Boson gas

In this lecture the method of thermal field theory will be applied to a weakly interacting boson gas.

Partition function

- We consider a boson gas described the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x), \tag{1}$$

where $\phi(x)$ is the real scalar, m is the mass parameter and λ is the coupling constant.

- As we remember from the Lecture II, the partition function defined as

$$Z \equiv \sum_\alpha \langle\alpha|e^{-\beta\hat{H}}|\alpha\rangle, \tag{2}$$

where $\beta \equiv T^{-1}$ is the inverse temperature and $\hat{H}$ is the system's Hamiltonian.

- The partition function of non-interacting boson gas derived in Lecture II is

$$Z_0 = \exp\Big[-V\int\frac{d^3k}{(2\pi)^3}\ln(1-e^{-\beta\omega_\mathbf{k}})\Big], \tag{3}$$

where V is the system's volume and $\omega_\mathbf{k} \equiv \sqrt{\mathbf{p}^2+m^2}$.

- One computes the partition function as

$$\ln Z_0 = -\frac{V}{2\pi^2}\int_0^\infty dk\,k^2\ln(1-e^{-\beta\sqrt{m^2+k^2}}) = -\frac{V}{6\pi^2}\int_0^\infty dk\,\frac{dk^3}{dk}\ln(1-e^{-\beta\sqrt{m^2+k^2}}). \tag{4}$$

- Performing the partial integration, one finds

$$\ln Z_0 = \frac{V}{6\pi^2 T}\int_0^\infty\frac{dk\,k^4}{\sqrt{m^2+k^2}}\frac{1}{e^{\beta\sqrt{m^2+k^2}}-1}. \tag{5}$$

- Further on, we will be mostly interested in the hot gas such that $T \gg m$. Then, the bosons can be treated as massless and for $m=0$ we will deal with simple analytical formulas. The partition function equals

$$\ln Z_0 = \frac{V\,T^3}{6\pi^2}\underbrace{\int_0^\infty\frac{dx\,x^3}{e^x-1}}_{=\frac{\pi^4}{15}} = \frac{\pi^2 V\,T^3}{90}. \tag{6}$$

- The system's energy U, free energy $F = U - TS$ and pressure p are

$$U \equiv -\frac{d}{d\beta}\ln Z(T), \qquad F \equiv -T\ln Z(T), \qquad p = -\Big(\frac{\partial F}{\partial V}\Big)_T. \tag{7}$$

- The partition function (6) gives

$$U_0 = \frac{\pi^2 V\,T^4}{30}, \qquad F_0 = -\frac{\pi^2 V\,T^4}{90}, \qquad p_0 = \frac{\pi^2 T^4}{90}. \tag{8}$$

First order correction to partition function

- As we remember from the Lecture III, the partition function, which is of the form appropriate for perturbative expansion, is

$$Z = \mathrm{Tr}\Big[e^{-\beta \hat{H}^0} \mathcal{T}\big[e^{-\int_0^\beta d\tau\, \hat{H}^I_{\rm int}(-i\tau)}\big]\Big]. \tag{9}$$

- The zeroth order contribution to $\mathcal{T}[e^{-\int_0^\beta d\tau\, \hat{H}^I_{\rm int}(-i\tau)}]$ is unity and the first order contribution, which corresponds to the diagram shown in Fig. 1, equals

$$Z_{(1)} = \mathrm{Tr}\Big[e^{-\beta \hat{H}^0}\, \mathcal{T}\big[e^{-\int_0^\beta d\tau\, \hat{H}^I_{\rm int}(-i\tau)}\big]\Big]_{(1)} = \mathrm{Tr}\big[e^{-\beta\hat{H}^0}\big] \frac{\mathrm{Tr}\Big[e^{-\beta \hat{H}^0} \mathcal{T}\big[e^{-\int_0^\beta d\tau\, \hat{H}^I_{\rm int}(-i\tau)}\big]\Big]}{\mathrm{Tr}\big[e^{-\beta\hat{H}^0}\big]}\Bigg|_{(1)}$$

$$= -Z_0 \frac{\lambda}{8} \int_0^\beta d^4x\, \Delta(0)\, \Delta(0) = -Z_0 \frac{\lambda}{8} \big(\Delta(0)\big)^2 \int_0^\beta d^4x. \tag{10}$$

- As we remember, the function $\Delta(0)$ is identified with the function $\Delta^>(0)$. Since the function $\Delta^>(\tau, \mathbf{x})$ is, see Eq. (24) of Lecture III,

$$\Delta^>(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \Big[\big(f(\omega_{\mathbf{k}})+1\big) e^{-\omega_{\mathbf{k}}\tau} e^{i\mathbf{k}\cdot\mathbf{x}} + f(\omega_{\mathbf{k}}) e^{\omega_{\mathbf{k}}\tau} e^{-i\mathbf{k}\cdot\mathbf{x}}\Big], \tag{11}$$

  where the boson distribution function equals

$$f(\omega_{\mathbf{k}}) \equiv \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1}, \tag{12}$$

  the function $\Delta^>(0)$ is

$$\Delta^>(0) = \int \frac{d^3k}{(2\pi)^3} \frac{2f(\omega_{\mathbf{k}})+1}{2\omega_{\mathbf{k}}}. \tag{13}$$

- After performing the trivial angular integral, Eq. (13) becomes

$$\Delta^>(0) = \frac{1}{4\pi^2} \int_0^\infty \frac{dk\, k^2}{\sqrt{m^2+k^2}} \frac{1+e^{-\beta\sqrt{m^2+k^2}}}{1-e^{-\beta\sqrt{m^2+k^2}}}. \tag{14}$$

- Since for $k \gg m$ and $k \gg T$ the integrand linearly grows with k, the integral in Eq. (14) is quadratically divergent. One observes that the divergence remains in the zero temperature limit that is when $\beta \to \infty$. Therefore, it is the ultraviolet divergence which is well known in vacuum QFT.

- To get a finite result one should subtract the vacuum contribution from the formula (13). Since $f(\omega_{\mathbf{k}}) = 0$ in vacuum, the subtraction is done as follows

$$\Delta^>_R(0) = \int \frac{d^3k}{(2\pi)^3} \frac{2f(\omega_{\mathbf{k}})+1}{2\omega_{\mathbf{k}}} - \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} = \int \frac{d^3k}{(2\pi)^3} \frac{f(\omega_{\mathbf{k}})}{\omega_{\mathbf{k}}} \tag{15}$$

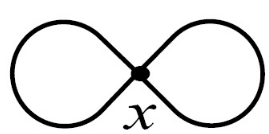

Figure 1: The first order contribution to the partition function (9)

and the renormalized Green's function, which is finite, equals

$$\Delta_R^{>}(0) = \frac{1}{2\pi^2}\int_0^\infty \frac{dk\,k^2}{\sqrt{m^2+k^2}}\frac{1}{e^{\beta\sqrt{m^2+k^2}}-1}. \tag{16}$$

- Assuming that $m = 0$, one finds

$$\Delta_R^{>}(0) = \frac{1}{2\pi^2}\int_0^\infty \frac{dk\,k}{e^{\beta k}-1} = \frac{T^2}{2\pi^2}\underbrace{\int_0^\infty \frac{dx\,x}{e^x-1}}_{=\frac{\pi^2}{6}} = \frac{T^2}{12}. \tag{17}$$

- The four-dimensional integral over x is

$$\int_0^\beta d^4x \equiv \int_0^\beta d\tau \int d^3x = \beta V, \tag{18}$$

where V is the system's volume.

- The first order correction to the partition function is

$$Z_{(1)} = -Z_0\frac{\lambda}{1152}V\,T^3. \tag{19}$$

- Using the expression (6) of Z_0, the partition function is found as

$$Z = \exp\Big(\frac{\pi^2 V\,T^3}{90}\Big)\Big[1-\frac{\lambda}{1152}V\,T^3\Big]. \tag{20}$$

- Since the second term in the square bracket in Eq. (20) should be, as a perturbative correction, much smaller than unity, the expression in the bracket can be approximated as

$$1-\frac{\lambda}{1152}V\,T^3 \approx \exp\Big(-\frac{\lambda}{1152}V\,T^3\Big), \tag{21}$$

which allows one to rewrite the partition function (20) in the following form

$$\ln Z = \frac{\pi^2 V\,T^3}{90} - \frac{\lambda}{1152}V\,T^3 = \frac{\pi^2 V\,T^3}{90}\Big[1-\frac{5\lambda}{64\pi^2}\Big]. \tag{22}$$

- The energy, free energy and pressure, which include the first order corrections, are

$$U = \frac{\pi^2 V\,T^4}{30}\Big[1-\frac{5\lambda}{64\pi^2}\Big], \tag{23}$$

$$F = -\frac{\pi^2 V\,T^4}{90}\Big[1-\frac{5\lambda}{64\pi^2}\Big], \tag{24}$$

$$p = \frac{\pi^2 T^4}{90}\Big[1-\frac{5\lambda}{64\pi^2}\Big]. \tag{25}$$

- Using the apparatus of thermal field theory, we have manged to go beyond the ideal gas approximation.
- Needless to say, the procedure of perturbative expansion can be systematically extended to higher orders.