Equations of motion of Green's functions

In Lecture V we have discussed the equations of motion of free Green's functions of the real-time formalism of statistical QFT. In this lecture the discussion is extended to interacting fields. The equations of motion play an important role in statistical QFT.

Equations of motion of contour Green's functions

- The equations of motion of contour Green's function are

$$\big(\Box_x + m^2\big)\Delta(x,y) = -\delta_C^{(4)}(x,y) + \int_C d^4x'\,\Pi(x,x')\,\Delta(x',y), \tag{1}$$

$$\big(\Box_y + m^2\big)\Delta(x,y) = -\delta_C^{(4)}(x,y) + \int_C d^4x'\,\Delta(x,x')\,\Pi(x',y), \tag{2}$$

where the contour Dirac delta is defined as

$$\delta_C^{(4)}(x,y) = \begin{cases} \delta^{(4)}(x-y) & \text{for} \;\; x_0, y_0 \;\; \text{from the upper branch,} \\ 0 & \text{for} \;\; x_0, y_0 \;\; \text{from the different branches,} \\ -\delta^{(4)}(x-y) & \text{for} \;\; x_0, y_0 \;\; \text{from the lower branch,} \end{cases} \tag{3}$$

and $\Pi(x,y)$ is the self energy which describes the effect of interaction.

- As we explain below, the equations (1, 2) can be treated as a definition of the self energy provided the free and interacting Green's functions are known.

- Keeping in that the free Green's functions satisfy the equations

$$\big(\Box_x + m^2\big)\Delta_0(x,y) = -\delta_C^{(4)}(x,y), \tag{4}$$

$$\big(\Box_y + m^2\big)\Delta_0(x,y) = -\delta_C^{(4)}(x,y), \tag{5}$$

the equations (1, 2) can be written symbolically as

$$\Delta_0^{-1}\,\Delta = \mathbb{1} - \Pi\,\Delta, \tag{6}$$

$$\Delta\,\Delta_0^{-1} = \mathbb{1} - \Delta\,\Pi. \tag{7}$$

- Eqs. (6, 7) show that

$$\Pi = \Delta^{-1} - \Delta_0^{-1}. \tag{8}$$

Equations of motion of real-time Green's functions

- The contour self energy is decomposed as

$$\Pi(x,y) = \delta_C^{(4)}(x,y)\Pi_\delta(x) + \Theta_C(x_0,y_0)\,\Pi^>(x,y) + \Theta_C(y_0,x_0)\,\Pi^<(x,y), \tag{9}$$

where the one-point contribution $\Pi_\delta(x)$, which is generated by tadpole diagrams, is singled out. One also observes that the contour self energy equals

$$\Pi(x,y) = \delta_C^{(4)}(x,y)\Pi_\delta(x) + \begin{cases} \Pi^>(x,y) & \text{for} \;\; x_0 \in C^- \;\;\&\;\; y_0 \in C^+, \\ \Pi^<(x,y) & \text{for} \;\; x_0 \in C^+ \;\;\&\;\; y_0 \in C^-, \\ \Pi^c(x,y) & \text{for} \;\; x_0 \in C^+ \;\;\&\;\; y_0 \in C^+, \\ \Pi^a(x,y) & \text{for} \;\; x_0 \in C^- \;\;\&\;\; y_0 \in C^-, \end{cases} \tag{10}$$

where C^+ and C^- denote the upper and the lower branch of the Keldysh contour.

Equations of motion of $\Delta^{\gtrless}$

- Eqs. (1, 2) provide the equations of motion of real-time Green's functions. Putting x_0 on the lower branch of the contour of the contour and y_0 on the upper branch, we obtain the equations of $\Delta^>(x,y)$

$$\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^>(x,y) = \int d^4x' \Big(\Pi^>(x,x')\,\Delta^c(x',y) - \Pi^a(x,x')\,\Delta^>(x',y)\Big), \tag{11}$$

$$\big(\Box_y + m^2 - \Pi_\delta(y)\big)\Delta^>(x,y) = \int d^4x' \Big(\Delta^>(x,x')\,\Pi^c(x',y) - \Delta^a(x,x')\,\Pi^>(x',y)\Big), \tag{12}$$

where the decomposition (9) has been used.

- The equations of motion of $\Delta^<(x,y)$ are

$$\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^<(x,y) = \int d^4x' \Big(\Pi^c(x,x')\,\Delta^<(x',y) - \Pi^<(x,x')\,\Delta^a(x',y)\Big), \tag{13}$$

$$\big(\Box_y + m^2 - \Pi_\delta(y)\big)\Delta^<(x,y) = \int d^4x' \Big(\Delta^c(x,x')\,\Pi^<(x',y) - \Delta^<(x,x')\,\Pi^a(x',y)\Big). \tag{14}$$

Equations of motion of $\Delta^{\pm}$

- The retarded and advanced Green's functions are, as we remember, defined in the following way

$$\Delta^+(x,y) \equiv \Theta(x_0-y_0)\big(\Delta^>(x,y) - \Delta^<(x,y)\big), \tag{15}$$

$$\Delta^-(x,y) \equiv -\Theta(y_0-x_0)\big(\Delta^>(x,y) - \Delta^<(x,y)\big). \tag{16}$$

- The equations of motion of the functions $\Delta^{\pm}$ can be found from Eq. (11 - 14), using the identities

$$\Delta^c(x,y) = \Delta^+(x,y) + \Delta^<(x,y), \qquad \Delta^a(x,y) = -\Delta^+(x,y) + \Delta^>(x,y), \tag{17}$$

$$\Delta^c(x,y) = \Delta^-(x,y) + \Delta^>(x,y), \qquad \Delta^a(x,y) = -\Delta^-(x,y) + \Delta^<(x,y), \tag{18}$$

which are found directly from the definitions (15, 16) combined with the definitions of the chronological (Feynman) and antichronological functions which can be written as

$$\Delta^c(x,y) \equiv \Theta(x_0-y_0)\,\Delta^>(x,y) + \Theta(y_0-x_0)\,\Delta^<(x,y), \tag{19}$$

$$\Delta^a(x,y) \equiv \Theta(y_0-x_0)\,\Delta^>(x,y) + \Theta(x_0-y_0)\,\Delta^<(x,y). \tag{20}$$

- Let us derive the first equation of motion of $\Delta^+(x,y)$. Starting with the formula (15) and using the equations (11, 13), one finds

$$\begin{aligned}\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) \qquad\qquad (21)\\ + \Theta(x_0-y_0)\int d^4x' \Big(\Pi^>(x,x')\,\Delta^c(x',y) - \Pi^a(x,x')\,\Delta^>(x',y)\\ - \Pi^c(x,x')\,\Delta^<(x',y) + \Pi^<(x,x')\,\Delta^a(x',y)\Big).\end{aligned}$$

- Using the identities (17, 18) and the analogous relations among the self energies, Eq. (11) provides

$$
\begin{aligned}
&\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) + \Theta(x_0-y_0) \qquad (22)\\
&\times \int d^4x' \Big(\Pi^>(x,x')\big(\Delta^-(x',y)+\Delta^>(x',y)\big) - \big(-\Pi^+(x,x')+\Pi^>(x,x')\big)\Delta^>(x',y)\\
&\quad - \big(\Pi^+(x,x')+\Pi^<(x,x')\big)\Delta^<(x',y) + \Pi^<(x,x')\big(-\Delta^-(x',y)+\Delta^<(x',y)\big)\Big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) \qquad (23)\\
&\quad + \Theta(x_0-y_0)\int d^4x' \Big(\Pi^>(x,x')\,\Delta^-(x',y) + \Pi^+(x,x')\,\Delta^>(x',y)\\
&\quad - \Pi^+(x,x')\,\Delta^<(x',y) - \Pi^<(x,x')\Delta^-(x',y)\Big),
\end{aligned}
$$

and it is further rewritten as

$$
\begin{aligned}
&\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) + \Theta(x_0-y_0) \qquad (24)\\
&\times \int d^4x' \Big(\big(\Pi^>(x,x')-\Pi^<(x,x')\big)\Delta^-(x',y) + \Pi^+(x,x')\big(\Delta^>(x',y)-\Delta^<(x',y)\big)\Big).
\end{aligned}
$$

- Using the identity

$$
\Delta^+(x,y) - \Delta^-(x,y) = \Delta^>(x,y) - \Delta^<(x,y), \qquad (25)
$$

the equation (24) becomes

$$
\begin{aligned}
&\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) + \Theta(x_0-y_0) \qquad (26)\\
&\times \int d^4x' \Big(\big(\Pi^+(x,x')-\Pi^-(x,x')\big)\Delta^-(x',y) + \Pi^+(x,x')\big(\Delta^+(x',y)-\Delta^-(x',y)\big)\Big),
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) \qquad (27)\\
&\quad + \Theta(x_0-y_0)\int d^4x' \Big(\Pi^+(x,x')\,\Delta^+(x',y) - \Pi^-(x,x')\,\Delta^-(x',y)\Big).
\end{aligned}
$$

- The second term in the r.h.s. of Eq. (27) does not contribute as $\Theta(x_0'-x_0)\,\Theta(y_0-x_0')=0$ for $x_0>y_0$. So, we finally obtain

$$
\big(\Box_x + m^2 - \Pi_\delta(x)\big)\Delta^+(x,y) = -\delta^{(4)}(x-y) + \int d^4x'\,\Pi^+(x,x')\,\Delta^+(x',y). \qquad (28)
$$

- One finds analogously the complete set of equations of motion of the functions $\Delta^{\pm}(x,y)$ which read

$$\boxed{\big(\Box_x + m^2\big)\Delta^{\pm}(x,y) = -\delta^{(4)}(x-y) + \int d^4x'\,\Pi^{\pm}(x,x')\,\Delta^{\pm}(x',y),} \tag{29}$$

$$\boxed{\big(\Box_y + m^2\big)\Delta^{\pm}(x,y) = -\delta^{(4)}(x-y) + \int d^4x'\,\Delta^{\pm}(x,x')\,\Pi^{\pm}(x',y),} \tag{30}$$

where the retarded and advance self energies have been redefined as

$$\Pi^{\pm}(x,y) + \delta^{(4)}(x-y)\Pi_\delta(x) \longrightarrow \Pi^{\pm}(x,y), \tag{31}$$

that is the one-point self energy has been included in both the retarded and advanced self energies.

- The retarded and advanced self energies are now related to the self energies $\Pi^{\gtrless}$ as

$$\Pi^{\pm}(x,y) = \delta^{(4)}(x-y)\Pi_\delta(x) \pm \Theta(\pm x_0 \mp y_0)\big(\Pi^{>}(x,y) - \Pi^{<}(x,y)\big). \tag{32}$$

Exercise: Derive Eqs. (29, 30).

Retarded and advanced functions of translationally invariant systems

- In case of tarnslationally systems, Eqs. (29, 30) become

$$\big(\Box_x + m^2\big)\Delta^{\pm}(x) = -\delta^{(4)}(x) + \int d^4x'\,\Pi^{\pm}(x-x')\,\Delta^{\pm}(x'), \tag{33}$$

$$\big(\Box_{-x} + m^2\big)\Delta^{\pm}(x) = -\delta^{(4)}(x) + \int d^4x'\,\Delta^{\pm}(x-x')\,\Pi^{\pm}(x'). \tag{34}$$

Changing the integration variable in Eq. (30) as $x - x' \to x'$ and observing that $\Box_{-x} = \Box_x$, one finds that the equations (33, 34) are identical to each other. So, we have a single equation

$$\boxed{\big(\Box + m^2\big)\Delta^{\pm}(x) = -\delta^{(4)}(x) + \int d^4x'\,\Pi^{\pm}(x-x')\,\Delta^{\pm}(x').} \tag{35}$$

- Performing the Fourier transformation, one finds

$$\big(-p^2 + m^2\big)\Delta^{\pm}(p) = -1 + \Pi^{\pm}(p)\,\Delta^{\pm}(p). \tag{36}$$

- The free functions obey the equation

$$\big(-p^2 + m^2\big)\Delta^{\pm}_0(p) = -1, \tag{37}$$

which is solved by

$$\Delta^{\pm}_0(p) = \frac{1}{p^2 - m^2 \pm i\mathrm{sgn}(p_0)0^+}, \tag{38}$$

where the infinitesimal imaginary element is introduced that the functions obey the appropriate initial conditions.

- Using the free function (38), the equation (36) can be written as

$$\boxed{\Delta^{\pm}(p) = \Delta_0^{\pm}(p) - \Delta_0^{\pm}(p)\,\Pi^{\pm}(p)\,\Delta^{\pm}(p).} \tag{39}$$

- Assuming that the self energy $\Pi^{\pm}(p)$ is known and using the formula (38), the Green's functions $\Delta^{\pm}(p)$ found as solutions of Eq. (39) are

$$\Delta^{\pm}(p) = \frac{1}{p^2 - m^2 + \Pi^{\pm}(p)}, \tag{40}$$

where the infinitesimal imaginary element included in the formula (38) is ignored because the self energy $\Pi^{\pm}(p)$ usually has a non-vanishing imaginary part. If not, the infinitesimal element must be included.

- The Green's function of the form (40) can be obtained performing a resummation of the first order contribution. At the first order, we have

$$\Delta_{(1)}^{\pm}(p) = \Delta_0^{\pm}(p) - \Delta_0^{\pm}(p)\,\Pi_{(1)}^{\pm}(p)\,\Delta_0^{\pm}(p). \tag{41}$$

The second order result, which, however, is incomplete, can be obtained as

$$\begin{aligned}
\Delta_{(2)}^{\pm}(p) &= \Delta_0^{\pm}(p) - \Delta_0^{\pm}(p)\,\Pi_{(1)}^{\pm}(p)\,\Delta_{(1)}^{\pm}(p) \\
&= \Delta_0^{\pm}(p) - \Delta_0^{\pm}(p)\,\Pi_{(1)}^{\pm}(p)\Big(\Delta_0^{\pm}(p) - \Delta_0^{\pm}(p)\,\Pi_{(1)}^{\pm}(p)\,\Delta_0^{\pm}(p)\Big) \\
&= \Delta_0^{\pm}(p)\Big(1 - \Pi_{(1)}^{\pm}(p)\Delta_0^{\pm}(p) + \Pi_{(1)}^{\pm}(p)\,\Delta_0^{\pm}(p)\,\Pi_{(1)}^{\pm}(p)\,\Delta_0^{\pm}(p)\Big).
\end{aligned}$$

Extending the iterative procedure to infinite order, one finds

$$\Delta^{\pm}(p) = \Delta_0^{\pm}(p)\sum_{n=0}^{\infty}\big(-\Pi_{(1)}^{\pm}(p)\,\Delta_0^{\pm}(p)\big)^n = \frac{\Delta_0^{\pm}(p)}{1 - \Pi_{(1)}^{\pm}(p)\,\Delta_0^{\pm}(p)}, \tag{42}$$

where the geometric series has been summed over. Using the formula (38), we get the resummed the Green's functions (42) as

$$\Delta^{\pm}(p) = \frac{1}{p^2 - m^2 + \Pi_{(1)}^{\pm}(p)}. \tag{43}$$

Although the result (43) looks as the full Green's function (40), it is, strictly speaking, valid only at the first order, as only the first order contribution has been resummed.