# Quasiparticles of boson gas

In Lecture IV we have discussed thermodynamic characteristics of a weakly interacting boson gas in equilibrium using the imaginary time formalism. The system has been described by means of the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x), \tag{1}$$

where $\phi(x)$ is the real scalar field, $m$ is the mass parameter and $\lambda$ is the coupling constant which is assumed to be a small number. We have found that the energy and pressure, which include the first order corrections, are

$$U = \frac{\pi^2 V\,T^4}{30}\Big[1 - \frac{5\lambda}{64\pi^2}\Big], \qquad\qquad p = \frac{\pi^2 T^4}{90}\Big[1 - \frac{5\lambda}{64\pi^2}\Big]. \tag{2}$$

In this lecture properties of gas constituents – *quasiparticles* – will be studied applying the real-time formalism of statistical QFT. We will use again the Lagrangian density (1).

## Dispersion equation

- Quasiparticles are either particles whose properties are modified due to particle interaction with a medium or they are particle-like collective excitations of a medium. In both cases the medium behaves as if it contained weakly interacting particles.
- The main characteristics of a quasiparticle is a dispersion relation which gives the energy as a function of momentum of a quasiparticle.
- In the absence of interaction, the dispersion equation is $p^2 - m^2 = 0$, and consequently, $E_{\mathbf{p}} = \pm\sqrt{m^2 + \mathbf{p}^2}$, where the sign $+$ is for particles and $-$ for antiparticles.
- The dispersion relation is determined by a position of a pole of the retarded Green's function. Since the retarded Green's function is, as we remember, of the form

$$\Delta^+(p) = \frac{1}{p^2 - m^2 + \Pi^+(p)}, \tag{3}$$

where $\Pi^+(p)$ is the self energy, the dispersion relation is a solution of the equation

$$\boxed{p^2 - m^2 + \Pi^+(p) = 0.} \tag{4}$$

- To grasp a physical meaning of the dispersion equation (4) it is useful to consider a field equation of motion

$$(\Box + m^2)\langle\hat{\phi}(x)\rangle = \frac{1}{3!}\langle\hat{\phi}^3(x)\rangle, \tag{5}$$

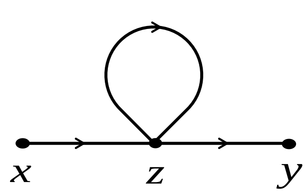


Figure 1: The first order contribution to the retarded Green's function

which is rewritten as

$$(\Box + m^2)\langle\hat{\phi}(x)\rangle = \int d^4x'\,\Pi^+(x-x')\langle\hat{\phi}(x')\rangle, \tag{6}$$

where we have introduced the retarded self-energy to solve the equation as an initial value problem. Performing the Fourier transformation, we get

$$\big(p^2 - m^2 + \Pi^+(p)\big)\langle\hat{\phi}(p)\rangle = 0. \tag{7}$$

- There is a nontrivial solution of the equation (7) if $p^2 - m^2 + \Pi^+(p) = 0$. So, we see that the dispersion equation provides a necessary condition for an existence of solutions to the field equation.
- The solution of Eq. (7) is of the form

$$\langle\hat{\phi}(p)\rangle \sim \delta\big(p^2 - m^2 + \Pi^+(p)\big), \tag{8}$$

and consequently,

$$\langle\hat{\phi}(x)\rangle \sim \sum_n C_n \exp\Big[-i\Big(\big(\omega_n(\mathbf{p}) + i\gamma_n(\mathbf{p})\big)t - \mathbf{p}\cdot\mathbf{r}\Big)\Big], \tag{9}$$

where $x \equiv (t, \mathbf{r})$ and $\omega_n(\mathbf{p}) + i\gamma_n(\mathbf{p})$ is a solution of the dispersion equation with the functions $\omega_n(\mathbf{p})$ and $\gamma_n(\mathbf{p})$ being both real.

- We note that the self energy $\Pi^+(p)$ is, in general, a complex valued function, and consequently a solution of the dispersion equation is complex.
- As one observes, the amplitude of the solution (9) is time dependent through the factor $e^{\gamma_n(\mathbf{p})t}$. When $\gamma_n(\mathbf{p}) < 0$, a quasiparticle excitation is damped or a quasiparticle has a finite lifetime $\tau = 1/\gamma_n(\mathbf{p})$. When $\gamma_n(\mathbf{p}) > 0$ the amplitude exponentially grows and we deal with an instability. When $\gamma_n(\mathbf{p}) = 0$, the quasiparticles are stable – the field amplitude is constant as a function of time.

## First order dispersion relation

- The first order correction to the retarded Green's function is represented by the diagram shown in Fig. 11 and it is given as

$$\Delta^+_{(1)}(x-y) = \frac{i\lambda}{2}\int d^4z\,\Delta^+_0(x,z)\,\Delta^>_0(z,z)\,\Delta^+_0(z,y), \tag{10}$$

which can be rewritten as

$$\Delta^+_{(1)}(x) = \frac{i\lambda}{2}\,\Delta^>_0(0)\int d^4z\,\Delta^+_0(x-z)\,\Delta^+_0(z). \tag{11}$$

- As we remember, only the connected diagrams should be taken into account.
- After the Fourier transformation, the equation (11) becomes

$$\Delta^+_{(1)}(p) = \frac{i\lambda}{2}\,\Delta^>_0(x=0)\,\Delta^+_0(p)\,\Delta^+_0(p). \tag{12}$$

- Comparing the formula (12) with
$$\Delta^+(p) = \Delta_0^+(p) - \Delta_0^+(p)\,\Pi_{(1)}^+(p)\,\Delta_0^+(p), \tag{13}$$
one finds that
$$\Pi_{(1)}^+(p) = -\frac{i\lambda}{2}\,\Delta_0^>(x=0). \tag{14}$$

- Since
$$i\Delta_0^>(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}}\Big[\big(f(\omega_{\mathbf{k}})+1\big)e^{-ikx} + f(\omega_{\mathbf{k}})e^{ikx}\Big], \tag{15}$$
where the boson distribution function equals
$$f(\omega_{\mathbf{k}}) \equiv \frac{1}{e^{\beta\omega_{\mathbf{k}}}-1}, \tag{16}$$
and $\omega_{\mathbf{k}} \equiv \sqrt{m^2+\mathbf{k}^2}$, we have
$$\Pi_{(1)}^+(p) = -\frac{\lambda}{2}\int \frac{d^3k}{(2\pi)^3}\frac{2f(\omega_{\mathbf{k}})+1}{2\omega_{\mathbf{k}}}. \tag{17}$$

- After performing the trivial angular integral, Eq. (17) becomes
$$\Pi_{(1)}^+(p) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dk\,k^2}{\sqrt{m^2+k^2}}\frac{1+e^{-\beta\sqrt{m^2+k^2}}}{1-e^{-\beta\sqrt{m^2+k^2}}}. \tag{18}$$

- Since for $k \gg m$ and $k \gg T$ the integrand linearly grows with $k$, the integral in Eq. (18) is quadratically divergent. One observes that the divergence remains in the zero temperature limit that is when $\beta \to \infty$. Therefore, it is the ultraviolet divergence which is well known in vacuum QFT.

- To get a finite result one should subtract the vacuum contribution from the formula (18). Since $f(\omega_{\mathbf{k}}) = 0$ in vacuum, the subtraction is done as follows
$$\Pi_R^+(p) \equiv -\frac{\lambda}{2}\int \frac{d^3k}{(2\pi)^3}\frac{2f(\omega_{\mathbf{k}})+1}{2\omega_{\mathbf{k}}} + \frac{\lambda}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{k}}} = -\frac{\lambda}{2}\int \frac{d^3k}{(2\pi)^3}\frac{f(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \tag{19}$$
and $\Pi_R^+(p)$ is the <u>renormalized</u> self energy, which is finite, and it equals
$$\Pi_R^+(p) = -\frac{\lambda}{4\pi^2}\int_0^\infty \frac{dk\,k^2}{\sqrt{m^2+k^2}}\frac{1}{e^{\beta\sqrt{m^2+k^2}}-1}. \tag{20}$$

- If $m \ll T$, we can put $m=0$ under the integral (20). Then, one finds
$$\Pi_R^+(p) = -\frac{\lambda}{4\pi^2}\int_0^\infty \frac{dk\,k}{e^{\beta k}-1} = -\frac{\lambda T^2}{4\pi^2}\underbrace{\int_0^\infty \frac{dx\,x}{e^x-1}}_{=\frac{\pi^2}{6}} = -\frac{\lambda T^2}{24}. \tag{21}$$

- Defining an effective mass of a quasiparticle as
$$m_*^2 \equiv m^2 - \Pi_R^+(m, \mathbf{p}=0), \tag{22}$$
one finds that
$$m_*^2 = \frac{\lambda T^2}{24}, \tag{23}$$
if $m \ll T$.

- We see that bosons, which are massless in vacuum, become massive in the gas that is they acquire the thermal mass (23).
- Since $\Pi^{+}_{(1)}(p)$ is pure real, the quasiparticles are stable at the first order of perturbative expansion.