Collective excitations in QGP

– small anisotropy makes a big difference

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Outline

- ✓ Motivation
- ✓ General dispersion equation
- ✓ Isotropic system
- ✓ Weakly anisotropic system
- ✓ Prolate vs. oblate
- ✓ Conclusions

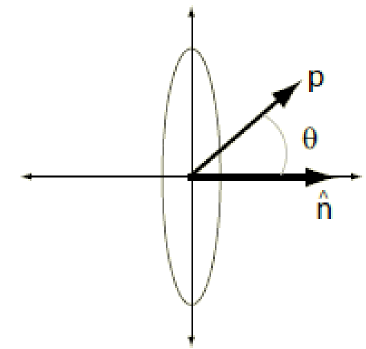
Motivation

- ✓ QGP from relativistic heavy-ion collisions is anisotropic.
- ✓ Anisotropic plasma is qualitatively different than the isotropic one.
- ✓ How big the anisotropy should be to matter?
- ✓ Spectrum of collective excitations is an important characteristics of any many body system.

Momentum distribution

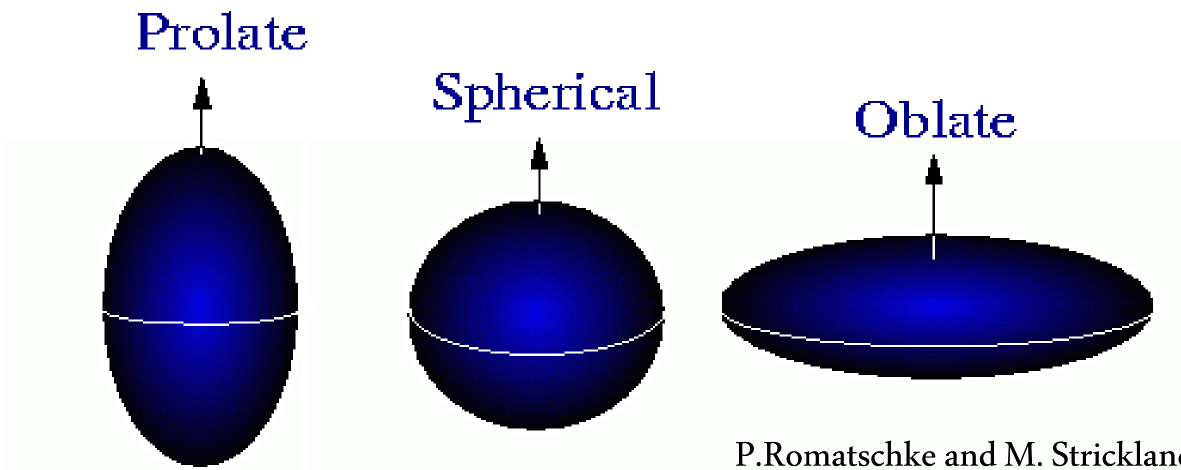
The anisotropic momentum distribution, which we use, is obtained from an isotropic one, by rescaling it in one direction

$$f_{\xi}(\mathbf{p}) = \sqrt{1+\xi}\; f_{\rm iso}\left(\sqrt{\mathbf{p}^2 + \xi(\mathbf{p}\cdot\mathbf{n})^2}\right)$$



$$\xi \in (-1, \infty)$$

$\xi < 0$ $\xi = 0$ $\xi > 0$



P.Romatschke and M. Strickland, Phys. Rev. D **68**, 036004 (2003)

General dispersion equation

The dispersion equation

$$\det\left[\Sigma(\omega,\mathbf{k})\right]=0$$

$\omega(\mathbf{k})$ - collective mode in a plasma system

Definition of the matrix sigma

$$\Sigma^{ij}(\omega,\mathbf{k})\equiv(\omega^2-\mathbf{k}^2)\delta^{ij}+k^ik^j-\Pi^{ij}(\omega,\mathbf{k})$$

inverse gluon propagator in temporal axial gauge

The dielectric tensor is related to the retarded gluon polarization tensor

$$\varepsilon^{ij}(\omega,\mathbf{k})=\delta^{ij}-\frac{1}{\omega^2}\Pi^{ij}(\omega,\mathbf{k})$$

Gluon polarization tensor can be written down, as

$$\Pi^{ij}(\omega,\mathbf{k})=\frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\frac{f(\mathbf{p})}{|\mathbf{p}|}\left[\delta^{ij}+\frac{k^iv^j+v^ik^j}{\omega-\mathbf{k}\cdot\mathbf{v}}+\frac{(\mathbf{k}^2-\omega^2)v^iv^j}{(\omega-\mathbf{k}\cdot\mathbf{v})^2}\right]$$

plasma constituents are assumed to be massless

Stable and unstable modes

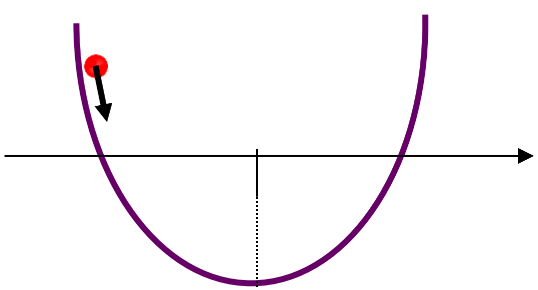
Dispersion equation

$$\det\left[\Sigma^{ij}(\omega,\mathbf{k})\right] = \det\left[-\mathbf{k}^2\delta^{ij} + k^i k^j + \omega^2\varepsilon^{ij}(\omega,\mathbf{k})\right] = 0$$

Collective mode - solution $\omega(\mathbf{k})$

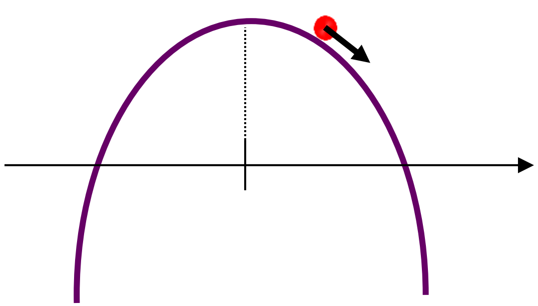
$$e^{-i\omega(\mathbf{k})t} = e^{-i[\mathrm{Re}\,\omega(\mathbf{k}) + i\,\mathrm{Im}\,\omega(\mathbf{k})]t} = e^{-i\,\mathrm{Re}\,\omega(\mathbf{k})}e^{\mathrm{Im}\,\omega(\mathbf{k})}$$

$\mathrm{Im}\,\omega(\mathbf{k}) < 0$



stable configuration

$\mathrm{Im}\,\omega(\mathbf{k}) > 0$



Unstable configuration

Unstable modes occur due to anisotropy of the momentum distribution

Method to inverse the matrix Σ

Inversion of the matrix Σ which depends on **k** and **n**

$$\Sigma = \alpha A + \beta B + \gamma C + \delta D$$

basis of matrices

$$\begin{cases} A^{ij} = \delta^{ij} - \dfrac{k^i k^j}{\mathbf{k}^2}, & B^{ij} = \dfrac{k^i k^j}{\mathbf{k}^2} \\ C^{ij} = \dfrac{n_T^i n_T^j}{\mathbf{n}_T^2}, & D^{ij} = n_T^i k^j + k^i n_T^j \end{cases}$$

$$n_T^i = \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) n^j$$

$$\Sigma^{-1} = \bar{\alpha} A + \bar{\beta} B + \bar{\gamma} C + \bar{\delta} D$$

$$\Sigma \Sigma^{-1} = \mathbf{1} \quad \Rightarrow \quad \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$$

The coefficients α, β, γ, δ are determined by the following contractions:

$$k^i \Sigma^{ij} k^j = k^2 \beta, \qquad n_T^i \Sigma^{ij} k^j = n_T^2 k^2 \delta,$$

$$n_T^i \Sigma^{ij} n_T^j = n_T^2 (\alpha + \gamma), \qquad \mathrm{Tr}\,\Sigma = 2\alpha + \beta + \gamma,$$

P. Romatschke and M. Srickland, Phys. Rev. D **68**, 036004 (2003)

Dispersion equation

The inverse matrix

$$\Sigma^{-1} = \frac{1}{\alpha} A + \frac{-\alpha(\alpha+\gamma)B + \left(-\delta^2 \mathbf{k}^2 \mathbf{n}_T^2 + \beta\gamma\right)C + \alpha\delta D}{\alpha\left(\delta^2 \mathbf{k}^2 \mathbf{n}_T^2 + \beta(\alpha+\gamma)\right)}$$

The dispersion relation:

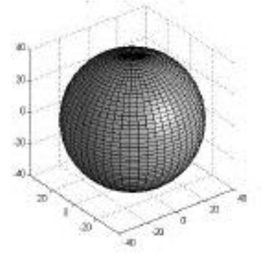
$$\alpha = 0; \qquad \delta^2 \mathbf{k}^2 \mathbf{n}_T^2 + \beta(\alpha+\gamma) = 0$$

where

$$\alpha = \Sigma^{ii} - \frac{n_T^i n_T^j}{\mathbf{n}_T^2}\Sigma^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\Sigma^{ij}; \qquad \beta = \frac{k^i k^j}{\mathbf{k}^2}\Sigma^{ij};$$

$$\gamma = -\Sigma^{ii} + 2\frac{n_T^i n_T^j}{\mathbf{n}_T^2}\Sigma^{ij} + \frac{k^i k^j}{\mathbf{k}^2}\Sigma^{ij}; \qquad \delta = \frac{n_T^i k^j}{\mathbf{n}_T^2 \mathbf{k}^2}\Sigma^{ij}$$

Collective mode in isotropic QGP



In isotropic plasma the matrix Σ is decomposed as: $\Sigma^{ij} = \alpha_{\text{iso}} A^{ij} + \beta_{\text{iso}} B^{ij}$

and the inverse propagator : $\Sigma^{-1} = \frac{1}{\alpha_{\text{iso}}} A + \frac{1}{\beta_{\text{iso}}} B$

with

$$\alpha_{iso}(\omega,\mathbf{k}) = \omega^2 - k^2 - \frac{m_D^2 \omega^2}{2k^2}\left[1 - \left(\frac{\omega}{2k} - \frac{k}{2\omega}\right)\ln\left(\frac{\omega+k}{\omega-k}\right)\right]$$

$$\beta_{iso}(\omega,\mathbf{k}) = \omega^2 + \frac{m_D^2 \omega^2}{k^2}\left[1 - \frac{\omega}{2k}\ln\left(\frac{\omega+k}{\omega-k}\right)\right]$$

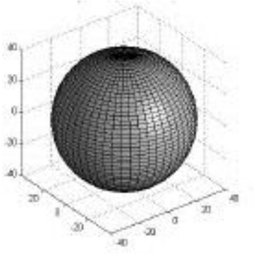
Debye mass

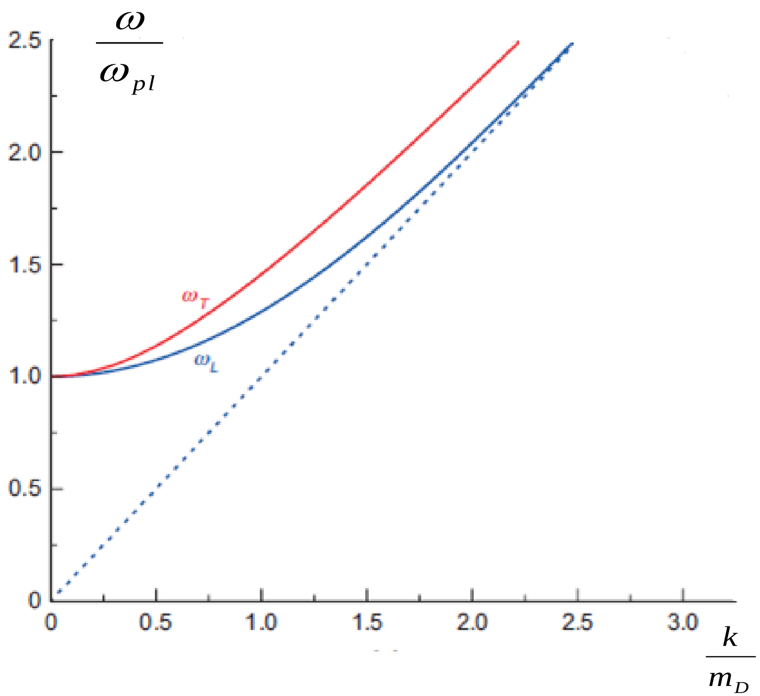
$$m_D = g^2 \int \frac{d^3 p}{(2\pi)^3} \frac{f_\xi(\mathbf{p})}{|\mathbf{p}|}$$

Dispersion relation for transverse/ longitudinal excitations

$$\alpha_{iso}(\omega,\mathbf{k}) = 0; \qquad \beta_{iso}(\omega,\mathbf{k}) = 0$$

Collective mode in isotropic QGP





- There are no UNSTABLE MODE
- There is longitudinal mode

$$\mathbf{k} \parallel \mathbf{E} \quad \Leftrightarrow \delta \boldsymbol{\rho} \approx e^{-i(\omega t - \mathbf{kr})}$$

- There is double transverse mode

$$\mathbf{k} \perp \mathbf{E} \Leftrightarrow \delta \mathbf{j} \approx e^{-i(\omega t - \mathbf{kr})}$$

- Both modes start at k = 0 with $\omega = \omega_{pl} \equiv \frac{m_D}{\sqrt{3}}$
- Both modes stay inside the light-cone

Only four real solutions - two positive & two negative

Weakly anisotropic system

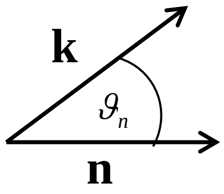
Weakly anisotropic distribution $|\xi| << 1$

$$f_\xi(\mathbf{p}) \approx \left(1+\frac{\xi}{2}\right) f_{\text{iso}}(p) + \frac{\xi}{2}\frac{d\ f_{\text{iso}}(p)}{d\ p}\, p(\mathbf{v}\cdot\mathbf{n})^2$$

Coefficients α, β, γ, δ are found in the analytic form

$$\alpha(\omega,\mathbf{k}) = \left(1+\frac{\xi}{2}\right)\alpha_{iso}(\omega,\mathbf{k}) + \xi\frac{m_D^2}{8}\left\{\frac{8}{3}\cos^2\vartheta_n + \frac{2}{3}\left(5-19\cos^2\vartheta_n\right)\frac{\omega^2}{k^2} - 2\left(1-5\cos^2\vartheta_n\right)\frac{\omega^4}{k^4}\right.$$
$$\left. + \left[1-3\cos^2\vartheta_n - \left(2-8\cos^2\vartheta_n\right)\frac{\omega^2}{k^2} + \left(1-5\cos^2\vartheta_n\right)\frac{\omega^4}{k^4}\right]\frac{\omega}{k}\ln\left(\frac{\omega+k}{\omega-k}\right)\right\},$$

$$\beta(\omega,\mathbf{k}) = \ldots,$$
$$\gamma(\omega,\mathbf{k}) = \ldots,$$
$$\delta(\omega,\mathbf{k}) = \ldots$$



Weakly anisotropic system

General dispersion equations:

1) $\alpha(\omega,\mathbf{k}) - \omega^2 + k^2 = 0$

2) $\delta^2(\omega,\mathbf{k})k^2 n_T^2 - \left(\beta(\omega,\mathbf{k}) - \omega^2\right)\left(\alpha(\omega,\mathbf{k}) + \gamma(\omega,\mathbf{k}) - \omega^2 + k^2\right) = 0$

In the limit of weak anisotropy, we have three dispersion equations because $\delta^2 = O(\xi^2)$

1) $\alpha(\omega,\mathbf{k}) - \omega^2 + k^2 = 0$

2) $\beta(\omega,\mathbf{k}) - \omega^2 = 0$

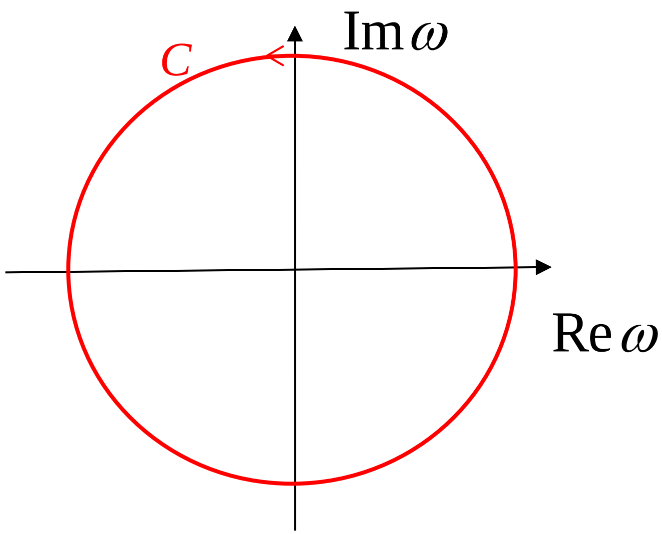
3) $\alpha(\omega,\mathbf{k}) + \gamma(\omega,\mathbf{k}) - \omega^2 + k^2 = 0$

Nyquist analysis

Nyquist analysis allows one to find the number of solution of the equation

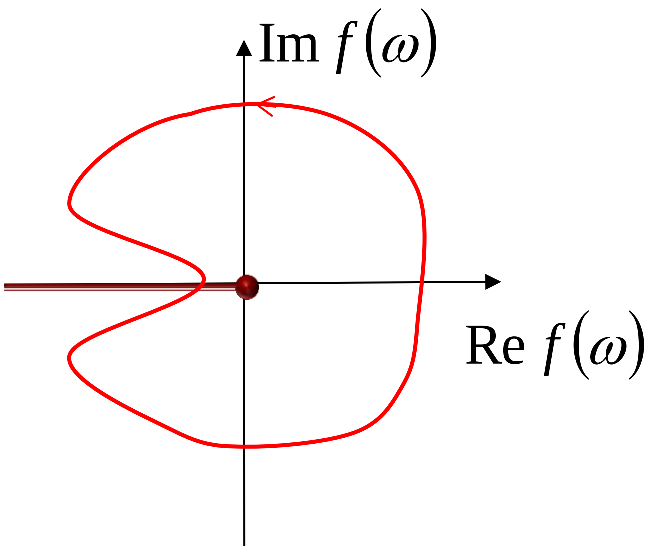
$$f(\omega)=0$$

$$\oint_C \frac{d\omega}{2\pi i}\frac{f'(\omega)}{f(\omega)}=\left.\frac{\ln f(\omega)}{2\pi i}\right|_{\omega_0^+}^{\omega_0^-}=number\ of\ \ solutions\ \ in\ C$$

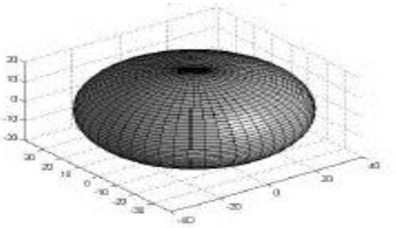


$f(\omega)$ *analytic in C*

$\mathrm{Re}\,f(\omega=0)<0$



Equation $\alpha(\omega,\mathbf{k})-\omega^2+k^2=0$



$k^2-\xi\frac{m_D^2}{3}\cos^2\vartheta_n \geq 0$ 2 solutions

$k^2-\xi\frac{m_D^2}{3}\cos^2\vartheta_n < 0$ 4 solutions

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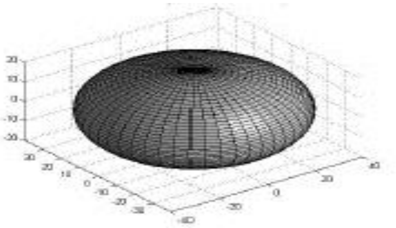

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$$\omega^2(\mathbf{k}) \underset{k<<m_D}{\approx} \frac{m^2}{3}\left(1+\frac{\xi}{10}\right) + \frac{6}{5}\left[1+\frac{\xi}{14}\left(\frac{4}{15}+\cos^2\vartheta_n\right)\right]k^2$$

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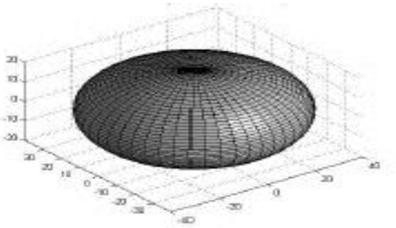

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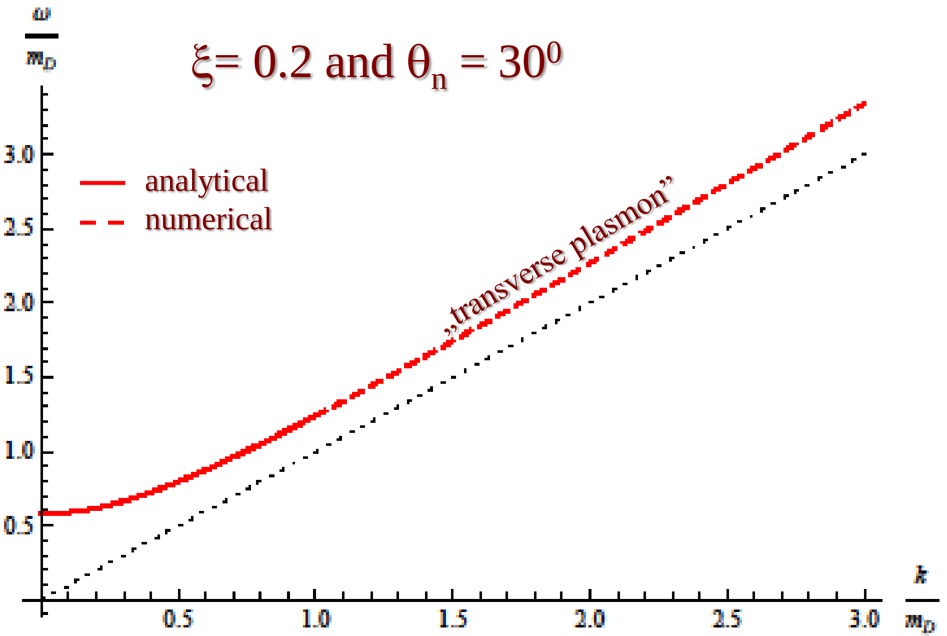

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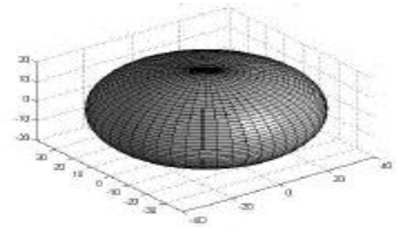

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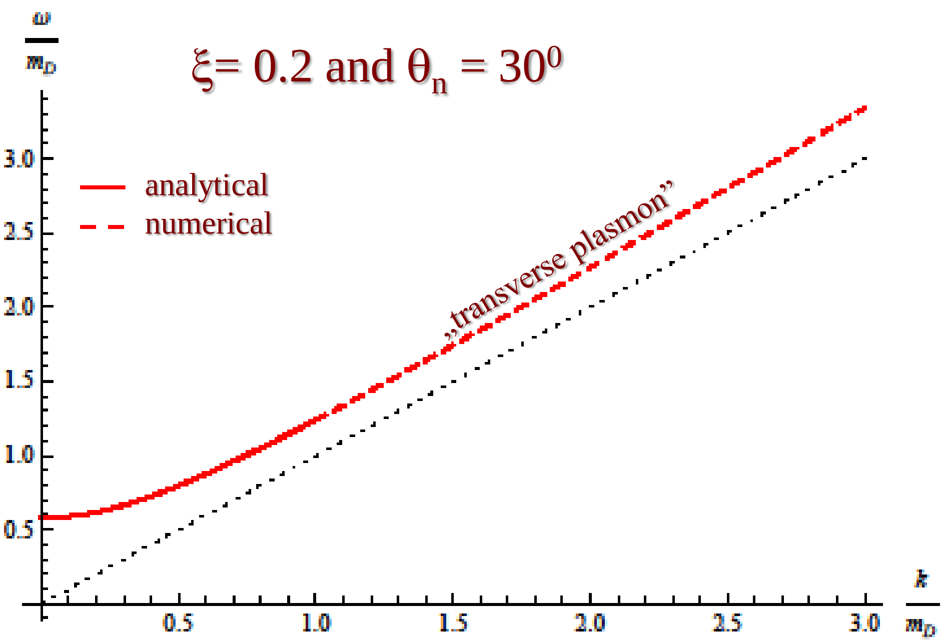
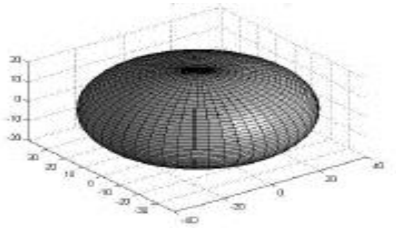
$$\omega(\mathbf{k})=\pm i\gamma(\mathbf{k})$$

$$\gamma(\mathbf{k})\underset{k>>\gamma}{\approx}\frac{1}{2}\left(\sqrt{\frac{\lambda^2}{k^2}+4\left(\eta-k^2\right)}-\frac{\lambda}{k}\right)$$



Equation $\alpha(\omega,\mathbf{k}) - \omega^2 + k^2 = 0$



$k^2 - \xi \frac{m_D^2}{3} \cos^2 \vartheta_n \geq 0$ 2 solutions

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$$\omega^2(\mathbf{k}) \underset{k<<m_D}{\approx} \frac{m^2}{3}\left(1+\frac{\xi}{10}\right) + \frac{6}{5}\left[1+\frac{\xi}{14}\left(\frac{4}{15}+\cos^2\vartheta_n\right)\right]k^2$$

$\omega(\mathbf{k}) = \pm i\gamma(\mathbf{k})$

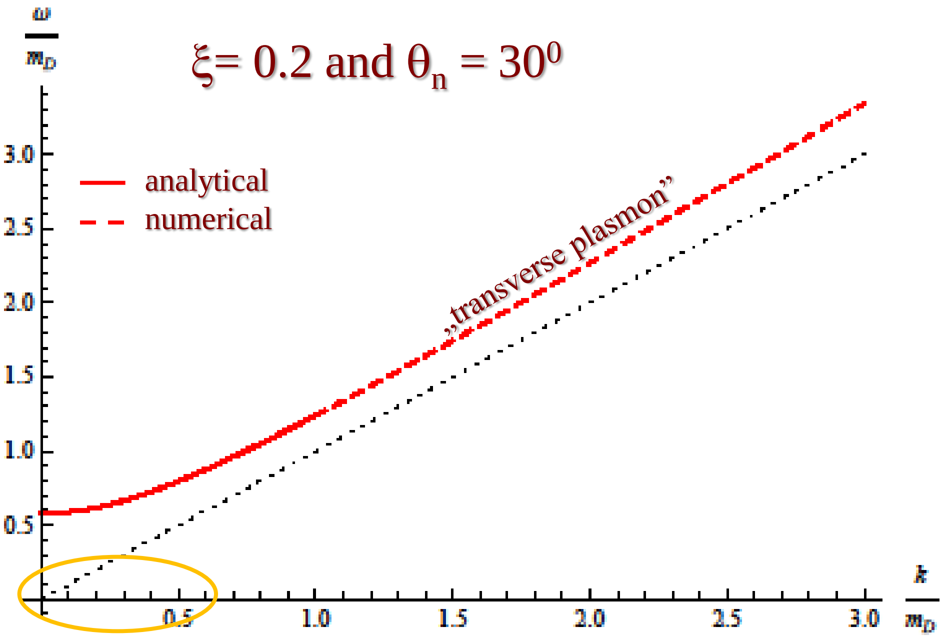
$$\gamma(\mathbf{k}) \underset{k>>\gamma}{\approx} \frac{1}{2}\left(\sqrt{\frac{\lambda^2}{k^2}+4\left(\eta-k^2\right)} - \frac{\lambda}{k}\right)$$

where

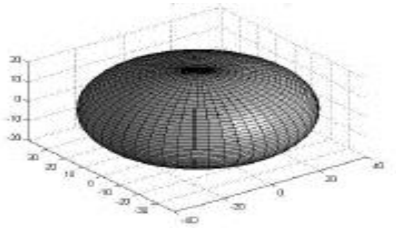
$$\lambda \equiv \frac{\pi}{4}\left(1+\frac{3}{2}\xi\cos^2\vartheta_n\right)m_D^2$$

$$\eta \equiv \frac{1}{3}\xi\cos^2\vartheta_n m_D^2$$

ξ= 0.2 and $\theta_n = 30^0$



Equation $\alpha(\omega,\mathbf{k})-\omega^2+k^2=0$



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$$\omega^2(\mathbf{k})\underset{k<<m_D}{\approx}\frac{m^2}{3}\left(1+\frac{\xi}{10}\right)+\frac{6}{5}\left[1+\frac{\xi}{14}\left(\frac{4}{15}+\cos^2\vartheta_n\right)\right]k^2$$

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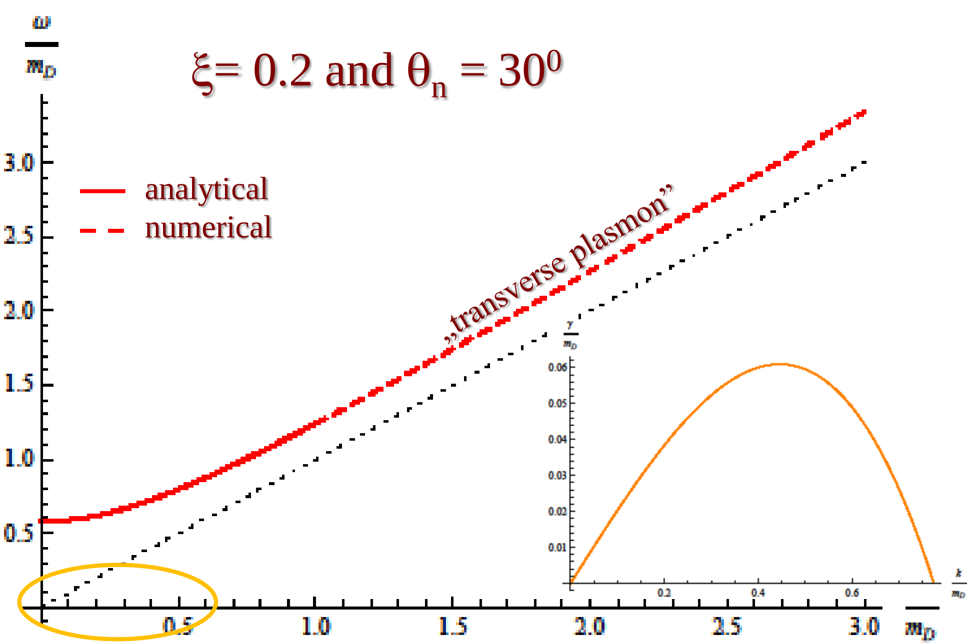
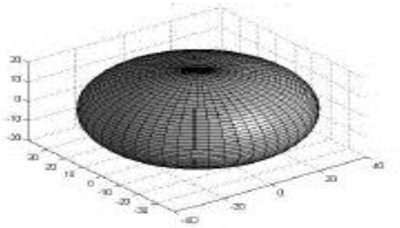


Equation $\beta(\omega, \mathbf{k}) - \omega^2 = 0$



There are always two solutions

$$\omega^2(\mathbf{k}) \underset{k<<m_D}{\approx} \frac{m_D^2}{3}\left(1+\frac{\xi}{5}\left(\frac{1}{2}+\cos^2\vartheta_n\right)\right)+\frac{3}{5}\left[1+\frac{\xi}{35}\left(4-12\cos^2\vartheta_n\right)\right]k^2$$

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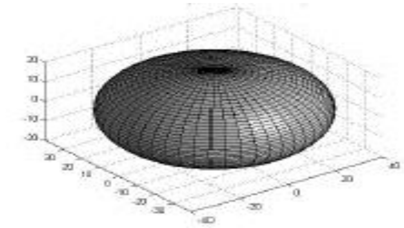

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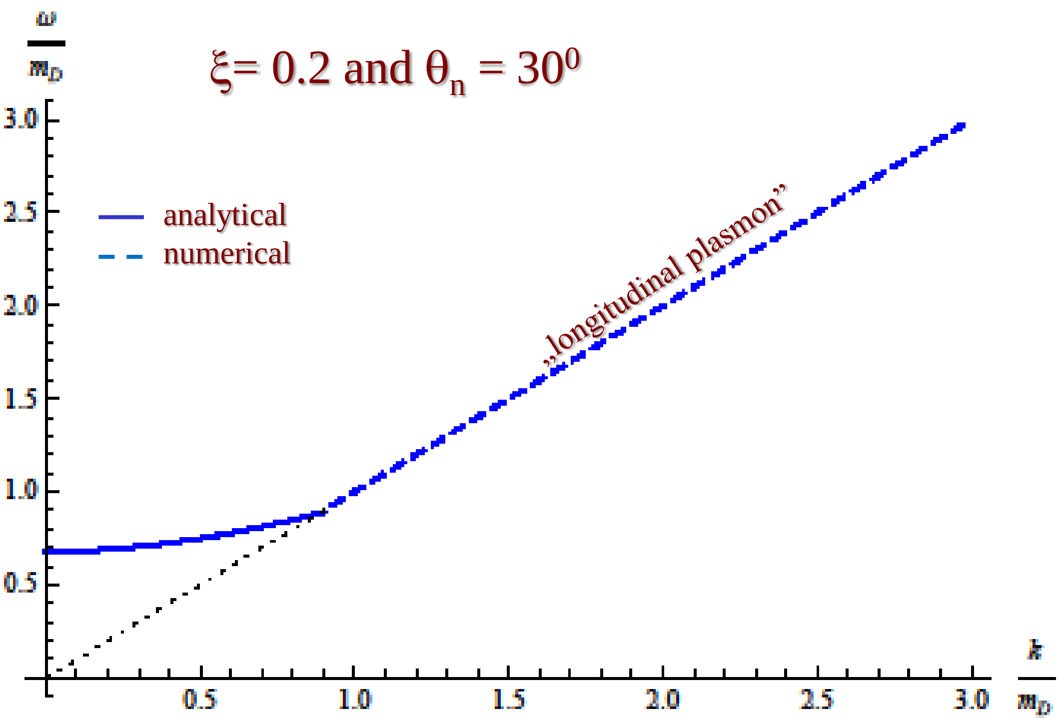
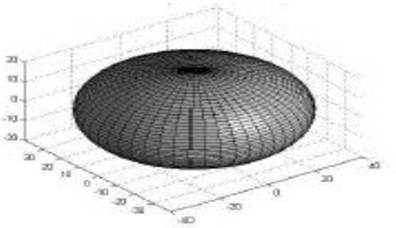


Equation $\alpha(\omega,\mathbf{k})+\gamma(\omega,\mathbf{k})-\omega^2+k^2$



$k^2+\xi\frac{m_D^2}{3}\left(1-2\cos^2\vartheta_n\right)\geq 0$ 2 solutions

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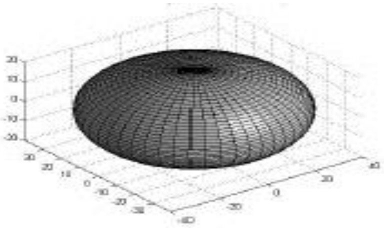

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$$\omega^2(\mathbf{k})\underset{k<<m_D}{\approx}\frac{m_D^2}{3}\left(1+\frac{\xi}{15}\left(\frac{3}{2}-\cos^2\vartheta_n\right)\right)+\frac{6}{5}\left[1-\frac{\xi}{30}\left(\frac{23}{7}-6\cos^2\vartheta_n\right)\right]k^2$$

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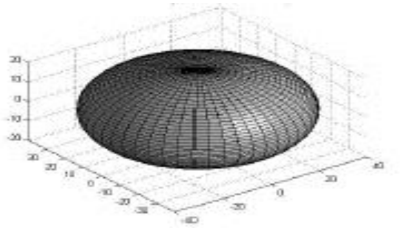

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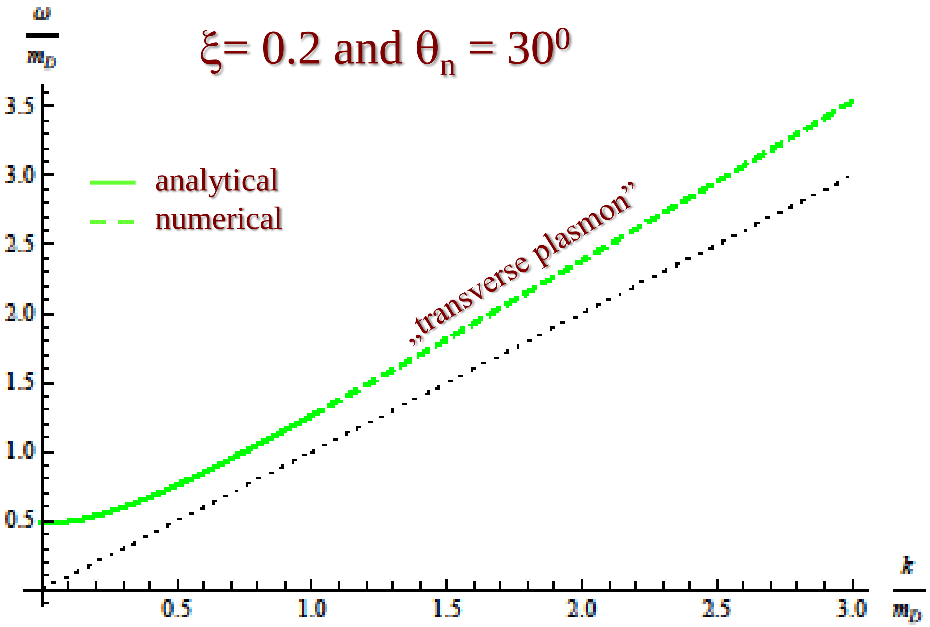
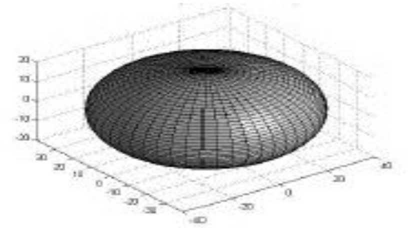


Equation $\alpha(\omega,\mathbf{k})+\gamma(\omega,\mathbf{k})-\omega^2+k^2$



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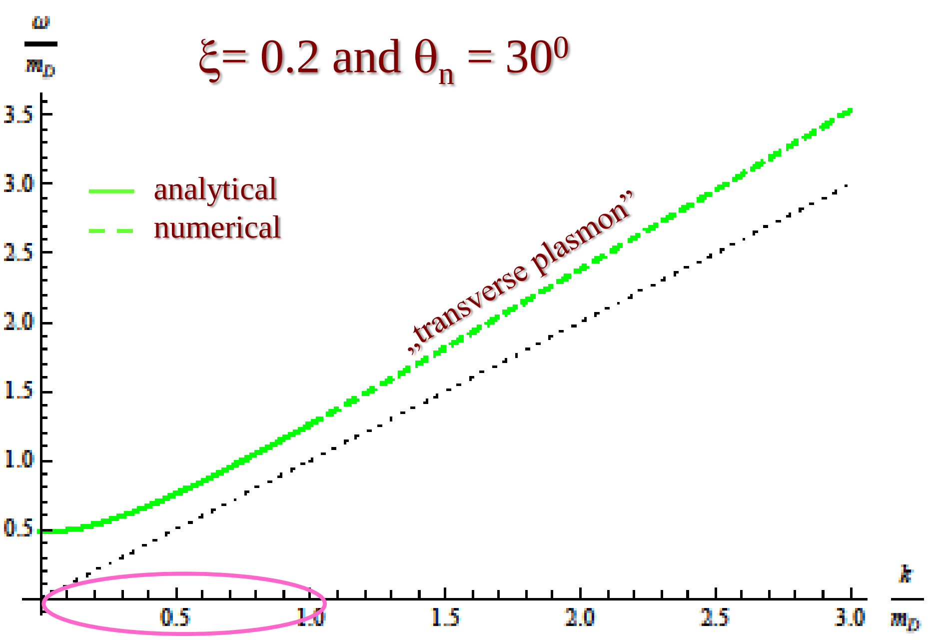
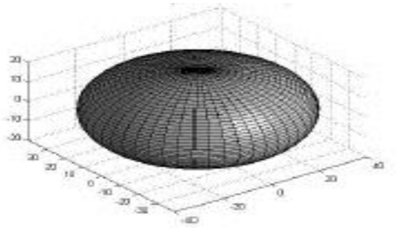
$$\omega(\mathbf{k})=\pm i\gamma(\mathbf{k})$$

$$\gamma(\mathbf{k})\underset{k>>\gamma}{\approx}\frac{1}{2}\left(\sqrt{\frac{\lambda^2}{k^2}+4\left(\eta-k^2\right)}-\frac{\lambda}{k}\right)$$



Equation $\alpha(\omega,\mathbf{k})+\gamma(\omega,\mathbf{k})-\omega^2+k^2$



$k^2+\xi\frac{m_D^2}{3}\left(1-2\cos^2\vartheta_n\right)\geq 0$ 2 solutions $\quad k^2+\xi\frac{m_D^2}{3}\left(1-2\cos^2\vartheta_n\right)<0$ 4 solutions

$$\omega^2(\mathbf{k})\underset{k<<m_D}{\approx}\frac{m_D^2}{3}\left(1+\frac{\xi}{15}\left(\frac{3}{2}-\cos^2\vartheta_n\right)\right)+\frac{6}{5}\left[1-\frac{\xi}{30}\left(\frac{23}{7}-6\cos^2\vartheta_n\right)\right]k^2$$

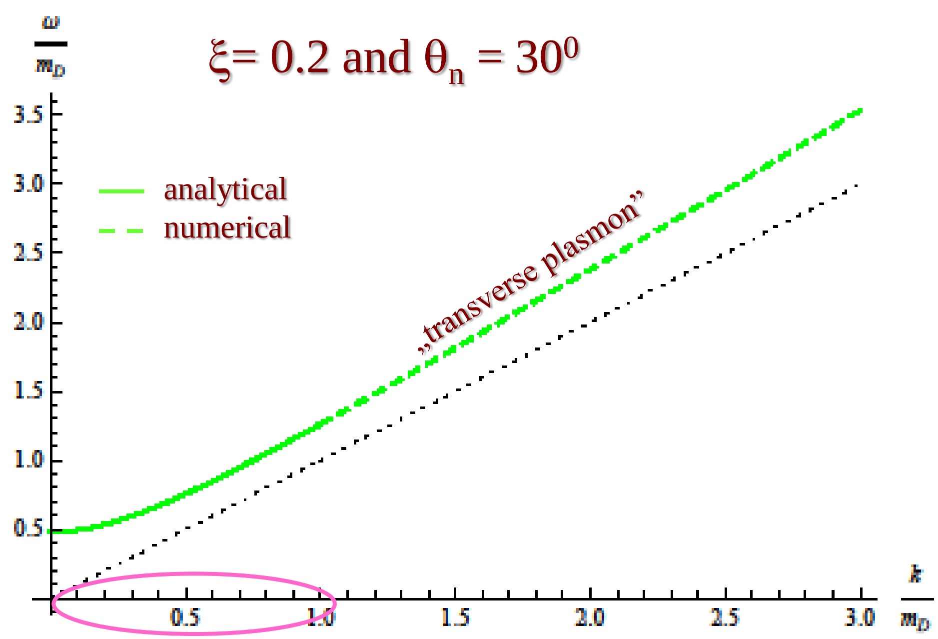
$$\omega(\mathbf{k})=\pm i\gamma(\mathbf{k})$$

$$\gamma(\mathbf{k})\underset{k>>\gamma}{\approx}\frac{1}{2}\left(\sqrt{\frac{\lambda^2}{k^2}+4\left(\eta-k^2\right)}-\frac{\lambda}{k}\right)$$

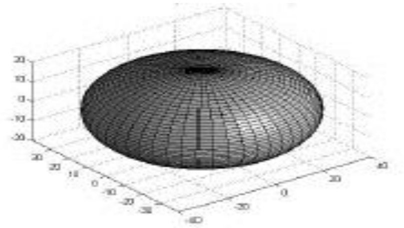
where

$$\lambda=\frac{\pi}{4}\left[1+\xi\left(-1+\frac{5}{2}\cos^2\vartheta_n\right)\right]m_D^2$$

$$\eta=\frac{1}{3}\xi\left(-1+2\cos^2\vartheta_n\right)m_D^2$$



Equation $\alpha(\omega,\mathbf{k})+\gamma(\omega,\mathbf{k})-\omega^2+k^2$



$k^2+\xi\frac{m_D^2}{3}\left(1-2\cos^2\vartheta_n\right)\geq 0$ 2 solutions $\qquad$ $k^2+\xi\frac{m_D^2}{3}\left(1-2\cos^2\vartheta_n\right)<0$ 4 solutions

$$\omega^2(\mathbf{k})\underset{k<<m_D}{\approx}\frac{m_D^2}{3}\left(1+\frac{\xi}{15}\left(\frac{3}{2}-\cos^2\vartheta_n\right)\right)+\frac{6}{5}\left[1-\frac{\xi}{30}\left(\frac{23}{7}-6\cos^2\vartheta_n\right)\right]k^2$$

$$\omega(\mathbf{k})=\pm i\gamma(\mathbf{k})$$

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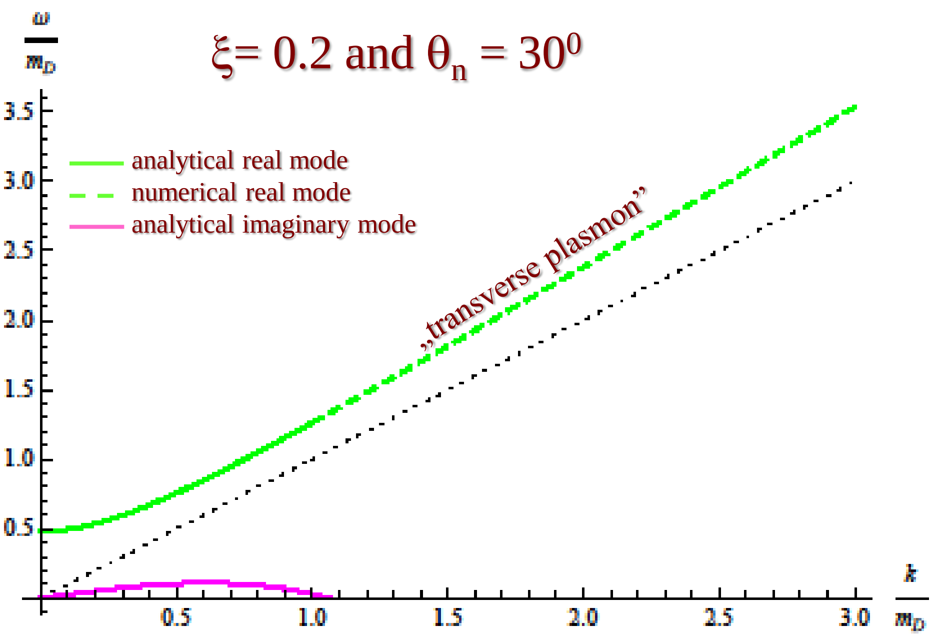
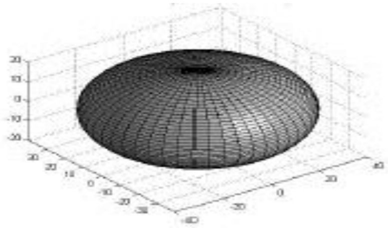
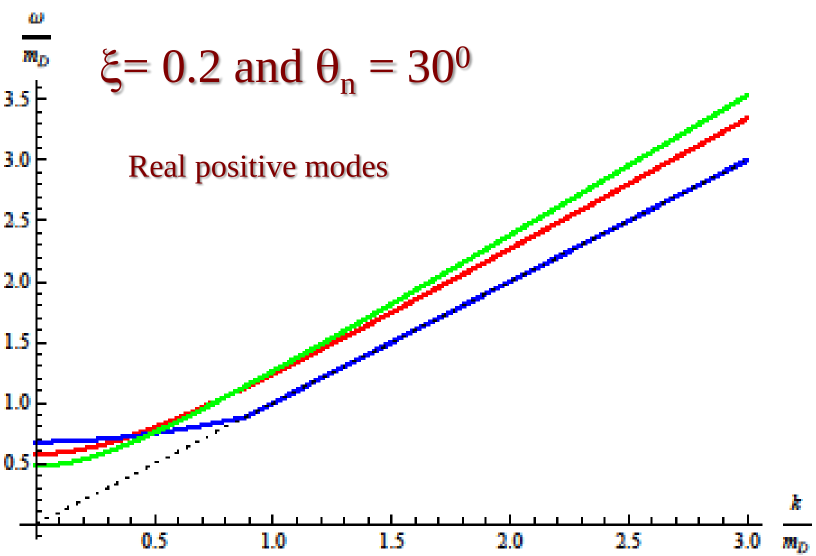


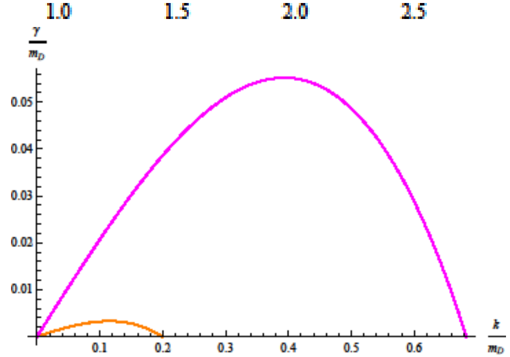
ξ - dependence

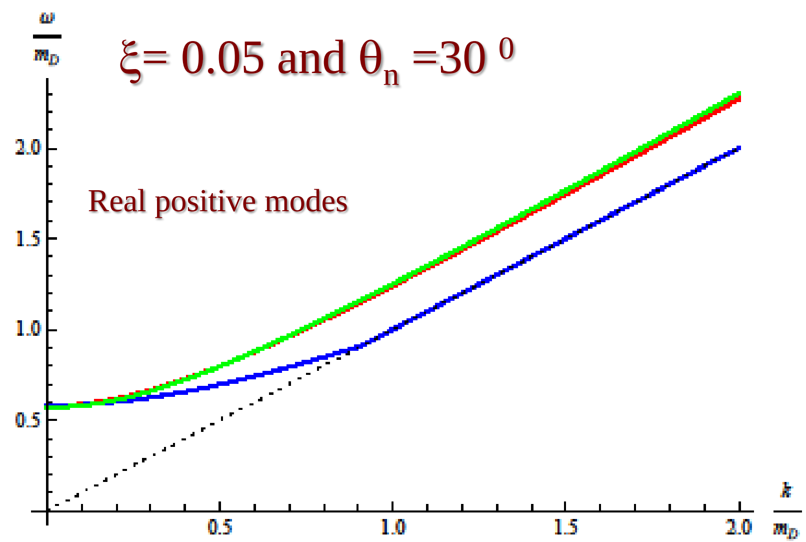


Even very small deviation from isotropy introduces a qualitative difference.

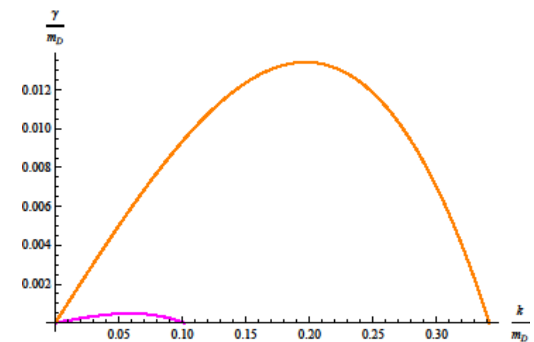


Imaginary positive modes

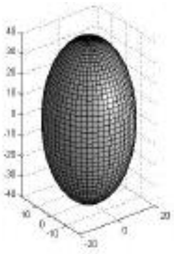




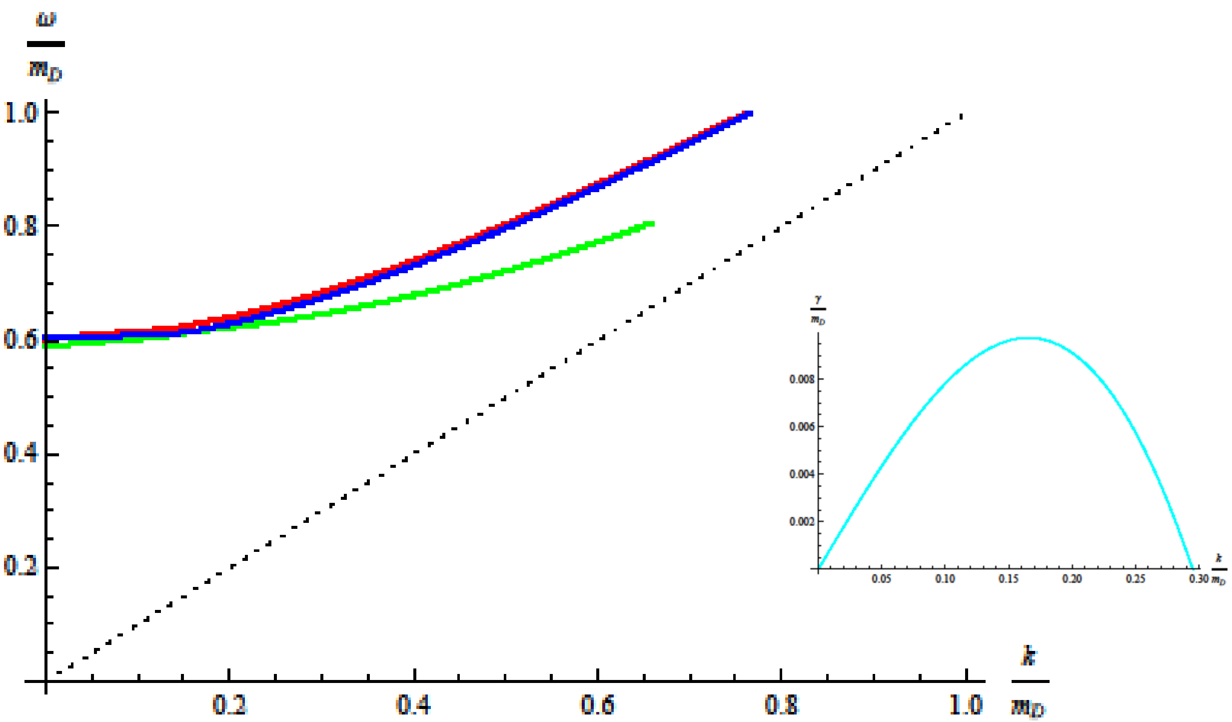
Imaginary positive modes



Prolate



$\xi = -0.2$ and $\theta_n = 30^0$



Conclusions

IF you allow even for a small anisotropy

→ you MUST consider plasma instabilities

(in a weak coupling regime)

Complete spectrum of modes is needed to compute various plasma characteristics e.g. the energy loss in anisotropic QGP.