 1. The values of the corresponding power indices are similar, indicating the existence of a common mechanism behind these processes [24]. The suggestion that power indices for jets are almost 2 times smaller than those for hadrons [21,22] comes mainly from a different parameterization of hadrons and jets spectra (not a simple Tsallis distribution) and corresponding distributions are very similar at the high $p_T$ limit, see Figure 2.
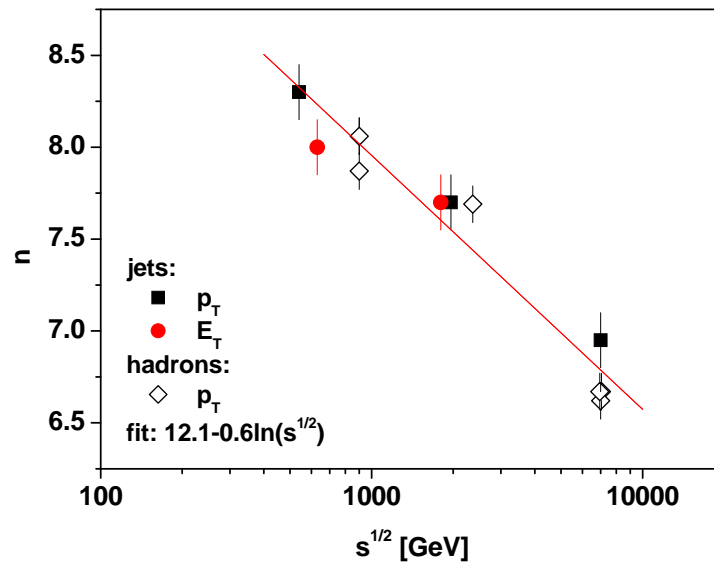


**Figure 1.** The slope parameters $n$ extracted from the fits by Equation (1) to the transverse spectra of jets [24] (full symbols) in comparison with the slope parameters for distributions of charged particles [22] (open symbols).
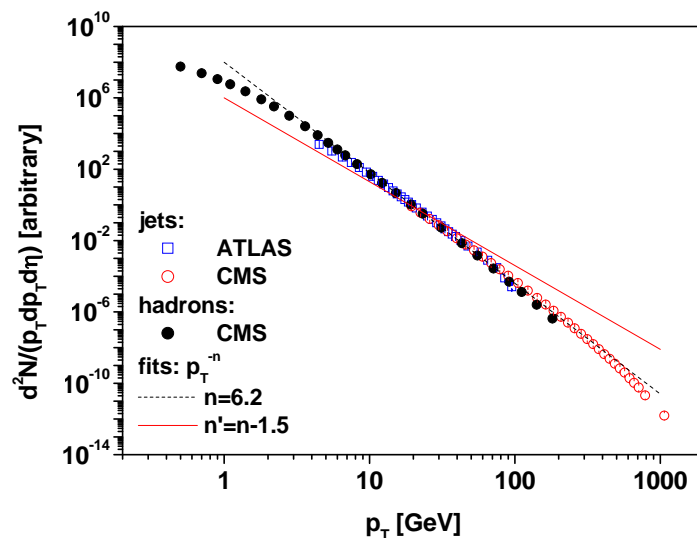


**Figure 2.** Transverse spectra for jets [17,26] (full symbol) in comparison with spectra for hadrons [13,14] (open symbols) steaming from proton-proton collisions at $\sqrt{s}$ = 7 TeV (with arbitrary normalization at $p_T$ = 20 GeV). Full and dashed line show power dependencies with slope parameters $n$ to be different by $\Delta n$ = 1.5.

Experimental studies of kaon to pion multiplicity ratio in $pp$ collisions have been reported by different measurements [30]. The experimental data on the $K/\pi$ ratio available for rather broad energy range show that it increases with $\sqrt{s}$ [30,31]. It seems natural to present it in a pure statistical phenomenological approach as a ratio of two Boltzmann–Gibbs (B-G) exponential factors

$$\left(\frac{K}{\pi}\right)_{B-G} = \frac{\exp\left(-\frac{m_K}{T}\right)}{\exp\left(-\frac{m_\pi}{T}\right)}, \tag{8}$$

where $m_K$ and $m_\pi$ are masses of charged kaon and charged pion, respectively. Observed energy dependence of $K/\pi$ ratio indicates that dynamical input is needed [30]. The non-extensive Tsallis statistics offers very economical way to account for the energy dependence of the experimentally observed $(K/\pi)_{\mathrm{exp}}$ ratio (We assumed that $K/\pi$ energy dependence comes from $q(s)$ dependence). Nonextensivity parameter roughly increase with energy as $q - 1 = 0.02\ln(\sqrt{s}/10)$ and nicely correspond with numerically evaluated energy dependence of $K/\pi = 0.075 + 0.009\ln(\sqrt{s}/10) = 0.075 + 0.45(q-1)$. On the other hand temperature parameter $T$ (hardly estimated experimentally) is a function of the chemical potential $\mu$ at fixed value of $q$ [32] (we assumed $\mu = 0$ and $T = m_\pi$). In such approach we have

$$\left(\frac{K}{\pi}\right)_q = \frac{\left[1 + (q-1)\frac{m_K}{T}\right]^{\frac{1}{1-q}}}{\left[1 + (q-1)\frac{m_\pi}{T}\right]^{\frac{1}{1-q}}}. \tag{9}$$

Due to the fact that the values of $q$ deduced from data remain close to unity, for quick estimations one can always use some approximate formulas obtained by expanding in $q - 1$ and keeping only linear terms. In our case Equation (9) is approximated by

$$\left(\frac{K}{\pi}\right)_q \simeq \left(\frac{K}{\pi}\right)_{B-G}\left[1 + \frac{1}{2}(q-1)\left(\frac{m_K^2 - m_\pi^2}{T^2}\right)\right]. \tag{10}$$

Identifying $(K/\pi)_q = (K/\pi)_{exp}$ we get, approximately,

$$q \simeq 1 + \frac{\left(\frac{K}{\pi}\right)_{exp} - \left(\frac{K}{\pi}\right)_{B-G}}{\left(\frac{K}{\pi}\right)_{B-G}\left(\frac{m_K^2 - m_\pi^2}{T^2}\right)}. \tag{11}$$

In Figure 3 we show values of $q - 1$ deduced from $(K/\pi)_{exp}$ and compared to those obtained from the $p_T$ distributions and compiled in [33] and to those obtained from the multiplicity $N$ distributions $P(N)$ [34,35]. In this case we use the fact that measured $P(N)$ are of the Negative Binomial (NBD) form ( The single NBD is usually the first choice of $P(N)$ in fitting experimental data. With growing energy and number of produced particles, the NBD deviates from data and is usually replaced by combinations of two, three or multicomponent NBDs [36,37].) for which parameter $k$ depends on the moments $\langle N \rangle$ and $Var(N)$,

$$\frac{1}{k} = \frac{Var(N)}{\langle N \rangle^2} - \frac{1}{\langle N \rangle}. \tag{12}$$

On the other hand, as shown in [38] the fact that $P(N)$ is of the NBD type can be related to that the nonextensivity of the production process characterized by nonextensivity parameter $q - 1 = 1/k$.

Actually, assuming that fluctuations of energy $U$ of the system are entirely given by its thermal part [39], i.e., $\omega_U = \omega_T$, where $\omega_X^2 = Var(X)/\langle X \rangle^2$ one gets in this case

$$q - 1 = \frac{1}{3k}. \tag{13}$$

The values of $q$ obtained from Equation (13) are plotted in Figure 3.
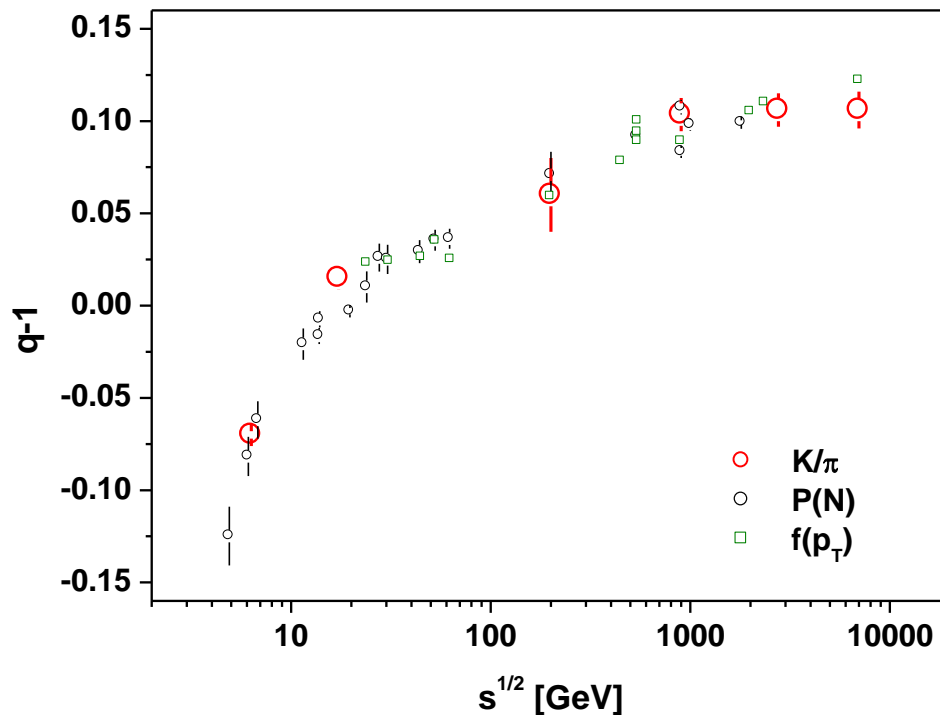
**Figure 3.** Energy dependence of $q - 1$ deduced from data on $(K/\pi)_{exp}$ ratio (charged particles) compared with values obtained from transverse momenta distributions, $f(p_T)$ [33] and from the non-diffractive samples of the multiplicity distributions $P(N)$ [34,35] of the negative binomial form.

The results presented here can be summarized in the following way:

(i)　　A Tsallis distribution successfully describes inclusive $p_T$ distributions of particles as well as jets produced in $pp$ interactions. The nonextensivity parameter in the case of jets, $q = 1.14$, is comparable to $q = 1.15$ describing inclusive distributions of transverse momenta of particles at the same energy 7 TeV.

(ii)　　The values of $q$ obtained in this case are roughly the same as those obtained from an analyzes of multiplicity distributions. In particular correspond to $q$ values evaluated from $K/\pi$ ratio as shown in Figure 3.

(iii)　　One observes a similarity of the corresponding nonextensivity parameters of multiplicity distributions $P(N)$ and transverse momentum distributions $f(p_T)$ in multiparticle production processes. Being confirmed, this strengthens the nonextensive Tsallis approach.

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