Phase transitions

In this lecture we will use the methods of effective potential introduced in Lecture IV to discuss phase transitions of systems of quantum fields.

Phase transitions

- A *phase transition* is the physical process which changes one phase (state) of a system to a qualitatively different phase. Each phase is characterized by a different value of an *order parameter*.

- An order parameter, which is a measure of the degree of order across the boundaries in a phase transition, typically changes from zero in one phase and nonzero in the other. An example of an order parameter is the magnetization in a ferromagnetic system undergoing a phase transition. For solid-liquid and liquid-gas transitions, the order parameter is the difference of the densities. In case of systems of quantum fields, an expectation value of the field is usually the order parameter.

- A phase transition is usually identified as a singularity of a partition function in the thermodynamic limit which is the limit $V \to \infty$, $C \to \infty$ at fixed C/V, where V is the system's volume and C is a conserved charge. In nonrelativistic systems C is usually identified with a particle number.

- If the thermodynamic limit is not taken, the partition function, which is a sum of exponential functions $e^{-\beta E_n}$, where E_n is the energy of the $n-$th state of the system under consideration, is expected to be an analytic function of V and T. So, there are no phase transitions.

- According to the classification introduced by Paul Ehrenfest, a phase transitions is characterized by the behavior of the free energy as a function of other thermodynamic variables (*e.g.* T or V) in the thermodynamical limit.

- A phase transition is labeled by the lowest derivative of the free energy that is discontinuous at the transition.

- The *first-order phase transitions* exhibit a discontinuity in the first derivative of the free energy with respect to some thermodynamic variables. Various solid-liquid and liquid-gas transitions are classified as first-order transitions. The transitions are associated with the energy transfer – release or absorption – due to the latent heat which results from the difference in energy density of the phases.

- The *second-order phase transitions* are continuous in the first derivative but exhibit discontinuity in a second derivative of the free energy. These include the ferromagnetic-paramagnetic transition in materials such as iron. The magnetization, which is the first derivative of the free energy with respect to the applied magnetic field, vanishes above the Curie temperature and it is finite below.

- There are phase transitions, like a conductor-insulator transition, which do not fit to the Ehrenfest classification.

Thermal effective potential

- As we remember, the information about a system of fields in thermodynamic equilibrium is encoded in the partition function Z. The path integral representation of Z is

$$Z(T,V) = \int \mathcal{D}\phi(x) \exp\left(S_E[\phi]\right), \tag{1}$$

where $S_E[\phi]$ is the Euclidean action which for the scalar field equals

$$S_E[\phi] \equiv \int_0^\beta d^4x\, \mathcal{L} = \int_0^\beta d^4x \left[-\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \right]. \tag{2}$$

- We rewrite the partition function (1) as

$$Z(T,V) = \int_{-\infty}^\infty d\bar\phi \int \mathcal{D}\phi(x) \exp\left(S_E[\phi + \bar\phi]\right), \tag{3}$$

where $\bar\phi$ is a constant and homogeneous field and $\phi'(x)$ is a fluctuation around it.

- As discussed in Lecture XIV, the effective action equals

$$\Gamma[\phi_0] = -i\ln\left[\int \mathcal{D}\phi(x) \exp\left(iS[\phi + \phi_0]\right) \right]. \tag{4}$$

- When $\phi_0(x)$ is independent of x that is $\phi_0(x) = \bar\phi$, the effective action is expressed through the effective potential $V_{\text{eff}}(\bar\phi)$ as

$$\Gamma[\bar\phi] = -\int d^4x\, V_{\text{eff}}(\bar\phi) = V\mathcal{T}\, V_{\text{eff}}(\bar\phi), \tag{5}$$

where

$$V\mathcal{T} \equiv \int d^4x. \tag{6}$$

- Combining Eqs. (4) and (5) and changing the action in the Minkowski space into the Euclidean action relevant for a system in thermal equilibrium, we have

$$V_{\text{eff}}(\bar\phi, T) = -\frac{T}{V} \ln\left[\int \mathcal{D}\phi(x) \exp\left(S_E[\phi + \bar\phi]\right) \right], \tag{7}$$

where the temperature T is explicitly shown as an argument of the effective potential to stress that we deal with the *thermal effective potential*. The factor T/V occurs in Eq. (7) because $\int d^4x$ is replaced by

$$\int_0^\beta d^4x \equiv \int_0^\beta d\tau \int d^3x = \beta V = \frac{V}{T}. \tag{8}$$

- Comparing to each other the formulas (3) and (7), we find

$$Z(T,V) = \int_{-\infty}^\infty d\bar\phi \, \exp\left[-\beta V\, V_{\text{eff}}(\bar\phi, T) \right]. \tag{9}$$

- Let us expand the effective potential around its absolute minimum at $\bar{\phi}_{\min}$ as

$$V_{\text{eff}}(\bar{\phi}, T) = V_{\text{eff}}(\bar{\phi}_{\min}, T) + \frac{1}{2} \underbrace{\left. \frac{d^2 V_{\text{eff}}(\bar{\phi}, T)}{d\bar{\phi}^2} \right|_{\bar{\phi}=\bar{\phi}_{\min}}}_{\equiv V''_{\text{eff}}(\bar{\phi}_{\min}, T)} (\bar{\phi} - \bar{\phi}_{\min})^2 + \ldots \tag{10}$$

The first derivative is absent in the expansion as it vanishes at $\bar{\phi}_{\min}$.

- Substituting the expansion (10) into Eq. (9), one finds

$$Z(T) = \exp\left[-\beta V \, V_{\text{eff}}(\bar{\phi}_{\min}, T) \right] \sqrt{\frac{2\pi T}{V''_{\text{eff}}(\bar{\phi}_{\min}, T) \, V}}. \tag{11}$$

- Since the free energy equals $F = -T \ln Z$, Eq. (11) gives

$$F(T, V) = V \, V_{\text{eff}}(\bar{\phi}_{\min}, T) - T \ln \sqrt{\frac{2\pi T}{V''_{\text{eff}}(\bar{\phi}_{\min}, T) \, V}}. \tag{12}$$

- The first term in the formula (12) is proportional to the macroscopic volume V while in the second term the volume enters under the logarithm. Consequently, the first term is much bigger than the second one because $V \gg \ln V$. So, the free energy density equals

$$f(T) \equiv \frac{F(T, V)}{V} = V_{\text{eff}}(\bar{\phi}_{\min}, T). \tag{13}$$

- In case of the first order phase transition at $T = T_c$, the free energy density is continuous at $T = T_c$ but the first derivative exhibits a discontinuity. Since

$$\frac{df(T)}{dT} = \left. \frac{\partial V_{\text{eff}}(\bar{\phi}, T)}{\partial T} \right|_{\bar{\phi}=\bar{\phi}_{\min}} + \left. \frac{\partial V_{\text{eff}}(\bar{\phi}, T)}{\partial \bar{\phi}} \right|_{\bar{\phi}=\bar{\phi}_{\min}} \frac{d\bar{\phi}_{\min}}{dT} = \left. \frac{\partial V_{\text{eff}}(\bar{\phi}, T)}{\partial T} \right|_{\bar{\phi}=\bar{\phi}_{\min}}, \tag{14}$$

where we have taken into account that $\bar{\phi}_{\min}$ minimizes V_{eff}.

- Since $V_{\text{eff}}(\bar{\phi}, T)$ is expected to be an analytic function of its arguments, Eq. (14) shows that a first order phase transition occurs when

$$\left. \bar{\phi}_{\min} \right|_{T=T_c^-} \neq \left. \bar{\phi}_{\min} \right|_{T=T_c^+}, \tag{15}$$

that is when a position of the minimum of V_{eff} changes in a discontinuous way at $T = T_c$.
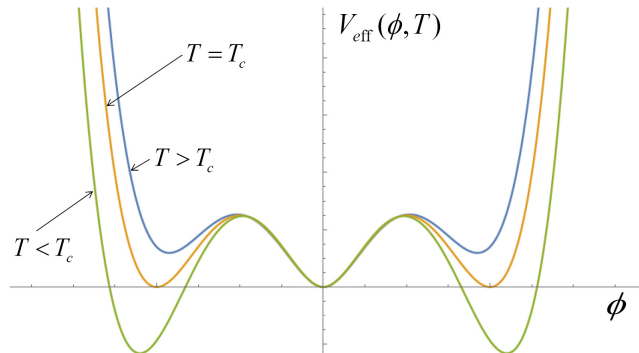

Figure 1: The effective potential corresponding to a first order phase transition

- The effective potential corresponding to a first order phase transition is shown in Fig. 1. At the critical temperature T_c the depth of minima at a finite and vanishing $\bar{\phi}$ are equal to each other. For $T < T_c$ the minimum at $\bar{\phi} \neq 0$ is absolute but the absolute minimum for $T > T_c$ is at vanishing $\bar{\phi}$. Therefore, when T_c is approached from below, we appear in the minimum at $\phi \neq 0$ but when we approach T_c from above, we find the minimum at $\bar{\phi} = 0$.

- The potential from Fig. 1 nicely illustrates an existence of metastable *overheated* and *super-cooled* phases. When we heat up the system, which is initially in the asymmetric phase with $\bar{\phi} \neq 0$, the system remains in the local minimum even above T_c – the phase is overheated – and it takes some times before the global minimum is reached. Analogously, when the system, which is initially in the symmetric phase with $\bar{\phi} = 0$, is cooled down, it stays in the supercooled phase below T_c before the system appears in the global minimum.

Computation of thermal effective potential

- We again consider to the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} \mu^2 \phi^2(x) - \frac{\lambda}{4!} \phi^4(x) \tag{16}$$

where the mass term has a 'wrong' sign ($\mu^2 > 0$) which causes the spontaneous symmetry breaking in vacuum.

- As we remember, the effective potential is expanded in powers of a constant field $\bar{\phi}$ as

$$V_{\text{eff}}(\bar{\phi}) = \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma^{(n)}(0, 0, \dots, 0) \, \bar{\phi}^n, \tag{17}$$

where $\Gamma^{(n)}(0, 0, \dots, 0)$ are one-particle irreducible vertex functions with vanishing external momenta.

- To identify the vertices to be used in the computation of the effective potential, we consider the Lagrangian which enters the action (4) of 'shifted field'. The Lagrangian reads

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} \mu^2 \big(\phi(x) + \bar{\phi} \big)^2 - \frac{\lambda}{4!} \big(\phi(x) + \bar{\phi} \big)^4 \\
&= \frac{1}{2} \mu^2 \bar{\phi}^2 - \frac{\lambda}{4!} \bar{\phi}^4 + \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} \mu^2 \phi^2(x) \\
&\quad + \Big(-\mu^2 \bar{\phi} + \frac{\lambda}{6} \bar{\phi}^3 \Big) \phi(x) - \frac{\lambda}{4} \bar{\phi}^2 \phi^2(x) - \frac{\lambda}{6} \bar{\phi} \phi^3(x) - \frac{\lambda}{24} \phi^4(x).
\end{aligned} \tag{18}
$$

- The zero-loop vertex functions, which occur due to the mass $(-\frac{1}{2}\mu^2\bar{\phi}^2)$ and interaction $(\frac{\lambda}{4!}\bar{\phi}^4)$ terms of the Lagrangian (18), are

$$\Gamma^{(2)}_{0-\text{loop}}(0,0) = -2! \frac{1}{2} \mu^2 = -\mu^2, \qquad \Gamma^{(4)}_{0-\text{loop}}(0,0,0,0) = 4! \frac{\lambda}{4!} = \lambda, \tag{19}$$

where the combinatoric factors 2! and 4! reflect in how many ways two or four fields $\bar{\phi}$ can be attached to the two- and four-point vertices.

- Substituting the vertices (19) into the expansion (17) we get

$$V_{\text{eff}}^{0-\text{loop}}(\bar{\phi}) = -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4, \tag{20}$$

which is just the <u>classical potential</u>.

One-loop contribution

- A derivation of a general form the one-loop contribution to the effective potential is fully analogous to the vacuum calculation discussed in detail in Lecture XIV. The derivation is even simpler as we do not need to perform the Wick rotation because we already work in the Euclidean space.

- As we remember, the one-loop contribution to the vacuum effective potential can be written as

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}) = \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\ln\left(1 + \frac{-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{k_E^2}\right), \tag{21}$$

which is the formula (90) of Lecture XIV with $m^2 = -\mu^2$, or as

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}) = \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\ln\left(\frac{k_E^2 - \mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{-\mu^2}\right), \tag{22}$$

which is the formula (92). The two expressions differ from each other by a contribution which is independent of $\bar{\phi}$.

- In case of fields in thermodynamic equilibrium, the formulas (21) and (22) change into

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T) = \frac{1}{2}T\sum_{n=-\infty}^{\infty}\int \frac{d^3k}{(2\pi)^3}\ln\left(1 + \frac{-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{(2\pi Tn)^2 + \mathbf{k}^2}\right), \tag{23}$$

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T) = \frac{1}{2}T\sum_{n=-\infty}^{\infty}\int \frac{d^3k}{(2\pi)^3}\ln\left(\frac{(2\pi Tn)^2 + \mathbf{k}^2 - \mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{-\mu^2}\right), \tag{24}$$

where $2\pi Tn$ is the Matsubara frequency.

- The formula (24) is more convenient for further calculations but one observes that the sum over Matsubara frequencies diverges in Eq. (24). However, the sum converges in Eq. (23). So, we rewrite the formula (23) as

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T) = \frac{1}{2}T\sum_{n=-\infty}^{\infty}\int \frac{d^3k}{(2\pi)^3}\left[\ln\left(\frac{(2\pi Tn)^2 + \mathbf{k}^2 - \mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{-\mu^2}\right)\right. \tag{25}$$

$$\left. - \ln\left(\frac{(2\pi Tn)^2 + \mathbf{k}^2 - \mu^2}{-\mu^2}\right)\right].$$

- Now, we can perform the sum over Matsubara frequencies, using the following trick. We define

$$S(a) = \sum_{n=-\infty}^{\infty}\left(\ln(n^2 + a^2 + c^2) - \ln(n^2 + a^2)\right) \tag{26}$$

and we observe that

$$\frac{dS(a)}{da} = 2a\sum_{n=-\infty}^{\infty}\left(\frac{1}{n^2 + a^2 + c^2} - \frac{1}{n^2 + a^2}\right). \tag{27}$$

- Knowing that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \operatorname{cth}(\pi a), \tag{28}$$

one finds

$$\frac{dS(a)}{da} = \frac{2\pi a}{\sqrt{a^2 + c^2}} \operatorname{cth}(\pi\sqrt{a^2 + c^2}) - 2\pi \operatorname{cth}(\pi a). \tag{29}$$

- Since the indefinite integral of $\operatorname{cth}x$ equals

$$\int dx \operatorname{cth}x = \ln(\operatorname{sh}x), \tag{30}$$

we obtain

$$S(a) = 2\ln\left(\operatorname{sh}(\pi\sqrt{a^2 + c^2})\right) - 2\ln\left(\operatorname{sh}(\pi\, a)\right). \tag{31}$$

- Keeping in mind that the parameters a and c from Eq. (26) are

$$a^2 \equiv \frac{\mathbf{k}^2 - \mu^2}{(2\pi T)^2}, \qquad c^2 \equiv \frac{\lambda\,\bar{\phi}^2}{2(2\pi T)^2}, \tag{32}$$

the potential (25) equals

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T) = T \int \frac{d^3k}{(2\pi)^3} \left[\ln\left(\operatorname{sh}\left(\frac{\beta}{2}\sqrt{\mathbf{k}^2 - \mu^2 + \frac{1}{2}\lambda\,\bar{\phi}^2}\right)\right) \right. \tag{33}$$

$$\left. - \ln\left(\operatorname{sh}\left(\frac{\beta}{2}\sqrt{\mathbf{k}^2 - \mu^2}\right)\right) \right].$$

- One observes that the second term in the formula (33) is independent of $\bar{\phi}$ and thus it can be ignored. So, we have

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T) = \frac{T}{2\pi^2} \int_0^\Lambda dk\, k^2 \ln\left(\operatorname{sh}\left(\frac{\beta}{2}\sqrt{k^2 - \mu^2 + \frac{1}{2}\lambda\,\bar{\phi}^2}\right)\right), \tag{34}$$

where the trivial angular integral is taken and the upper cut-off Λ is introduced to regularize the integral.

- Since

$$\ln(\operatorname{sh}x) = x + \ln(1 - e^{-2x}) - \ln 2, \tag{35}$$

the formula (34) is rewritten as

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T) = \frac{1}{2\pi^2} \int_0^\Lambda dk\, k^2 \left[\frac{1}{2}\sqrt{k^2 - \mu^2 + \frac{1}{2}\lambda\,\bar{\phi}^2} \right. \tag{36}$$

$$\left. + T\ln\left(1 - \exp\left(-\beta\sqrt{k^2 - \mu^2 + \frac{1}{2}\lambda\,\bar{\phi}^2}\right)\right) \right],$$

where again the term proportional to $\ln 2$, which is independent of $\bar{\phi}$, is neglected.

One-loop vacuum contribution

- With the formula (36), one can easily take the limit $T \to 0$ which is simply the first term in Eq. (36). Consequently,

$$V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T = 0) = \frac{1}{4\pi^2} \int_0^{\Lambda} dk \, k^2 \sqrt{k^2 - \mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}. \tag{37}$$

- To compute the real part of the potential (37), we split the integral as

$$\Re V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T = 0) = \frac{1}{4\pi^2} \left[\Theta(M^2) \int_0^{\Lambda} dk \, k^2 \sqrt{k^2 + M^2} \right. \tag{38}$$

$$\left. + \Theta(-M^2) \left(\int_0^{\sqrt{-M^2}} dk \, k^2 \sqrt{-k^2 - M^2} + \int_{\sqrt{-M^2}}^{\Lambda} dk \, k^2 \sqrt{k^2 + M^2} \right) \right],$$

where $M^2 \equiv -\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2$.

- Since the indefinite integral of interest equals

$$\int dx \, x^2 \sqrt{1 + x^2} = \frac{1}{4} x(1 + x^2)^{3/2} - \frac{1}{8} x(1 + x^2)^{1/2} - \frac{1}{8} \ln\left(x + \sqrt{1 + x^2}\right), \tag{39}$$

the effective potential (38) for $M^2 > 0$ is

$$\Re V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T = 0) = \frac{1}{4\pi^2} \int_0^{\Lambda} dk \, k^2 \sqrt{k^2 + M^2} \tag{40}$$

$$= \frac{1}{16\pi^2} \left[\Lambda(\Lambda^2 + M^2)^{3/2} - \frac{1}{2} \Lambda M^2 (\Lambda^2 + M^2)^{1/2} - \frac{1}{2} M^4 \ln\left(\frac{\Lambda + (\Lambda^2 + M^2)^{1/2}}{M} \right) \right]$$

$$\approx \frac{1}{16\pi^2} \left[\Lambda^4 - \frac{1}{2} \Lambda^2 M^2 - \frac{1}{4} M^4 \ln\left(\frac{\Lambda^2}{M^2} \right) \right],$$

where the terms, which vanish in the limit $\Lambda \to \infty$, are neglected.

- When $M^2 < 0$, one finds, as expected, the result (40) with M^2 replaced by $-M^2$. So, we write

$$\Re V_{\text{eff}}^{1-\text{loop}}(\bar{\phi}, T = 0) = \frac{1}{16\pi^2} \left[\Lambda^4 - \frac{1}{2} \Lambda^2 |M^2| - \frac{1}{4} M^4 \ln\left(\frac{\Lambda^2}{|M^2|} \right) \right]. \tag{41}$$

Renormalization

- Combining the zero- and one-loop results (20) and (41), the effective potential becomes

$$\Re V_{\text{eff}}(\bar{\phi}, T = 0) = -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 + \frac{1}{2}\delta m^2\bar{\phi}^2 + \frac{\delta\lambda}{4!}\bar{\phi}^4 \tag{42}$$

$$+ \frac{\Lambda^4}{16\pi^2} - \frac{\Lambda^2}{32\pi^2} \left| -\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2 \right| - \frac{1}{64\pi^2} \left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2 \right)^2 \ln\left| \frac{\Lambda^2}{-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2} \right|,$$

where we have included the mass and coupling constant counterterms to implement a renormalization procedure which allows one to eliminate the dependence of the effective potential on the cut-off parameter Λ.

- We adopt, as previously, the following renormalization conditions

$$\frac{d^2 V_{\text{eff}}(\bar{\phi}, T=0)}{d\bar{\phi}^2}\bigg|_{\bar{\phi}=0} = -\mu^2, \tag{43}$$

$$\frac{d^4 V_{\text{eff}}(\bar{\phi}, T=0)}{d\bar{\phi}^4}\bigg|_{\bar{\phi}=0} = \lambda. \tag{44}$$

- Since

$$\frac{d^2 V_{\text{eff}}(\bar{\phi}, T=0)}{d\bar{\phi}^2}\bigg|_{\bar{\phi}=0} = -\mu^2 + \delta m^2 + \frac{\lambda}{32\pi^2}\left[\mu^2\left(\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{1}{2}\right) - \Lambda^2\right], \tag{45}$$

the condition (43) gives

$$\delta m^2 = -\frac{\lambda}{32\pi^2}\left[\mu^2\left(\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{1}{2}\right) - \Lambda^2\right]. \tag{46}$$

- Because

$$\frac{d^4 V_{\text{eff}}(\bar{\phi}, T=0)}{d\bar{\phi}^4}\bigg|_{\bar{\phi}=0} = \lambda + \delta\lambda - \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{3}{2}\right], \tag{47}$$

the condition (44) provides

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{3}{2}\right]. \tag{48}$$

- Substituting the results (46) and (48) into Eq. (42), one finds

$$\Re V_{\text{eff}}(\bar{\phi}, T=0) = -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 \tag{49}$$

$$- \frac{\lambda}{64\pi^2}\left[\mu^2\left(\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{1}{2}\right) - \Lambda^2\right]\bar{\phi}^2 + \frac{\lambda^2}{256\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) - \frac{3}{2}\right]\bar{\phi}^4$$

$$+ \frac{\Lambda^4}{16\pi^2} - \frac{\Lambda^2}{32\pi^2}\left| -\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right| - \frac{1}{64\pi^2}\left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right)^2 \ln\left|\frac{\Lambda^2}{-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}\right|,$$

which is manipulated to the form

$$\Re V_{\text{eff}}(\bar{\phi}, T=0) = -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 \tag{50}$$

$$+ \frac{1}{64\pi^2}\left[\left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right)^2 \ln\left|\frac{-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{\mu^2}\right| + \frac{\lambda}{2}\mu^2\bar{\phi}^2 - \frac{3\lambda^2}{8}\bar{\phi}^4\right],$$

where the terms, which are independent of $\bar{\phi}$, are neglected. Needless to say, the effective potential (50) fully agrees with the analogous result discussed in Lecture XIV which was obtained somewhat differently.

One-loop thermal contribution

- The thermal one-loop contribution, which is given by the second term in Eq. (36), equals

$$V_{\text{eff}}^{1-\text{loop}-\text{th}}(\bar{\phi}, T) = \frac{T}{2\pi^2}\int_0^\infty dk\, k^2 \ln\left(1 - e^{-\beta\sqrt{\mathbf{k}^2 + M^2}}\right), \tag{51}$$

where, as previously, $M^2 \equiv -\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2$. The upper cut-off is not needed as the integral is ultraviolet convergent.

- We have already encountered the integral (54) – it determines the free partition function $Z_0(T, V)$. Specifically,

$$\ln Z_0(T, V) = -\frac{V}{2\pi^2} \int_0^\infty dk \, k^2 \ln \left(1 - e^{-\beta\sqrt{\mathbf{k}^2 + M^2}} \right). \tag{52}$$

- In Lecture XIII we have discussed $\ln Z_0(T, V)$ when $M^2 \ll T^2$. The expansion around $M^2 = 0$ is

$$\frac{1}{V} \ln Z_0(T, V) = \frac{\pi^2 T^3}{90} - \frac{T M^2}{24} + \frac{M^3}{12\pi} + \frac{M^4}{64\pi^2} \ln(\beta^2 M^2) + \mathcal{O}\left(\frac{M^4}{T}\right). \tag{53}$$

- Using the expansion (53), one finds the one-loop effective thermal potential as

$$V_{\text{eff}}^{1-\text{loop}-\text{th}}(\bar\phi, T) = -\frac{\pi^2 T^4}{90} + \frac{T^2 M^2}{24} - \frac{T M^3}{12\pi} - \frac{M^4}{64\pi^2} \ln(\beta^2 M^2) + \mathcal{O}\left(M^4\right). \tag{54}$$

Further on, the first term, which is independent of $\bar\phi$, is ignored.

Complete effective potential

- Combing the results (50) and (54), we get the complete effective potential

$$\Re V_{\text{eff}}(\bar\phi, T) = -\frac{1}{2}\mu^2 \bar\phi^2 + \frac{\lambda}{4!} \bar\phi^4 \tag{55}$$

$$+ \frac{1}{64\pi^2} \left[\left(-\mu^2 + \frac{1}{2}\lambda \bar\phi^2 \right)^2 \ln \left| \frac{-\mu^2 + \frac{1}{2}\lambda \bar\phi^2}{\mu^2} \right| + \frac{\lambda}{2} \mu^2 \bar\phi^2 - \frac{3\lambda^2}{8} \bar\phi^4 \right]$$

$$+ \frac{T^2}{24} \left(-\mu^2 + \frac{1}{2}\lambda \bar\phi^2 \right) - \frac{T}{12\pi} \left| -\mu^2 + \frac{1}{2}\lambda \bar\phi^2 \right|^{3/2}$$

$$- \frac{1}{64\pi^2} \left(-\mu^2 + \frac{1}{2}\lambda \bar\phi^2 \right)^2 \ln \left| \beta^2 \left(-\mu^2 + \frac{1}{2}\lambda \bar\phi^2 \right) \right|,$$

which includes the zero- and one-loop vacuum contributions and one-loop thermal one.
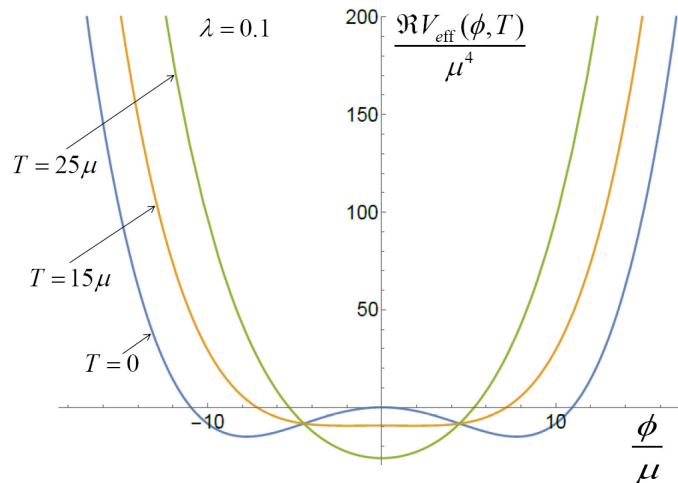

Figure 2: The real part of the effective potential (55) for three temperatures

- In the formula (55) we observe an interesting cancellation between the vacuum and thermal contributions (from the 2-nd and the 4-th lines of Eq. (55)) which eliminates the field $\bar{\phi}$ from the logarithms. Specifically,

$$\ln\left|\frac{-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2}{\mu^2}\right| - \ln\left|\beta^2\left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right)\right| = -\ln(\beta^2\mu^2). \tag{56}$$

- Taking into account the cancellation (56), the effective potential reads

$$\Re V_{\text{eff}}(\bar{\phi}, T) = -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 + \frac{1}{64\pi^2}\left(\frac{\lambda}{2}\mu^2\bar{\phi}^2 - \frac{3\lambda^2}{8}\bar{\phi}^4\right) + \frac{T^2}{24}\left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right)$$
$$- \frac{T}{12\pi}\left|-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right|^{3/2} - \frac{1}{64\pi^2}\left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right)^2\ln(\beta^2\mu^2). \tag{57}$$

- The real part of the effective potential (57) is shown in Fig. 2 for three temperatures. As one sees, there is the symmetry breaking at low temperatures – the field expectation value $\bar{\phi}$, which minimizes the potential, is finite. At high temperatures the symmetry is restored – the field expectation value $\bar{\phi}$, which minimizes the potential, vanishes.

Simplified model

- An analysis of the potential (57) is rather complex and its conclusions depend on how the potential is modified by higher order corrections. The problem is discussed in M. Carrington, Phys. Rev. D **45**, 2933 (1992). Here we limit our discussion to mostly qualitative analysis based on the potential which approximates (57).

- To understand a mechanism of the symmetry restoration, we roughly approximate the potential (57) as

$$V_{\text{eff}}(\bar{\phi}, T) = -\frac{1}{2}\mu^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 + \frac{T^2}{24}\left(-\mu^2 + \frac{1}{2}\lambda\bar{\phi}^2\right), \tag{58}$$

which holds for $\mu^2 \neq 0$ and $\lambda \ll 1$. Since the potential (58) is real, there is no need to keep the symbol $\Re$.

- Although the approximation (58) includes, except the classical potential, only the dominant thermal correction, one checks numerically that the potential (58) reproduces well the curves in Fig. 2.

- Rewriting the potential (58) as

$$V_{\text{eff}}(\bar{\phi}, T) = \frac{1}{2}\left(-\mu^2 + \frac{\lambda T^2}{24}\right)\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4, \tag{59}$$

where the term independent of $\bar{\phi}$ is neglected, one realizes that this is the effect of mass generation of the field which is responsible for the symmetry restoration. When the thermal mass

$$m_{\text{eff}}^2 \equiv \frac{\lambda T^2}{24}, \tag{60}$$

which is discussed in Lectures 8 and 13, is bigger than μ^2, the symmetry is restored.

- The critical temperature equals

$$T_c = \sqrt{\frac{24\mu^2}{\lambda}}, \tag{61}$$

and for $\lambda = 0.1$ it is $T_c \approx 15\mu$, as can be inferred from Fig. 2.

- Let us now discuss what is the order of the phase transition. The derivative of the simplified potential (59) equals

$$\frac{\partial V_{\text{eff}}(\bar{\phi}, T)}{\partial \bar{\phi}} = \bar{\phi}\left(-\mu^2 + \frac{\lambda T^2}{24} + \frac{\lambda}{6}\bar{\phi}^2\right). \tag{62}$$

So, there are at most three different real values of $\bar{\phi}$ which extremize the potential.

- When $T < T_c$, the potential (59) has one local maximum at $\bar{\phi} = 0$ and two minima at

$$\bar{\phi}_{\pm} = \pm\sqrt{\frac{6}{\lambda}\left(\mu^2 - \frac{\lambda T^2}{24}\right)}. \tag{63}$$

- When $T \to T_c$, the points $\bar{\phi}_{\pm} \to 0$ and at $T = T_c$, the three extreme points merge into one at $\bar{\phi} = 0$. When $T > T_c$, there is only one minimum at $\bar{\phi} = 0$. Such a behavior of the effective potential suggests a second order or a smooth phase transition also called a *crossover*.

- A character of a phase transition is determined by a behavior of the free energy and its derivatives at a critical temperature. The effective potential at $\bar{\phi}_{\text{min}}$ equals the free energy, see Eq. (13), but computing the potential we have ignored the contributions independent of $\bar{\phi}$, including those which are infinite. So, the effective potentials (57) and (58) do not provide the complete free energy. Therefore, instead of analyzing the free energy and its derivatives as functions of temperature, we rather consider the effective potential at $\bar{\phi}_{\text{min}}$ and its derivative.

- The effective potential $\bar{\phi}_{\text{min}}$ and its first two temperature derivatives are

$$V_{\text{eff}}(\bar{\phi}_{\text{min}}, T) = \begin{cases} -\frac{3\mu^4}{2\lambda} + \frac{\mu^2 T^2}{8} - \frac{\lambda T^4}{384} & \text{for } T < T_c, \\ 0 & \text{for } T > T_c, \end{cases} \tag{64}$$

$$\frac{dV_{\text{eff}}(\bar{\phi}_{\text{min}}, T)}{dT} = \begin{cases} \frac{\mu^2 T}{4} - \frac{\lambda T^3}{96} & \text{for } T < T_c, \\ 0 & \text{for } T > T_c, \end{cases} \tag{65}$$

$$\frac{d^2 V_{\text{eff}}(\bar{\phi}_{\text{min}}, T)}{dT^2} = \begin{cases} \frac{\mu^2}{4} - \frac{\lambda T^2}{32} & \text{for } T < T_c, \\ 0 & \text{for } T > T_c. \end{cases} \tag{66}$$

- One finds that the effective potential and its first derivative are smooth functions of T at $T = T_c$. The second derivative, however, is discontinuous. So, we conclude that in case of the simplified potential (59), we deal with the <u>second order phase transition</u>.