Simplest collisional processes

The lecture is devoted to a discussion of two simplest collisional processes: the binary interaction of real bosons and electron scattering on a Coulomb field.

Binary collisions of scalar bosons

We are going to compute the cross section of binary collision of self-interacting scalar bosons.

Perturbative expansion

- As we remember the $\hat{S}$ operator in the interaction picture is

$$\hat{S}^I = \mathbb{1} - i \int_{-\infty}^{\infty} dt\, \hat{H}^I_{\rm int}(t) + \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, T \hat{H}^I_{\rm int}(t_1)\, \hat{H}^I_{\rm int}(t_2) + \dots \tag{1}$$

  and the interaction Hamiltonian of self-interacting real scalar field is

$$\hat{H}_{\rm int} = \frac{\lambda}{4!}\, \hat{\phi}^4(x). \tag{2}$$

- Assuming that the coupling constant λ is small that is $\lambda \ll 1$, we approximate the operator (28) as

$$\hat{S} \approx \mathbb{1} - i\frac{\lambda}{4!} \int d^4x\, \hat{\phi}^4(x), \tag{3}$$

  which gives

$$\hat{T} = -i(\hat{S} - \mathbb{1}) \approx -\frac{\lambda}{4!} \int d^4x\, \hat{\phi}^4(x). \tag{4}$$

- The field $\hat{\phi}$ is treated as the free one.

States

- A single particle state and its conjugate are

$$|\mathbf{p}\rangle = \frac{1}{\sqrt{V}}\, \hat{a}^\dagger(\mathbf{p})|0\rangle, \qquad \langle \mathbf{p}| = \frac{1}{\sqrt{V}}\, \langle 0|\hat{a}(\mathbf{p}), \tag{5}$$

  where V is the normalization volume introduced to satisfy the normalization condition

$$\langle \mathbf{p}|\mathbf{p}\rangle = 1. \tag{6}$$

- To understand the origin of V let us consider the scalar product

$$\langle \mathbf{p}|\mathbf{q}\rangle = \frac{1}{V} \langle 0|\hat{a}(\mathbf{p})\hat{a}^\dagger(\mathbf{q})|0\rangle. \tag{7}$$

- Using the commutation relation

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{8}$$

  and the property $\hat{a}(\mathbf{q})|0\rangle = 0$, one finds

$$\langle \mathbf{p}|\mathbf{q}\rangle = \frac{1}{V}\, (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}), \tag{9}$$

  which for $\mathbf{p} = \mathbf{q}$ equals $(2\pi)^3\delta^{(3)}(\mathbf{p} = 0)/V$. As already explained, $(2\pi)^3\delta^{(3)}(\mathbf{p} = 0) = V$.

- The two-particle state is

$$|\mathbf{p}_1, \mathbf{p}_2\rangle = \frac{1}{V}\, \hat{a}^\dagger(\mathbf{p}_1)\, \hat{a}^\dagger(\mathbf{p}_2)|0\rangle, \tag{10}$$

  We assume here that $\mathbf{p}_1 \neq \mathbf{p}_2$.

- One finds that

$$\langle \mathbf{p}_1, \mathbf{p}_2 | \mathbf{q}_1, \mathbf{q}_2 \rangle = \frac{1}{V^2} \langle 0 | \hat{a}(\mathbf{p}_1)\, \hat{a}(\mathbf{p}_2)\, \hat{a}^\dagger(\mathbf{q}_1)\, \hat{a}^\dagger(\mathbf{q}_2) | 0 \rangle \tag{11}$$
$$= \frac{(2\pi)^6}{V^2} \Big(\delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_1)\, \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_2) + \delta^{(3)}(\mathbf{p}_1 - \mathbf{q}_2)\, \delta^{(3)}(\mathbf{p}_2 - \mathbf{q}_1) \Big),$$

which equals unity if $\mathbf{p}_1 = \mathbf{q}_1$ and $\mathbf{p}_2 = \mathbf{q}_2$ or $\mathbf{p}_1 = \mathbf{q}_2$ and $\mathbf{p}_2 = \mathbf{q}_1$. So, the two-particle state is properly normalized.

Amplitude

- The transition amplitude of the binary process $\mathbf{p}_1, \mathbf{p}_2 \to \mathbf{p}'_1, \mathbf{p}'_2$ is

$$T_{fi} \equiv \langle \mathbf{p}'_1, \mathbf{p}'_2 | \hat{T} | \mathbf{p}_1, \mathbf{p}_2 \rangle = -\frac{\lambda}{4!} \int d^4x\, \langle \mathbf{p}'_1, \mathbf{p}'_2 | \hat{\phi}^4(x) | \mathbf{p}_1, \mathbf{p}_2 \rangle, \tag{12}$$

where the operator $\hat{T}$ is given by Eq. (4). All momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2$ are assumed to be different from each other.

- Substituting the field decomposed into plane waves

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}}}} \big[e^{-ikx} \hat{a}(\mathbf{k}) + e^{ikx} \hat{a}^\dagger(\mathbf{k}) \big], \tag{13}$$

into Eq. (12) and using the two-particles states (10), the amplitude (12) equals

$$T_{fi} = -\frac{\lambda}{4!} \frac{1}{V^2} \int d^4x \int \frac{d^3k_1}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_1}}} \frac{d^3k_2}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_2}}} \frac{d^3k_3}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_3}}} \frac{d^3k_4}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_4}}} \tag{14}$$
$$\times \langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}(\mathbf{p}'_2) \big[e^{-ik_1x} \hat{a}(\mathbf{k}_1) + e^{ik_1x} \hat{a}^\dagger(\mathbf{k}_1) \big] \big[e^{-ik_2x} \hat{a}(\mathbf{k}_2) + e^{ik_2x} \hat{a}^\dagger(\mathbf{k}_2) \big]$$
$$\times \big[e^{-ik_3x} \hat{a}(\mathbf{k}_3) + e^{ik_3x} \hat{a}^\dagger(\mathbf{k}_3) \big] \big[e^{-ik_4x} \hat{a}(\mathbf{k}_4) + e^{ik_4x} \hat{a}^\dagger(\mathbf{k}_4) \big] \hat{a}^\dagger(\mathbf{p}_1)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle.$$

- The computation of the amplitude (14) using solely the commutation relations satisfied by the creation and annihilation operators and the fact that $\hat{a}(\mathbf{p})|0\rangle = 0$ is elementary but very tedious. A few observations greatly simplify the the problem.

- There are $2^4 = 16$ terms of the amplitude (14) but only those of equal number of creation and annihilation operators contribute. So, we take into account only 6 terms with 4 creation and 4 annihilation operators.

- We call the operators, which depend on $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2$, as <u>external</u> and those which depend on $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ as <u>internal</u>. The vacuum expectation value is nonzero if each external creation (annihilation) operator is paired with the internal annihilation (creation) operator.

- Since the creation and annihilation operators of different momenta commute with each other and can be interchanged there are 6 identical terms which contribute to the amplitude (14). The terms give

$$T_{fi} = -6 \frac{\lambda}{4!} \frac{1}{V^2} \int d^4x \int \frac{d^3k_1}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_1}}} \frac{d^3k_2}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_2}}} \frac{d^3k_3}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_3}}} \frac{d^3k_4}{(2\pi)^3 \sqrt{2\omega_{\mathbf{k}_4}}} \tag{15}$$
$$\times\; e^{i(k_1+k_2-k_3-k_4)x} \langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}(\mathbf{p}'_2)\, \hat{a}^\dagger(\mathbf{k}_1)\, \hat{a}^\dagger(\mathbf{k}_2)\, \hat{a}(\mathbf{k}_3)\, \hat{a}(\mathbf{k}_4)\, \hat{a}^\dagger(\mathbf{p}_1)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle.$$

- Since the operators of different momenta act independently from each other on the vacuum state the vacuum expectation value can be written as

$$\langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}(\mathbf{p}'_2)\, \hat{a}^\dagger(\mathbf{k}_1)\, \hat{a}^\dagger(\mathbf{k}_2)\, \hat{a}(\mathbf{k}_3)\, \hat{a}(\mathbf{k}_4)\, \hat{a}^\dagger(\mathbf{p}_1)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle \tag{16}$$
$$= \langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}^\dagger(\mathbf{k}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{p}'_2)\, \hat{a}^\dagger(\mathbf{k}_2) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_3)\, \hat{a}^\dagger(\mathbf{p}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_4)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle$$
$$+ \langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}^\dagger(\mathbf{k}_2) | 0 \rangle \langle 0 | \hat{a}(\mathbf{p}'_2)\, \hat{a}^\dagger(\mathbf{k}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_3)\, \hat{a}^\dagger(\mathbf{p}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_4)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle$$
$$+ \langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}^\dagger(\mathbf{k}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{p}'_2)\, \hat{a}^\dagger(\mathbf{k}_2) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_4)\, \hat{a}^\dagger(\mathbf{p}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_3)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle$$
$$+ \langle 0 | \hat{a}(\mathbf{p}'_1)\, \hat{a}^\dagger(\mathbf{k}_2) | 0 \rangle \langle 0 | \hat{a}(\mathbf{p}'_2)\, \hat{a}^\dagger(\mathbf{k}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_4)\, \hat{a}^\dagger(\mathbf{p}_1) | 0 \rangle \langle 0 | \hat{a}(\mathbf{k}_3)\, \hat{a}^\dagger(\mathbf{p}_2) | 0 \rangle,$$

where the 4 terms correspond 4 possible pairings of the external and internal operators.

- Since

$$\int \frac{d^3k\ e^{ikx}}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\langle 0|\hat{a}(\mathbf{p})\,\hat{a}^\dagger(\mathbf{k})|0\rangle = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}\ e^{ikx}\langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{p}) + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k})|0\rangle = \frac{e^{ipx}}{\sqrt{2\omega_{\mathbf{p}}}}$$

  and all terms in the right-hand-side of Eq. (16) give the same contribution, the amplitude (15) equals

$$T_{fi} = -\frac{\lambda}{2^2}\frac{1}{V^2}\int d^4x \frac{e^{i(p_1'+p_2'-p_1-p_2)x}}{\sqrt{E_1E_2E_1'E_2'}} = -\frac{\lambda}{2^2V^2}\frac{(2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2')}{\sqrt{E_1E_2E_1'E_2'}}, \tag{17}$$

  where $E_1 \equiv \sqrt{m^2+\mathbf{p}_1^2}$ and $E_2,\ E_1',\ E_2'$ are analogously defined. The energy-momentum conservation has appeared automatically.

- The amplitude $\mathcal{M}_{fi}$, which is defined as

$$T_{fi} = (2\pi)^4\delta^{(4)}(P_i - P_f)\,\frac{\mathcal{M}_{fi}}{\sqrt{V^{n+2}E_1E_2E_1'E_2'\dots E_n'}}, \tag{18}$$

  equals

$$\mathcal{M}_{fi} = -\frac{\lambda}{2^2}. \tag{19}$$

- The combinatorial factor which comes from Eqs. (15) and (16) is $6\cdot 4 = 4!$. It explains why λ is traditionally divided by 4! in the Lagrangian.

Cross section

- Substituting the amplitude (19) into the cross-section formula

$$d\sigma = \frac{(2\pi)^4\delta^{(4)}(P_f - P_i)}{\sqrt{(p_1\cdot p_2)^2-(m_1m_2)^2}}\,|\mathcal{M}_{fi}|^2\,\frac{d^3p_1'}{(2\pi)^3E_1'}\,\frac{d^3p_2'}{(2\pi)^3E_2'}, \tag{20}$$

  we get

$$d\sigma = \frac{(2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2')}{\sqrt{(p_1\cdot p_2)^2-m^4}}\,\frac{\lambda^2}{2^4}\,\frac{d^3p_1'}{(2\pi)^3E_1'}\,\frac{d^3p_2'}{(2\pi)^3E_2'}, \tag{21}$$

  which is, as expected, the Lorentz scalar.

- In the center-of-mass frame, where all particles from the initial and final states have the same energy E, the cross section equals

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^2}{2^8\pi^2E^2}, \tag{22}$$

  $d\Omega$ is the element of the solid scattering angle but actually there is no angular dependence – the scattering is isotropic.

<u>Exercise</u>: Derive the formula (22) from Eq. (21).

- Due to no angular dependence of the differential cross section (22) the total cross section is found performing trivial angular integration. However, due to indistinguishability of final state bosons the final result must be divided by 2. Effectively we find the total cross section multiplying the differential cross section (22) by 2π. Thus, we get

$$\sigma = \frac{\lambda^2}{2^7\pi E^2}. \tag{23}$$

- The cross section (22) expressed through the Mandelstam invariant is

$$\frac{d\sigma}{dt} = -\frac{\lambda^2}{2^4\pi\,s(s-4m^2)}. \tag{24}$$

- The total cross section is
$$\sigma = \frac{1}{2}\int_{t_{\min}}^{t_{\max}} dt\, \frac{d\sigma}{dt} = \frac{\lambda^2}{2^5\pi\, s}, \tag{25}$$
where the $t_{\min} = 0$ and it occurs when $p' = p = (E, \mathbf{p})$ and $t_{\max} = -\mathbf{p}^2 = -s + 4m^2$ when in the center-of-mass frame $p' = (E, -\mathbf{p})$. The cross sections (23) and (25) are obviously equal to each other.

Mott scattering

Mott scattering is a scattering of electron in the Coulomb field generated by atomic nucleus of charge $-Ze$. The nucleus is assumed to be infinitely heavy.

- Assuming that the nucleus is at $\mathbf{x} = 0$, the Coulomb potential is
$$A^0(x) = -\frac{Ze}{4\pi\,|\mathbf{x}|}. \tag{26}$$

- The Hamiltonian density equals
$$\hat{\mathcal{H}}^I(x) = e\hat{\bar{\psi}}(x)\gamma^\mu\hat{\psi}(x)A_\mu(x) = -\frac{Ze^2}{4\pi\,|\mathbf{x}|}\,\hat{\bar{\psi}}(x)\gamma^0\hat{\psi}(x). \tag{27}$$

- Since the fine structure constant $\alpha \equiv \frac{e^2}{4\pi} = \frac{1}{137}$ is small, the series
$$\hat{S}_{\rm int} = \mathbb{1} - i\int_{-\infty}^{\infty} dt\, \hat{H}^I_{\rm int}(t) + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, T\hat{H}^I_{\rm int}(t_1)\,\hat{H}^I_{\rm int}(t_2) + \dots \tag{28}$$
can be approximated as
$$\hat{S}_{\rm int} \approx \mathbb{1} - i\int_{-\infty}^{\infty} dt\, \hat{H}^I_{\rm int}(t) \tag{29}$$
and the reaction operator equals
$$\hat{T} = -i(\hat{S} - \mathbb{1}) \approx \frac{Ze^2}{4\pi}\int \frac{d^4x}{|\mathbf{x}|}\,\hat{\bar{\psi}}(x)\gamma^0\hat{\psi}(x). \tag{30}$$

- We write the single electron state of momentum $\mathbf{p}$ and spin s as
$$|\mathbf{p}, s\rangle = \frac{1}{\sqrt{V}}\,\hat{a}^\dagger(\mathbf{p}, s)|0\rangle, \tag{31}$$
where V is the normalization volume.

- The states $|\mathbf{p}, s\rangle$, $|\mathbf{p}', s'\rangle$ are orthonormal that is
$$\langle s, \mathbf{p}|\mathbf{p}', s'\rangle = \frac{1}{V}\,(2\pi)^3\delta^{ss'}\delta^{(3)}(\mathbf{p} - \mathbf{p}'). \tag{32}$$

- The transition amplitude of the process $|\mathbf{p}, s\rangle \to |\mathbf{p}', s'\rangle$ is
$$T_{fi} \equiv \langle s', \mathbf{p}'|\hat{T}|\mathbf{p}, s\rangle = \frac{Ze^2}{4\pi}\int \frac{d^4x}{|\mathbf{x}|}\,\langle s', \mathbf{p}'|\hat{\bar{\psi}}(x)\gamma^0\hat{\psi}(x)|\mathbf{p}, s\rangle. \tag{33}$$

- Substituting the fields $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$ decomposed into plane waves as
$$\hat{\psi}(x) = \sum_{\pm s}\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{m}{E_{\mathbf{p}}}}\,\Big[e^{-ipx}\hat{a}(\mathbf{p}, s)\,u(\mathbf{p}, s) + e^{ipx}\hat{b}^\dagger(\mathbf{p}, s)\,v(\mathbf{p}, s)\Big], \tag{34}$$
$$\hat{\bar{\psi}}(x) = \sum_{\pm s}\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{m}{E_{\mathbf{p}}}}\,\Big[e^{-ipx}\hat{b}(\mathbf{p}, s)\,\bar{v}(\mathbf{p}, s) + e^{ipx}\hat{a}^\dagger(\mathbf{p}, s)\,\bar{u}(\mathbf{p}, s)\Big] \tag{35}$$

into Eq. (33), one finds

$$T_{fi} = \frac{Ze^2}{4\pi}\frac{1}{V}\int\frac{d^4x}{|\mathbf{x}|}\sum_{\pm s_1}\sum_{\pm s_2}\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\sqrt{\frac{m}{E_1}}\sqrt{\frac{m}{E_2}}\,e^{i(p_1-p_2)x} \quad (36)$$

$$\times\ \bar{u}(\mathbf{p}_1,s_1)\,\gamma^0 u(\mathbf{p}_2,s_2)\,\langle 0|\hat{a}(\mathbf{p}',s')\,\hat{a}^\dagger(\mathbf{p}_1,s_1)\,a(\mathbf{p}_2,s_2)\,\hat{a}^\dagger(\mathbf{p},s)|0\rangle,$$

where $E_1 \equiv \sqrt{m^2+\mathbf{p}_1^2}$ and $E_2 \equiv \sqrt{m^2+\mathbf{p}_2^2}$.

- A non-zero contribution to the amplitude (36) comes only from the components of fields $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$ which contain spinors u and $\bar{u}$. The components with v and $\bar{v}$ do not contribute to the amplitude because the Hamiltonian is normally ordered.

- Using the anticommutation relations of $\hat{a}(\mathbf{p},s)$ and $\hat{a}^\dagger(\mathbf{p},s)$, one finds

$$\langle 0|\hat{a}(\mathbf{p}',s')\,\hat{a}^\dagger(\mathbf{p}_1,s_1)\,a(\mathbf{p}_2,s_2)\,\hat{a}^\dagger(\mathbf{p},s)|0\rangle = (2\pi)^3\delta^{s's_1}\delta^{(3)}(\mathbf{p}'-\mathbf{p}_1)\,(2\pi)^3\delta^{s_2s}\delta^{(3)}(\mathbf{p}-\mathbf{p}_2). \quad (37)$$

- Using the result (37), the amplitude (36) becomes

$$T_{fi} = \frac{Ze^2}{4\pi}\frac{1}{V}\sqrt{\frac{m}{E}}\sqrt{\frac{m}{E'}}\int\frac{d^4x}{|\mathbf{x}|}\,e^{i(p'-p)x}\,\gamma^0\bar{u}(\mathbf{p}',s')\,\gamma^0 u(\mathbf{p},s), \quad (38)$$

where $E \equiv \sqrt{m^2+\mathbf{p}^2}$ and $E' \equiv \sqrt{m^2+\mathbf{p}'^2}$ is the initial and final electron energy, respectively.

- Performing the integral

$$\int dx_0\, e^{i(E'-E)x_0} = 2\pi\,\delta(E-E'), \quad (39)$$

we find that the electron energy is conserved.

- The integral over $\mathbf{x}$ gives

$$\int\frac{d^3x}{|\mathbf{x}|}\,e^{-i\mathbf{q}\cdot\mathbf{x}} = \frac{4\pi}{\mathbf{q}^2}, \quad (40)$$

where $\mathbf{q} \equiv \mathbf{p}'-\mathbf{p}$ is the momentum transfer.

- Substituting the results (39, 40) into Eq. (38) and using the amplitude V_{fi} related to T_{fi}

$$T_{fi} = 2\pi\,\delta(E_i-E_f)\,V_{fi}, \quad (41)$$

we find

$$V_{fi} = \frac{1}{V}\frac{Ze^2}{\mathbf{q}^2}\frac{m}{E}\,\bar{u}(\mathbf{p}',s')\,\gamma^0 u(\mathbf{p},s). \quad (42)$$

- Since the cross section of scattering on infinitely heavy target is

$$d\sigma = \frac{V}{|\mathbf{v}|}\,2\pi\,\delta(E-E')\,|V_{fi}|^2\frac{V\,d^3p'}{(2\pi)^3}, \quad (43)$$

where $\mathbf{v}$ is the projectile velocity, the amplitude (42) provides

$$d\sigma = \frac{1}{|\mathbf{v}|}\,2\pi\,\delta(E-E')\,\frac{Z^2e^4}{(\mathbf{q}^2)^2}\frac{m^2}{E^2}\,|\bar{u}(\mathbf{p}',s')\,\gamma^0 u(\mathbf{p},s)|^2\frac{d^3p'}{(2\pi)^3}, \quad (44)$$

where the normalization volume V has disappeared, as it should be.

- Writing down d^3p' as $d\Omega\, d|\mathbf{p}'|\mathbf{p}'^2$ and performing the integral over $|\mathbf{p}'|$, we get rid of the Dirac delta and the cross section equals

$$\frac{d\sigma}{d\Omega} = \frac{4Z^2\alpha^2m^2}{(\mathbf{q}^2)^2}\,|\bar{u}(\mathbf{p}',s')\,\gamma^0 u(\mathbf{p},s)|^2, \quad (45)$$

where $\alpha \equiv \frac{e^2}{4\pi}$.

- Assuming that the initial electron is not polarized and that the polarization of final-state electron is not measured, we sum the cross section over the initial polarizations and average over final polarization . Then,

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{1}{2}\sum_{ss'} \frac{4\alpha^2 m^2}{(\mathbf{q}^2)^2}\, |\bar{u}(\mathbf{p}', s')\,\gamma^0 u(\mathbf{p}, s)|^2. \tag{46}$$

- Now we compute

$$\begin{aligned} \sum_{ss'} |\bar{u}(\mathbf{p}', s')\,\gamma^0 u(\mathbf{p}, s)|^2 &= \sum_{ss'} \left[\bar{u}(\mathbf{p}', s')\,\gamma^0 u(\mathbf{p}, s)\right]^\dagger \bar{u}(\mathbf{p}', s')\,\gamma^0 u(\mathbf{p}, s) \\ &= \sum_{ss'} \bar{u}_\alpha(\mathbf{p}, s)\,\gamma^0_{\alpha\beta} u_\beta(\mathbf{p}', s')\;\bar{u}_\gamma(\mathbf{p}', s')\,\gamma^0_{\gamma\delta} u_\delta(\mathbf{p}, s), \end{aligned} \tag{47}$$

where we have taken into account that $\bar{u} = u^\dagger \gamma^0$ and $(\gamma^0)^\dagger = \gamma^0$.

- Using the completeness relation

$$\sum_{\pm s} u_\alpha(\mathbf{p}, s)\,\bar{u}_\beta(\mathbf{p}, s) = \left(\frac{\gamma\cdot p + m}{2m}\right)_{\alpha\beta}, \tag{48}$$

one finds

$$\sum_{ss'} |\bar{u}(\mathbf{p}', s')\,\gamma^0 u(\mathbf{p}, s)|^2 = \mathrm{Tr}\Big[\gamma^0 \frac{\gamma\cdot p' + m}{2m}\,\gamma^0 \frac{\gamma\cdot p + m}{2m}\Big] = \frac{1}{4m^2}\,\mathrm{Tr}\big[\gamma^0(\gamma\cdot p' + m)\,\gamma^0(\gamma\cdot p + m)\big], \tag{49}$$

where the trace is taken over the spinor indices.

Traces of gamma matrices

Using the anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \tag{50}$$

one proves the following formulas

$$\mathrm{Tr}[\gamma_\mu] = 0, \tag{51}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu] = 4g_{\mu\nu}, \tag{52}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho] = 0, \tag{53}$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = 4g_{\mu\sigma}\,g_{\nu\rho} - 4g_{\mu\rho}\,g_{\nu\sigma} + 4g_{\mu\nu}\,g_{\rho\sigma}. \tag{54}$$

Exercise: Derive the relations (51, 52, 53, 54).

- Using the relations (52, 53, 54), one obtains

$$\mathrm{Tr}\big[\gamma^0(\gamma\cdot p' + m)\,\gamma^0(\gamma\cdot p + m)\big] = 4m^2 + 8EE' - 4(p'\cdot p). \tag{55}$$

- Substituting the result (55) in Eq. (49), Eq. (46) gives the well-known Mott cross section

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{2Z^2\alpha^2}{(\mathbf{q}^2)^2}\,\big(m^2 + E^2 + \mathbf{p}'\cdot\mathbf{p}\big), \tag{56}$$

where $p'\cdot p = E^2 - \mathbf{p}'\cdot\mathbf{p}$.

- The formula (56) can be rewritten as

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{4Z^2\alpha^2 E^2}{(\mathbf{q}^2)^2}\left(1 - \frac{\mathbf{q}^2}{4E^2}\right). \tag{57}$$

- Defining the scattering angle θ as the angle between $\mathbf{p}$ and $\mathbf{p}'$ and keeping in mind that $v \equiv \frac{|\mathbf{p}|}{E}$, the Mott cross section is written as

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4\sin^4\frac{\theta}{2}}\frac{1}{v^2\mathbf{p}^2}\left(1 - v^2\sin^2\frac{\theta}{2}\right). \tag{58}$$

Exercise: Derive the Mott cross section given as (57) and (58), starting with the formula (56).

- In non-relativistic approximation $v^2 \ll 1$, the Mott cross section changes into the famous Rutherford cross section.

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{Z^2\alpha^2}{4\sin^4\frac{\theta}{2}}\frac{1}{m^2v^4}. \tag{59}$$

- The cross section (58), similarly to (59), has a strong maximum as $\theta \to 0$. Actually it is singular at $\theta = 0$ and the total cross section is infinite which reflects an infinite range of electromagnetic interactions.
- In reality the Coulomb potential of a nucleus is screened by electrons beyond the atomic radius and consequently the differential cross section is not singular at $\theta = 0$ and the total cross section is finite.