Spin and statistics

The scalar and electromagnetic fields have been quantized postulating commutation relations of field operators and the spinor field has been quantized by means of the anticommutation relations. One asks whether the scalar and electromagnetic fields can be quantized using the anticommutation relations and the spinor field with the commutation relations. The answer, which is given by the fundamental spin-statistics theorem, is negative. We are not going to present a general formulation of the theorem, neither its formal proof. Instead we will discuss two simplified arguments.

Hamiltonian

- As discussed in Lecture III, the Hamiltonian of the discretized scalar field is

$$\hat{H} = \sum_i \frac{\omega_i}{2} \Big(\hat{a}_i\, \hat{a}_i^\dagger + \hat{a}_i^\dagger\, \hat{a}_i\Big). \tag{1}$$

- Using the commutation relation $[\hat{a}_i, \hat{a}_j^\dagger] = \delta^{ij}$, one finds

$$\hat{H} = \sum_i \omega_i \Big(\hat{a}_i^\dagger\, \hat{a}_i + \frac{1}{2}\Big). \tag{2}$$

- Using the anticommutation relation $\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta^{ij}$, we get the result

$$\hat{H} = \frac{1}{2} \sum_i \omega_i \tag{3}$$

  is infinite and makes no sense. So, we cannot quantize the scalar field with the anticommutation relations.

- As discussed in Lecture IV, the Hamiltonian of the discretized spinor field is

$$\hat{H} = \sum_{\pm s} \sum_i E_i \Big[\hat{a}_i^\dagger(s)\, \hat{a}_i(s) - \hat{b}_i(s)\, \hat{b}_i^\dagger(s)\Big]. \tag{4}$$

- Using the anticommutation relation $\{\hat{b}_i(s), \hat{b}_j^\dagger(s')\} = \delta^{ss'}\delta^{ij}$, one finds

$$\hat{H} = \sum_{\pm s} \sum_i E_i \Big[\hat{a}_i^\dagger(s)\, \hat{a}_i(s) + \hat{b}_i^\dagger(s)\, \hat{b}_i(s)\Big], \tag{5}$$

  which is positive definite and allows one to define the vacuum state.

- Using the commutation relation $[\hat{b}_i(s), \hat{b}_j^\dagger(s')] = \delta^{ss'}\delta^{ij}$, one finds

$$\hat{H} = \sum_{\pm s} \sum_i E_i \Big[\hat{a}_i^\dagger(s)\, \hat{a}_i(s) - \hat{b}_i^\dagger(s)\, \hat{b}_i(s)\Big], \tag{6}$$

  which is not positive definite and the vacuum state cannot be defined.

- We conclude that a structure of Hamiltonians strongly suggests that the scalar and electromagnetic fields must be quantized with the commutators while the spinor field with the anticommutators. Consequently, as already discussed, the scalar and electromagnetic fields describe bosons while the spinor field fermions.

Microcausality

- Let $\hat{O}(x)$ be a position-dependent observable *e.g.* the operator of charge density. Then, the measurements of $\hat{O}(x)$ and $\hat{O}(x')$ must be independent from each other if the space-time points x and x' are causally disconnected that when the vector $x - x'$ is space-like. Consequently, we expect that

$$[\hat{O}(x), \hat{O}(x')] = 0, \quad \text{if} \quad (x - x')^2 < 0, \tag{7}$$

where $[\dots, \dots]$ denotes a commutator. Eq. (7) is known as the condition of microcausality.

- Observables are usually quadratic functions of fields. So, we assume that

$$\hat{O}(x) \equiv \hat{\varphi}_r(x)\, \hat{\varphi}_s(x), \tag{8}$$

where $\hat{\varphi}_r(x)$, $\hat{\varphi}_s(x)$ are linear functions of field operators.

- One shows that the microcausality condition (7) is satisfied by the observable (8) when either the fields $\hat{\varphi}_r(x)$ and $\hat{\varphi}_s(x')$ commute with each other or anticummute for $(x - x')^2 < 0$ that is when $\hat{\varphi}_r(x)$ and $\hat{\varphi}_s(x')$ satisfy the relation

$$[\hat{\varphi}_r(x), \hat{\varphi}_s(x')]_\pm = 0, \quad \text{if} \quad (x - x')^2 < 0, \tag{9}$$

where $[\dots, \dots]_+$ denotes the anticommutator and $[\dots, \dots]_-$ the commutator.

Exercise: Prove that the condition (7) holds for both signs in Eq. (9).

- It appears that the spinor field satisfies the condition (9) only with the anticommutator and the scalar field only with the commutator.
- A general analysis of the microcausality condition (7) is rather advanced. We will limit our discussion to noninteracting fields when the problem is much simplified.
- We are going to show that for the scalar fields $\hat{\varphi}_r(x)$ and $\hat{\varphi}_s(x)$ the condition (7) is satisfied when the fields $\hat{\varphi}_r(x)$ and $\hat{\varphi}_s(x)$ obey the commutation relations but not the anticommutation ones.
- Actually, we are going to consider not the commutator or anticommutator of the fields $\hat{\phi}(x)$ and $\hat{\phi}(x')$ but the vacuum expectation value

$$\Delta_\pm(x - x') \equiv \langle 0|[\hat{\phi}(x), \hat{\phi}(x')]_\pm|0\rangle. \tag{10}$$

- Using the field $\hat{\phi}(x)$ decomposed into the plane waves as

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}} \Big[e^{-ikx}\hat{a}(\mathbf{k}) + e^{ikx}\hat{a}^\dagger(\mathbf{k})\Big], \tag{11}$$

and assuming that the operators $\hat{a}(\mathbf{k})$ and $\hat{a}^\dagger(\mathbf{k})$ obey the commutation or anticommutation relations

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')]_\pm = (2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{12}$$

$$[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')]_\pm = 0 = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')]_\pm, \tag{13}$$

the expression (10) equals

$$\Delta_\pm(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}} \Big[e^{-ik(x-x')} \pm e^{ik(x-x')}\Big]. \tag{14}$$

Exercise: Derive the formula (14).

- Since the expression is the Lorentz scalar we can without a loss of generality to consider the equal-time points $x^\mu = (t, \mathbf{x})$ i $x'^\mu = (t, \mathbf{x}')$ as then the four-vector $x^\mu - x'^\mu = (0, \mathbf{x} - \mathbf{x}')$ is evidently space-like that is $(x - x')^2 = -(\mathbf{x} - \mathbf{x}')^2 < 0$. Then,

$$\Delta_\pm(x - x') = \int \frac{d^3k}{(2\pi)^3 2\omega_\mathbf{k}} \Big[e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \pm e^{-i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\Big]. \tag{15}$$

- The integrand of the integral (15) is even function of $\mathbf{k}$ for the sign plus and odd for the sign minus. Consequently,

$$\Delta_-(x - x') = 0, \quad \text{for} \quad (x - x')^2 < 0, \tag{16}$$

and

$$\Delta_+(x - x') \neq 0, \quad \text{for} \quad (x - x')^2 < 0, \tag{17}$$

- We see that the scalar field must be quantized with the commutation not anticommutation relations as otherwise the microcausality condition (7) is violated.
- We can repeat analogous analysis for the spinor filed finding that it must be quantized with the commutation not anticommutation relations to satisfy the microcausality condition (7).
- The presented arguments can be extended to interacting fields with the same result.
- The fact that we have considered only the observable which is quadratic in the field does not diminish a value of the result as the microcausality condition must be satisfied for any observables.

Interacting fields

Till now we have considered noninteracting fields and now we take into account interactions which are responsible for the extraordinary wealth of phenomena. We are going to discuss the self-interacting scalar real field and the electromagnetic interaction. In both cases we assume that the interaction is sufficiently weak. Then, one expects that the interacting fields are qualitatively similar to the free ones and we can use some results of non-interacting fields.

Self-interacting scalar field

- As we remember, the Lagrangian density of the scalar real field is

$$\mathcal{L}(x) = \frac{1}{2}\partial^\mu\phi(x)\partial_\mu\phi(x) - \frac{1}{2}m^2\phi^2(x). \tag{18}$$

- To go beyond the free field the Lagrangian density must include a term which is at least cubic in the field. Then, the equation of motion is no longer linear in the field.
- If the term proportional to ϕ^3 is included in (18), we get the theory which is sometimes discussed but it is fundamentally flawed. The Hamiltonian which contains the term ϕ^3 is not positive definite.

- So, the Lagrangian density (18) is supplemented by

$$\mathcal{L}^I(x) = -\frac{\lambda}{4!}\phi^4(x), \tag{19}$$

where the dimensionless parameter $\lambda \geqslant 0$ is called the coupling constant. The combinatorial factor 4!, which is merely a matter of convention, simplifies some formulas. As we will see, the negative sign in Eq. (19) guarantees that the Hamiltonian is positive definite.

- The equation of motion which takes into the interaction term (19) is

$$[\partial_\mu \partial^\mu + m^2]\,\phi(x) = -\frac{\lambda}{3!}\phi^3(x). \tag{20}$$

- Because of its nonlinear character a general solution of Eq. (20) is not known. However, one can solve the equation by means of perturbative methods provided $\lambda \ll 1$.

- Since the time derivative does not enter the interaction term (19), the canonical momentum conjugate to ϕ is as in the free theory and it equals $\pi = \dot{\phi}$. Consequently, to get the Hamiltonian density of the interacting theory one has to add the term

$$\mathcal{H}^I(x) = -\mathcal{L}^I(x) = \frac{\lambda}{4!}\phi^4(x) \tag{21}$$

to the free Hamiltonian.

- Quantization of the interacting scalar fields is performed in a way fully analogous to that of the free fields assuming that the interaction is sufficiently weak.

Quantum electrodynamics

- As discussed in Lectures I and IV, there is a conserved current of the spinor field which satisfies the Dirac equation. So, we identify the current multiplied by the elementary charge e that is $j^\mu \equiv e\overline{\psi}\gamma^\mu\psi$ with the electromagnetic current. Then, the Lagrangian density of the spinor field interacting with the electromagnetic field is

$$\mathcal{L}(x) = \frac{1}{4}F^{\mu\nu}(x)\,F_{\nu\mu}(x) + \overline{\psi}(x)\left(i\partial^\mu\gamma_\mu - m\right)\psi(x) - e\overline{\psi}(x)\gamma_\mu\psi(x)\,A^\mu(x), \tag{22}$$

where the Lagrangians of free spinor and electromagnetic fields are included.

- The equations of motion become

$$\left[i\gamma_\mu\Big(\partial^\mu + ieA^\mu(x)\Big) - m\right]\psi(x) = 0, \tag{23}$$

$$\partial_\mu F^{\mu\nu}(x) = e\overline{\psi}(x)\gamma^\nu\psi(x). \tag{24}$$

- Using the equation of the Dirac conjugate spinor

$$\overline{\psi}(x)\left[i\gamma_\mu\Big(\overleftarrow{\partial^\mu} - ieA^\mu(x)\Big) + m\right] = 0, \tag{25}$$

one easily checks that the current $j^\mu = e\overline{\psi}\gamma^\mu\psi$ satisfies the continuity equation $\partial_\mu j^\mu = 0$.

Exercise: Prove that the current $j^\mu = e\overline{\psi}\gamma^\mu\psi$ satisfies the continuity equation.

- The Lagrangian (22) is invariant under the gauge transformation

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x), \qquad \psi(x) \to e^{-ie\Lambda(x)}\psi(x). \tag{26}$$

- It is easier to prove the invariance when the Lagrangian is written as

$$\mathcal{L}(x) = \frac{1}{4}F^{\mu\nu}(x)\,F_{\nu\mu}(x) + \overline{\psi}(x)\left(i\Big(\partial^\mu + ieA^\mu(x)\Big)\gamma_\mu - m\right)\psi(x). \tag{27}$$

  Exercise: Prove that the Lagrangian (27) is invariant under the gauge transformation (26).

- One observes that the Lagrangian of free spinor field is not invariant under the gauge transformation. The interaction term makes the Lagrangian invariant. Therefore, the gauge invariance dictates a structure of the interaction term.
- Since the field derivatives do not enter the interaction term of the Lagrangian, the conjugate momenta remain as in the free theory.
- In the Hamiltonian density appears the extra term

$$\mathcal{H}^I(x) = -\mathcal{L}^I(x) = e\overline{\psi}(x)\gamma_\mu\psi(x)\,A^\mu(x). \tag{28}$$

- Quantization of the interacting fields is performed in a way fully analogous to that of the free fields assuming that the interaction is sufficiently weak. The electromagnetic interaction is indeed weak, the fine structure constant $\alpha \equiv \frac{e^2}{4\pi} = \frac{1}{137}$.

Temporal evolution

We discuss here a temporal evolution of quantum fields.

Evolution operator

- The evolution operator $\hat{U}(t_f, t_i)$ is the operator which transforms a state $|\psi(t_i)\rangle$ known at the moment of time t_i to the state $|\psi(t_f)\rangle$ at the time t_f that is

$$|\psi(t_f)\rangle = \hat{U}(t_f, t_i)|\psi(t_i)\rangle. \tag{29}$$

- In the Schrödinger picture

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \tag{30}$$

  and

$$|\psi(t)\rangle = e^{-i\hat{H}(t-t_0)}|\psi(t_0)\rangle, \tag{31}$$

  if the Hamiltonian $\hat{H}$ is time independent. Then,

$$\hat{U}(t_f, t_i) = e^{-i\hat{H}(t_f-t_i)}. \tag{32}$$

- The problem becomes more complicated if the Hamiltonian is time dependent as it happens in field theory. One is tempted to write down the solution (31) as

$$|\psi(t)\rangle \stackrel{?}{=} e^{-i\int_{t_0}^{t} dt' \hat{H}(t')}|\psi(t_0)\rangle \tag{33}$$

and

$$U(t_f, t_i) \stackrel{?}{=} e^{-i\int_{t_i}^{t_f} dt \hat{H}(t)}. \tag{34}$$

- The formulas (33, 34) would be correct if the Hamiltonian were an ordinary function of time, not the operator. The point is that, in general, $\hat{H}(t)$ does not commute with $\hat{H}(t')$.
- To understand the problem let us consider the equation

$$\hat{U}(t_f, t_i) = \hat{U}(t_f, t_m)\,\hat{U}(t_m, t_i), \tag{35}$$

with $t_i \leqslant t_m \leqslant t_f$ which must be obeyed by the evolution operator.

- We note that the exponential function of any operator $\hat{A}$ is defined as

$$e^{\hat{A}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\,\hat{A}^n. \tag{36}$$

Consequently, the well-known formula

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}}, \tag{37}$$

holds if the operators $\hat{A}$ and $\hat{B}$ commute with each other that is $[\hat{A}, \hat{B}] = 0$. Then, we can interchange the operators $\hat{A}$ and $\hat{B}$ to prove the formula.

- The operator (34) satisfies the relation (35) if $[\hat{H}(t_1), \hat{H}(t_2)] = 0$ but not in general.
- We modify the expression (34), considering small intervals of time. Then,

$$|\psi(t+\Delta t)\rangle = \Big(1 - i\hat{H}(t)\Delta t\Big)|\psi(t)\rangle, \tag{38}$$

and

$$\hat{U}(t+\Delta t, t) \approx 1 - i\hat{H}(t)\Delta t \approx e^{-i\hat{H}(t)\Delta t}. \tag{39}$$

- To satisfy the relation (35), the evolution operator $\hat{U}(t_f, t_i)$ can be written as

$$\hat{U}(t_f, t_i) \approx \hat{U}(t_f, t_f - \Delta t)\,\ldots\,\hat{U}(t_i + 2\Delta t, t_i + \Delta t)\,\hat{U}(t_i + \Delta t, t_i), \tag{40}$$

where the time interval $t_f - t_i > 0$ is split into N small pieces $\Delta t = (t_f - t_i)/N$.

- Substituting the approximate formula (39) into Eq. (40), we get

$$\hat{U}(t_f, t_i) \approx e^{-i\hat{H}(t_f - \Delta t)\Delta t} \ldots\; e^{-i\hat{H}(t_i + \Delta t)\Delta t}\; e^{-i\hat{H}(t_i)\Delta t}. \tag{41}$$

We observe that the operator of the latest time is on the left side, those of the earlier times are more on the right and finally at the end there is the operator of the earliest time. In other words, the operators are ordered chronologically.

- The evolution operator is properly defined by means of the chronological exponent

$$\hat{U}(t_f, t_i) \equiv Te^{-i\int_{t_i}^{t_f} dt\,\hat{H}(t)} \equiv 1 - i\int_{t_i}^{t_f} dt\,\hat{H}(t) + \frac{(-i)^2}{2!}\int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2\, T\hat{H}(t_1)\,\hat{H}(t_2) + \ldots, \quad (42)$$

where T denotes the operation of chronological ordering that is

$$T\hat{A}(t_1)\,\hat{B}(t_2) = \Theta(t_1 - t_2)\,\hat{A}(t_1)\,\hat{B}(t_2) \pm \Theta(t_2 - t_1)\,\hat{B}(t_2)\,\hat{A}(t_1). \quad (43)$$

The sign plus is chosen when the operators $\hat{A}$ and $\hat{B}$ are bosonic and sign minus for the fermionic operators. If the operators commute the chronologization is simply not needed as the operators can be interchanged.

- The evolution operator (42) satisfies the relation (35).

- In the interaction picture the temporal evolution is driven by the interaction term of the Hamiltonian and then

$$\begin{aligned}\hat{U}_{\rm int}(t_f, t_i) &= Te^{-i\int_{t_i}^{t_f} dt\,\hat{H}^I_{\rm int}(t)} \quad (44)\\ &= 1 - i\int_{t_i}^{t_f} dt\,\hat{H}^I_{\rm int}(t) + \frac{(-i)^2}{2!}\int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_f} dt_2\, T\hat{H}^I_{\rm int}(t_1)\,\hat{H}^I_{\rm int}(t_2) + \ldots\end{aligned}$$

Interaction picture

In Lecture II we have already discussed the Schrödinger and Heisenberg pictures of quantum theory. To formulate a perturbative expansion of interacting fields the interaction picture appears to be very useful.

- in the Schrödinger picture a state obeys the equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_{\rm S} = \hat{H}|\psi(t)\rangle_{\rm S}, \quad (45)$$

which is solved by

$$|\psi(t)\rangle_{\rm S} = e^{-i\hat{H}t}|\psi(0)\rangle_{\rm S}, \quad (46)$$

provided the Hamiltonian is time independent. The matrix element of a time-independent observable $\hat{\Omega}_{\rm S}$ is

$${}_{\rm S}\langle\phi(t)|\hat{\Omega}_{\rm S}|\psi(t)\rangle_{\rm S} = {}_{\rm S}\langle\phi(0)|e^{i\hat{H}t}\hat{\Omega}_{\rm S}e^{-i\hat{H}t}|\psi(0)\rangle_{\rm S}. \quad (47)$$

- The Heisenberg picture is obtained from the Schrödinger one by means of the unitary transformation $\hat{U}(t) \equiv e^{-i\hat{H}t}$. (Since $\hat{H}^\dagger$ we have $\hat{U}^{-1}(t) = \hat{U}^\dagger(t)$.) The states and observables are defined as

$$|\psi\rangle_{\rm H} \equiv |\psi(0)\rangle_{\rm S} = e^{i\hat{H}t}|\psi(t)\rangle_{\rm S} \quad (48)$$

$$\hat{\Omega}_{\rm H}(t) \equiv e^{i\hat{H}t}\hat{\Omega}_{\rm S}\,e^{-i\hat{H}t}. \quad (49)$$

- The equation of motion of an observable $\hat{\Omega}_{\rm H}$ is

$$\frac{d}{dt}\hat{\Omega}_{\rm H}(t) = i[\hat{H}, \hat{\Omega}_{\rm H}(t)], \quad (50)$$

where $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$.

- In the <u>interaction picture</u> the states and observables are defined as

$$|\psi(t)\rangle_{\rm int} \equiv e^{i\hat{H}^0_{\rm S}t}|\psi(t)\rangle_{\rm S}, \tag{51}$$

$$\hat{\Omega}_{\rm int}(t) \equiv e^{i\hat{H}^0_{\rm S}t}\hat{\Omega}_{\rm S}\, e^{-i\hat{H}^0_{\rm S}t}, \tag{52}$$

  where $\hat{H}^0_{\rm S}$ is the free Hamiltonian in the Schrödinger picture that is the Hamiltonian is split as $\hat{H}_{\rm S} = \hat{H}^0_{\rm S} + \hat{H}^I_{\rm S}$ with $\hat{H}^0_{\rm S}$ and $\hat{H}^I_{\rm S}$ being, respectively, the free and interaction part of the Hamiltonian. We note that $\hat{H}^0_{\rm S} = \hat{H}^0_{\rm int}$.

- The equation of motion of $\hat{\Omega}_{\rm int}(t)$ is

$$\frac{d}{dt}\hat{\Omega}_{\rm int}(t) = i[\hat{H}^0_{\rm int}, \hat{\Omega}_{\rm int}(t)]. \tag{53}$$

- In the interaction picture the temporal evolution of states is driven by $\hat{H}^I_{\rm int}(t)$ that is the states obey the equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_{\rm int} = \hat{H}^I_{\rm int}(t)|\psi(t)\rangle_{\rm int}. \tag{54}$$

<u>Exercise</u>: Derive Eq. (53).