# STATISTICAL CLUSTERS

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# STATISTICAL MODEL (as I understand it)

1. STATISTICAL MODEL: HADRONIZATION PROCEEDS THROUGH FORMATION OF INDEPENDENT CLUSTERS WHICH DECAY STATISTICALLY INTO OBSERVED HADRONS (BECATTINI).
2. ALTHOUGH DISTRIBUTION OF CLUSTERS IN MOMENTUM SPACE IS NOT FIXED BY THE MODEL, THIS APPROACH ALLOWS TO EVALUATE RATIOS OF VARIOUS PARTICLE YIELDS AND EVEN TRANSVERSE MOMENTUM SPECTRA (ASSUMING CLUSTER'S TRANSVERSE MOMENTA ARE NEGLIGIBLE). THE RESULTS AGREE WELL WITH DATA.
3. SINCE THIS IDEA SEEMS TO WORK EVERYWHERE, IT LOOKS THAT CLUSTERS REPRESENT A UNIVERSAL FEATURE OF HADRONIZATION. THEREFORE IT IS URGENT TO STUDY THEIR STRUCTURE AND PROPERTIES.

# CLUSTERS AND CORRELATIONS

1. **A VERY GENERAL CONSEQUENCE OF THE CLUSTER FORMATION IS THE PRESENCE OF** *MULTIPARTICLE CORRELATIONS***.**

2. **MORE: THE STRUCTURE AND PROPERTIES OF CLUSTERS ARE REFLECTED IN THE STRUCTURE AND PROPERTIES OF CORRELATIONS.**

3. **THEREFORE, IN SEARCH OF THE DYNAMICAL ORIGIN OF THE STATISTICAL MODEL IT IS NECESSARY TO STUDY CORRELATIONS BETWEEN PRODUCED PARTICLES.**

4. **PROBABLY THE MOST IMPORTANT QUESTION IS THAT OF** *UNIVERSALITY***.**

# TWO-PARTICLE CORRELATIONS

**1. EVEN IF PARTICLES FROM ONE CLUSTER ARE EMITTED INDEPENDENTLY, THEY ARE CORRELATED IN MOMENTUM SPACE.**

**TAKE A CLUSTER AT A POSITION $Y$ IN RAPIDITY. CONSIDER TWO PARTICLES EMITTED FROM THIS CLUSTER AT POSITIONS $y_1$ and $y_2$. THE DISTRIBUTION OF $y_1, y_2$ IS**

$$\rho(y_1, y_2) = \int dYF(Y)f(y_2 - Y)f(y_1 - Y) \tag{1}$$

**TAKING GAUSSIANS: $F = e^{-Y^2/\Omega^2}$; $f = e^{-(y-Y)^2/\Delta^2}$ WE OBTAIN**

$$\rho(y_1, y_2) \sim e^{-(y_1-y_2)^2/2\Delta^2} e^{-(y_1+y_2)^2/2(\Delta^2+2\Omega^2)} \tag{2}$$

**ONE SEES THAT DISTRIBUTION IN $(y_1 - y_2)$ MEASURES THE WIDTH OF THE CLUSTER.**

# STATISTICAL CLUSTERS

**NEGLECTING QUANTUM STATISTICS, THE DECAY OF THE CLUSTER CENTERED AT $\vec{P} = 0$ IS DESCRIBED BY**

$$\rho(\vec{p})d^3p/E \sim e^{-E/T}d^3p/E = e^{-E/T}d^2p_\perp dy \tag{3}$$

**ASSUME THE CLUSTER MOVES IN TRANSVERSE DIRECTION WITH VELOCITY $V$. ONE CAN EVALUATE VARIOUS DISTRIBUTION OF EMITTED PARTICLES:**

$$\begin{aligned} \rho(p_\perp) \sim K_0\left[\gamma m_\perp/T\right] I_0[\gamma V p_\perp/T]; \\ \rho(y) \sim \int p_\perp dp_\perp e^{-\gamma m_\perp \cosh y/T} I_0[\gamma V p_\perp/T]. \end{aligned} \tag{4}$$

**ONE SEES THAT, WITH INCREASING $V$, $\rho(p_\perp)$ BROADENS, WHILE $\rho(y)$ GETS NARROWER.**

**AT** $V = 0:\quad \rho(y) \sim \frac{1+(m/T)\cosh y}{\cosh^2 y} e^{-(m/T)\cosh y}$

# RAPIDITY CORRELATIONS 1

**CONSIDER TWO PARTICLES EMITTED FROM A CLUSTER AT REST (NO TRANSVERSE MOMENTUM) THE RAPIDITY DISTRIBUTION IS THE PRODUCT OF THE SINGLE-PARTICLE DISTRIBUTIONS. THEREFORE THE DISTRIBUTION OF THE DIFFERENCE $y_- = (y_1 - y_2)/2$ IS**

$$\rho(y_-) = \int dy_+ \rho(y_1)\rho(y_2) = \int dy_+ \rho(y_+ + y_-)\rho(y_+ - y_-) \quad (5)$$

**WITH**

$$\rho(y) \sim \frac{1 + (m/T)\cosh y}{\cosh^2 y} e^{-(m/T)\cosh y} \quad (6)$$

**THE RESULT DOES NOT DEPEND ON THE POSITION OF THE CLUSTER BUT DEPENDS ON THE RATIO $m/T$.**

# RAPIDITY CORRELATIONS 2

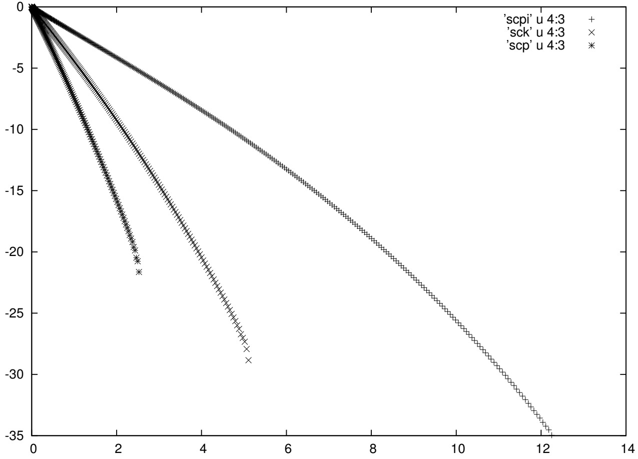


Figure: Distribution of $y_- = (y_1 - y_2)/2$ from decay of a statistical cluster at rest with $T = 160$ MeV, plotted versus $y_-^2$. The three lines correspond to pion, kaon and proton mass (proton is steepest).

# RAPIDITY CORRELATIONS 3: CLUSTERS MOVING IN TRANSVERSE DIRECTION

**CONSIDER A CLUSTER MOVING IN TRANSVERSE DIRECTION WITH VELOCITY $V$ AND THE LORENTZ FACTOR $\gamma$.**

$$\rho(y)dy = dy\frac{Am^2}{B^2}\left[1+1/B\right]e^{-B} \tag{7}$$

$$A = (m/T)\sqrt{1+\gamma^2v^2}\cosh(y-Y);$$
$$B = (m/T)\sqrt{1+(1+\gamma^2v^2)\sinh^2(y-Y)} \tag{8}$$

**AGAIN, THE DISTRIBUTION OF THE DIFFERENCE $y_- = (y_1 - y_2)/2$ IS**

$$\rho(y_-) = \int dy_+\rho(y_1)\rho(y_2) = \int dy_+\rho(y_+ + y_-)\rho(y_+ - y_-) \tag{9}$$

# RAPIDITY CORRELATIONS 4

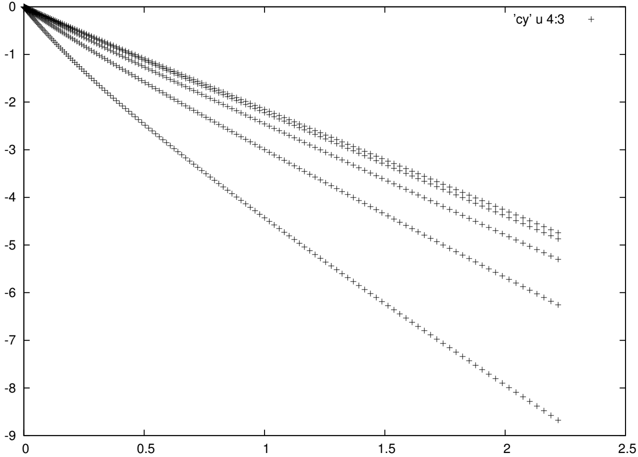


Figure: Distribution of $y_- = (y_1 - y_2)/2$, plotted vs $y_-^2$, from the pion decay of a statistical cluster moving in transverse direction with velocity $v = 0., 0.2, 0.4, 0.6, 0.8$. $T = 160$ MeV. The slope increases with increasing $v$.

# RAPIDITY CORRELATIONS 5

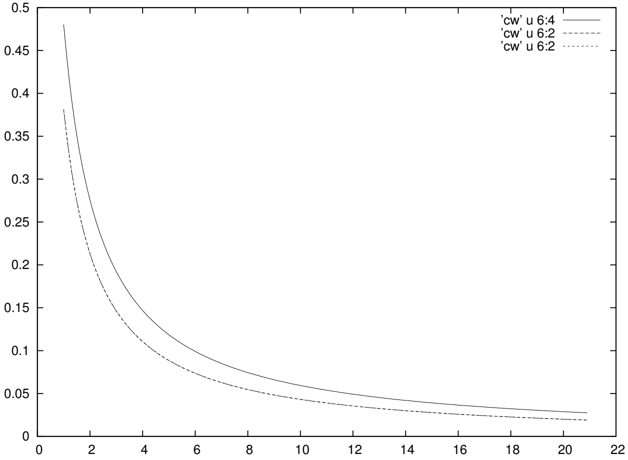


Figure: The width of rapidity distribution in cluster decay versus transverse Lorentz factor. Lower line: $<|y|>$; Upper line: $\sqrt{< y^2 >}$

# JET AS A STATISTICAL CLUSTER

**CONSIDER A CLUSTER AT RAPIDITY = 0, MOVING IN THE TRANSVERSE DIRECTION WITH TRANSVERSE ENERGY DISTRIBUTED ACCORDING TO POWER LAW:**

$$dw = (k-1)\left(\frac{E_c}{M_c}\right)^{-k} d\left(\frac{E_c}{M_c}\right) = (k-1)\gamma^{-k} d\gamma; \quad \int dw = 1. (10)$$

**THE DISTRIBUTION OF TRANSVERSE MOMENTUM OF PARTICLES FROM CLUSTER DECAY IS THEN**

$$\frac{dn}{dm_\perp^2} \sim \int_1^\infty d\gamma \gamma^{-k} I_0[\gamma V p_\perp / T] K_0[\gamma m_\perp / T] \qquad (11)$$

**IT TURNS OUT THAT THIS DISTRIBUTION IS VERY CLOSE TO THE TSALLIS DISTRIBUTION, AS SEEN IN THE NEXT SLIDE.**

# $p_\perp$ DISTRIBUTION

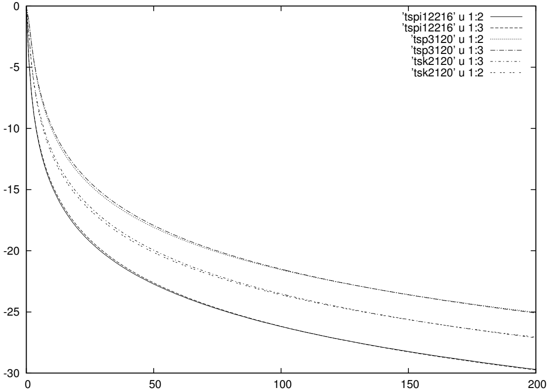


Figure: Transverse momentum distribution $dn/dp_\perp^2$ of a statistical cluster for $p_\perp \leq 200$ GeV. $T = 160$ MeV. Distribution of the cluster energy is taken to be $dN/dE \sim 1/E^k$ with $k = 4$. The result is compared with the Tsallis distribution $\sim 1/[1 + (m_\perp - m)/\kappa T_{ts}]^\kappa$. Pions: with $\kappa = 5.16$ and $T_{ts} = 122$ MeV. Kaons: $\kappa = 5.20$ and $T_{ts} = 200$ MeV. Protons: $\kappa = 5.20$ and $T_{ts} = 310$ MeV. The Tsallis formula is almost perfectly recovered

# $p_\perp$ DISTRIBUTION

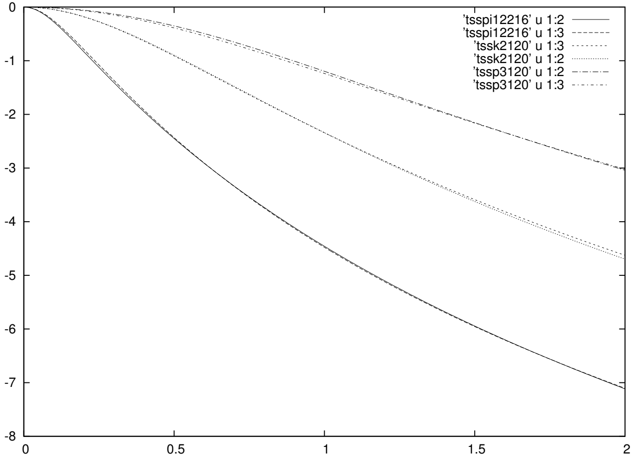


Figure: The same as in Fig. 4 in the region $p_\perp \leq 2$ GeV.

# MULTI-PARTICLE CORRELATIONS

**SELECT A BIN IN RAPIDITY OF SIZE $\delta$. CONSIDER CONTRIBUTION TO THIS BIN FROM ONE CLUSTER AT A POSITION $Y$ DISTRIBUTED WITH $F(Y)$. IF PARTICLES ARE EMITTED INDEPENDENTLY (POISSON), THE GENERATING FUNCTION OF THE DISTRIBUTION IN THE BIN $\delta$ IS**

$$\Phi(z,\delta) \equiv \sum_n P(n) z^n = \int_{-\infty}^{\infty} dY F(Y) e^{-\bar{n} g(\delta,Y)(z-1)} \tag{12}$$

**WITH**

$$g(\delta, Y) = \frac{\int_\delta dy' \, f(y' - Y)}{\int du \, f(u)} \tag{13}$$

**WHERE $f(u)$ IS THE DECAY DISTRIBUTION AND $\bar{n}$ IS THE AVERAGE MULTIPLICITY IN CLUSTER DECAY.**

# MANY CLUSTERS

**CONSIDER MANY CLUSTERS LOCATED AT THE POINTS $Y_m$ DISTRIBUTED AROUND THE POINTS $\bar{Y}_m = m\Delta$, $m = -N, ..., N$ ACCORDING TO**

$$F_m(Y_m) = (Y_0\sqrt{\pi})^{-1} e^{-(Y_m - \bar{Y}_m)^2/Y_0^2} \quad (14)$$

**IF THE CLUSTERS ARE INDEPENDENT, THE GENERATING FUNCTION IN THE BIN $\delta$ IS**

$$\Phi(z,\delta) = \prod_{m=-N}^{N} \Phi_m(y,\delta) \quad (15)$$

**WITH $\Phi_m(z,\delta) = \int_{-\infty}^{\infty} dY F_m(Y) e^{-\bar{n}g(\delta,Y)(z-1)}$.**

$$\Phi(z,\delta) = \int \left[\prod_{m=-N}^{N} dY_m F(Y_m)\right] e^{(z-1)\sum_{m=-N}^{N} \bar{n}g(\delta,Y_m)} \quad (16)$$

# FACTORIAL MOMENTS

**FROM THE GENERATING FUNCTION ONE CAN EVALUATE THE FACTORIAL MOMENTS**

$$F_k(\delta) = \frac{d^k\Phi(z,\delta)}{dz^k}[z=1]; \quad f_k \equiv F_k/F_1^k; \quad F_1 =< n > \qquad (17)$$

**FOR NEGATIVE BINOMIAL** $\phi(z) = [1+ < n > (1-z)/\kappa]^{-\kappa}$

$$f_2 = 1 + 1/\kappa\ ; \quad f_3 = (1+1/\kappa)(1+2/\kappa) \qquad (18)$$

**I.E.** $1/\kappa_2 = f2 - 1$ **AND** $1/\kappa_3 = [\sqrt{1+8f_3} - 3]/4$.

**THE DIFFERENCE BETWEEN** $1/\kappa_2$ **AND** $1/\kappa_3$ **MEASURES DEVIATION FROM THE NEGATIVE BINOMIAL.**

**I HAVE EVALUATED NUMERICALLY** $f_2$ **and** $f_3$ **OF THE DISTRIBUTION IN THE BIN** $\delta$ **FOR VARYING BIN SIZE AND VARIOUS VALUES OF** $\Delta$ **(AVERAGE DISTANCE BETWEEN CLUSTERS) AND** $Y_0$ **(THE WIDTH OF THE CLUSTER POSITION AROUND ITS AVERAGE).**

# NEGATIVE BINOMIAL ?

$\Delta = Y_0 = 2,\ 1,\ 0.5$

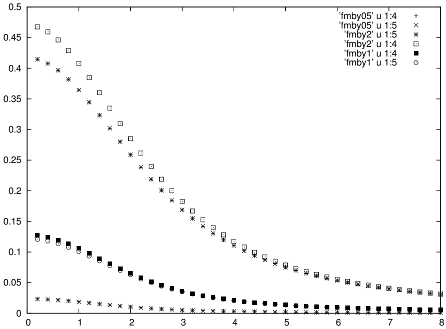


Figure: The parameter $1/k$ determined from $f_2$ and from $f_3$ plotted versus the length of the rapidity bin. The three groups of curves correspond to $\Delta = 2, \Delta = 1$ and $\Delta = 0.5$ ($\Delta$ is the distance between clusters). The distribution of clusters is $\sim e^{-(Y-Y_i)^2/Y_0^2}$ with $Y_0 = \Delta$.

# NEGATIVE BINOMIAL ?

$\Delta = 1 \quad Y_0 = \ 2,\ 1,\ 0.5$

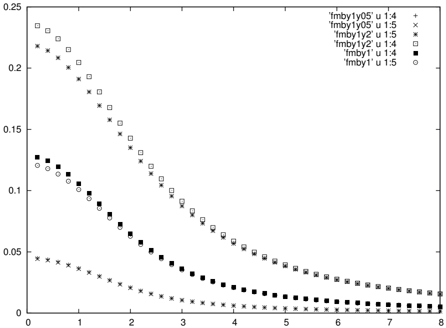


Figure: The parameter $1/k$ determined from $f_2$ and from $f_3$ plotted versus the length of the rapidity bin, for $\Delta = 1$. The three groups of curves correspond to $Y_0 = 2$, $Y_0 = 1$ and $Y_0 = 0.5$

# NEGATIVE BINOMIAL ?

$Y_0 = 1 \quad \Delta = \; 2,\; 1,\; 0.5$

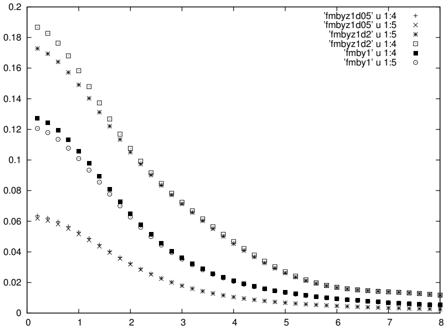


Figure: The parameter $1/k$ determined from $f_2$ and from $f_3$ plotted versus the length of the rapidity bin, for $Y_0 = 1$. The three groups of curves correspond to $\Delta = 2, \Delta = 1$ and $\Delta = 0.5$

# CONCLUSIONS AND OUTLOOK

1. IN MY OPINION, FORMATION OF CLUSTERS PROVIDES THE BEST ANSWER TO THE QUESTION WHY THE STATISTICAL MODEL WORKS ALMOST EVERYWHERE.
2. IT IS ENCOURAGING THAT THE OBSERVED (TSALLIS) TRANSVERSE MOMENTUM DISTRIBUTION CAN BE RECONSTRUCTED USING THIS IDEA.
3. IF CLUSTERS DO EXIST, THERE ARE SIMPLE CONSEQUENCES FOR MULTI-PARTICLE SPECTRA. MOST INTERESTING ARE (i) TWO PARTICLE CORRELATIONS BOTH IN LONGITUDINAL AND IN TRANSVERSE DIRECTIONS AND (ii) MULTIPLICITY DISTRIBUTIONS IN VARIOUS RAPIDITY BINS.
4. SUCH MEASUREMENTS SHOULD ALLOW TO LEARN ABOUT PROPERTIES OF THE CLUSTERS.

# APPENDIX 1

**We start from the decay of a cluster centered at $\vec{P} = 0$:**

$$\rho(\vec{k})d^3k/\epsilon \sim e^{-\epsilon/T} d^3k/\epsilon \tag{19}$$

**If this distribution is regarded from a system moving with transverse velocity $V$ in, say, $y$-direction, we have**

$$\rho(\vec{p})d^3p/E \sim= e^{-\gamma[Vp_y+E]/T} d^2p_\perp dy =$$
$$= e^{-\gamma[Vp_\perp \cos\phi + m_\perp \cosh y]/T} p_\perp dp_\perp d\phi dy \tag{20}$$

**Integration over $\phi$ gives**

$$\rho(p_\perp, y)p_\perp dp_\perp dy = 2\pi I_0[\gamma V p_\perp/T] e^{-\gamma m_\perp \cosh y/T} p_\perp dp_\perp dy \tag{21}$$

**Also integration over $y$ can be explicitly performed:**

$$\rho(p_\perp, y)p_\perp dp_\perp = 2\pi I_0[\gamma V p_\perp/T] K_0[\gamma m_\perp/T] p_\perp dp_\perp \tag{22}$$

**where we have used the identities**

$$\int_0^{2\pi} e^{-z\cos\phi} d\phi = I_0(z); \quad \int_0^{\infty} e^{-z\cosh y} dy = K_0(z) \tag{23}$$

# APPENDIX 2

**Denoting**

$$v_m^{(k)} = \int dY F_m(Y) \left[\bar{n}\, g(\delta, Y)\right]^k \qquad (24)$$

**we obtain**

$$F_1 = < n > = \sum_{m=-N}^{N} v_m^{(1)} \qquad (25)$$

$$F_2 = \sum_{m=-N}^{N} \left\{ v_m^{(2)} - [v_m^{(1)}]^2 \right\} + [F_1]^2 \qquad (26)$$

$$F_3 = \sum_{i=-N}^{N} v_i^{(3)} + 3 \sum_{i} v_i^{(2)} \sum_{j \neq i} v_j^{(1)} + \sum_{i \neq j \neq k} v_i^{(1)} v_j^{(1)} v_k^{(1)} =$$

$$= \sum_{i=-N}^{N} \left\{ v_i^{(3)} + 2[v_i^{(1)}]^3 - 3 v_i^{(2)} v_i^{(1)} \right\} + 3 F_2 F_1 - 2 F_1^3 \qquad (27)$$