Spinor field

This lecture is devoted to the canonical quantization of a free spinor field which obeys the Dirac equation. The field was introduced in the Lecture I.

Classical description of a spinor field

- Since the Lagrangian density of any field should be a Lorentz scalar we have to rewrite the Dirac equation known for the Lecture I in the covariant form which remains unchanged in any reference frame.
- Let us consider as an example the equation
$$x^\mu x_\mu = t^2 - \mathbf{x}^2, \tag{1}$$
where $x^\mu = (t, \mathbf{x})$ is the four-vector. The left-hand-side of the equation has a covariant form but the right-hand-side hasn't. A covariant form immediately reveals whether a given quantity is a scalar, vector or tensor.

Covariant form of Dirac equation

- In the Lecture I the Dirac equation was written down as
$$\Big[i\frac{\partial}{\partial t} + i\boldsymbol{\alpha}\cdot\nabla - \beta m\Big]\psi(x) = 0. \tag{2}$$
- To get the covariant form we multiply the equation (2) form the left by β keeping in mind that $\beta^2 = 1$. Then, we get
$$[i\gamma_\mu\partial^\mu - m]\psi(x) = 0, \tag{3}$$
where instead of the matrices α^i and β there are the matrices γ^μ such that
$$\gamma^0 \equiv \beta, \qquad \gamma^i \equiv \beta\alpha^i. \tag{4}$$
- Since α^i and β are Hermitian, one finds $\gamma^{0\dagger} = \gamma^0$ but
$$\gamma^{i\dagger} = (\beta\alpha^i)^\dagger = \alpha^{i\dagger}\beta^\dagger = \alpha^i\beta = -\beta\alpha^i = -\gamma^i.$$
So, γ^0 is Hermitian but γ^i are antiHermitian. The matrices γ^μ obey the identity
$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{5}$$
- Since the matrices $\alpha^i,\ \beta$ satisfy the relations
$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}, \qquad \alpha^i\beta + \beta\alpha^i = 0, \qquad \beta^2 = 1, \tag{6}$$
the analogous relations of gamma matrices read
$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{7}$$
where $g_{\mu\nu}$ is the metric tensor
$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

- In the Pauli-Dirac representation the matrices γ^μ are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{9}$$

- Taking the Hermitian conjugation of Eq. (3), one finds

$$\psi^\dagger(x)[-i(\gamma_0^\dagger \overleftarrow{\partial^0} + \gamma_i^\dagger \overleftarrow{\partial^i}) - m] = \psi^\dagger(x)[-i(\gamma_0 \overleftarrow{\partial^0} - \gamma_i \overleftarrow{\partial^i}) - m] = 0, \tag{10}$$

which multiplied by γ^0 from the right side gives

$$\begin{aligned} \psi^\dagger(x)[-i(\gamma_0^2 \overleftarrow{\partial^0} - \gamma_i\gamma_0 \overleftarrow{\partial^i}) - \gamma_0 m] &= \psi^\dagger(x)\gamma_0[-i(\gamma_0 \overleftarrow{\partial^0} + \gamma_i \overleftarrow{\partial^i}) - m] \\ &= \psi^\dagger(x)\gamma_0[-i\gamma_\mu \overleftarrow{\partial^\mu} - m] = 0. \end{aligned} \tag{11}$$

- Introducing the so-called <u>Dirac conjugate field</u>

$$\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma_0, \tag{12}$$

the Hermitian conjugated Dirac equation is written as

$$\overline{\psi}(x)[i\gamma_\mu \overleftarrow{\partial^\mu} + m] = 0. \tag{13}$$

Lorentz Transformation of Dirac spinor

- To determine transformation properties of spinors we postulate in accordance with the Principle of Relativity that if a spinor $\psi(x)$ satisfies the Dirac equation in the reference frame O the spinor transformed to the reference frame O'

$$\psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x), \tag{14}$$

where $S(\Lambda)$ is the transformation matrix, satisfies the transformed equation of the same form that is

$$[i\gamma_\mu \partial'^\mu - m]\psi'(x') = 0. \tag{15}$$

We assumed here that the mass m is the Lorentz invariant and that the matrices γ^μ remain unchanged. (It shows that it somewhat misleading to write down the matrices as a four-vector.)

- Keeping in mind that $\partial'^\mu = \Lambda^\mu{}_\nu \partial^\nu$ and multiplying Eq. (15) from the left side by $S^{-1}(\Lambda)$ we get

$$[iS^{-1}(\Lambda)\gamma_\mu S(\Lambda)\Lambda^\mu{}_\nu \partial^\nu - m]\psi(x) = 0. \tag{16}$$

- Eq. (16) coincides with the Dirac equation (3) if

$$S^{-1}(\Lambda)\gamma_\mu S(\Lambda)\Lambda^\mu{}_\nu = \gamma_\nu. \tag{17}$$

The transformation matrix $S(\Lambda)$ is found as a unique solution of Eq. (17).

- One shows that

$$S(\Lambda) = \exp\Big[\frac{i}{4}\sigma^{\mu\nu}\omega_{\nu\mu}\Big], \tag{18}$$

where $\omega^{\mu\nu}$ is antisymmetric real tensor ($\omega^{\mu\nu} = -\omega^{\nu\mu}$) which for infinitesimal Lorentz transformation $\Lambda^{\mu\nu}$ equals $\omega^{\mu\nu} = \Lambda^{\mu\nu} - g^{\mu\nu}$, and $\sigma^{\mu\nu}$ is the matrix

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \tag{19}$$

- One checks that

$$(\sigma^{ij})^\dagger = \sigma^{ij}, \qquad (\sigma^{0i})^\dagger = -\sigma^{0i} = \sigma^{i0}, \qquad \sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0.$$

- The matrix of the inverse transformation is

$$S^{-1}(\Lambda) = \exp\Big[-\frac{i}{4}\sigma^{\mu\nu}\omega_{\nu\mu}\Big]. \tag{20}$$

- We note that the matrix $S(\Lambda)$ is not unitary that is $S^\dagger(\Lambda) \neq S^{-1}(\Lambda)$. The reason is that the matrix jest $\sigma^{\mu\nu}$ is not Hermitian. However, the following relation holds

$$S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0 \tag{21}$$

which allows one to prove that the product $\overline{\psi}(x)\psi(x)$ is a Lorentz scalar.

Exercise: Prove the relation (21).

Exercise: Prove that the matrices (18, 20) satisfy Eq. (17) in case of infinitesimal Lorentz transformation.

- How does the conjugate spinor $\overline{\psi}(x)$ transform? Since $\psi(x)$ transforms as

$$\psi(x) \to S(\Lambda)\psi(x) \tag{22}$$

and $\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$, we have

$$\overline{\psi}(x) \to \Big(S(\Lambda)\psi(x)\Big)^\dagger\gamma^0 = \psi^\dagger(x)S^\dagger(\Lambda)\gamma^0. \tag{23}$$

Using the relation (21) one finds

$$\overline{\psi}(x) \to \overline{\psi}(x)S^{-1}(\Lambda). \tag{24}$$

- Due to the transformation rules (22, 24), the scalar product $\overline{\psi}(x)\psi(x)$ is a Loretz scalar and $\overline{\psi}(x)\gamma^\mu\psi(x)$ is a four-vector that is

$$\overline{\psi}(x)\gamma^\mu\psi(x) \to \Lambda^\mu{}_\nu\overline{\psi}(x)\gamma^\nu\psi(x). \tag{25}$$

Lagrange formalism of spinor field

- Knowing the transformation rules of the spinor field one immediately writes the Lagrangian density which is a Lorentz scalar and gives the Dirac equation of the Euler-Lagrange equation. Specifically, the Lagrangian density is

$$\mathcal{L}(x) = \overline{\psi}(x)\Big(i\partial^\mu\gamma_\mu - m\Big)\psi(x). \tag{26}$$

- Since the action $S \equiv \int d^4x\mathcal{L}$ is dimensionless the spinor field $\psi(x)$ is of the dimension $m^{3/2}$.
- We note that this is the action not the Lagrangian density which really matters. Therefore, if the field $\psi(x)$ vanishes at the boundary of Minkowski space, we can perform the partial integration as

$$S = \int d^4x\,\overline{\psi}(x)\Big(i\partial^\mu\gamma_\mu - m\Big)\psi(x) = \int d^4x\,\overline{\psi}(x)\Big(-i\overleftarrow{\partial^\mu}\gamma_\mu - m\Big)\psi(x) \tag{27}$$

and we get the the Lagrangian density equivalent to (26) which is

$$\mathcal{L}(x) = \overline{\psi}(x)\Big(-i\overleftarrow{\partial^\mu}\gamma_\mu - m\Big)\psi(x). \tag{28}$$

- If one chooses as independent variable the field ψ, the Euler-Lagrange equation is

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0, \tag{29}$$

and the Lagrangian density (26) leads to the conjugate Dirac equation

$$\overline{\psi}(x) \Big(i \overleftarrow{\partial^\mu} \gamma_\mu + m \Big) = 0. \tag{30}$$

- If the independent variable is $\overline{\psi}$, the Euler-Lagrange equation reads

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \overline{\psi})} - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0, \tag{31}$$

and the Lagrangian density (28) provides the Dirac equation

$$\Big(i\partial^\mu \gamma_\mu - m \Big) \psi(x) = 0. \tag{32}$$

- Eq. (30) can be, obviously, obtained from Eq. (32) taking the Hermitian conjugation.
- Since the Lagrangian density (26) is invariant under the transformation

$$\psi(x) \to e^{ia}\, \psi(x), \qquad \overline{\psi}(x) \to e^{-ia}\, \overline{\psi}(x), \tag{33}$$

where $a \in \mathbb{R}$, there is – according to the famous Noether theorem – a conserved charge in the system described by the Lagrangian density (26).

- The current, which obeys a continuity equation, can be easily found with no reference to the Noether theorem. Multiplying Eq. (32) from the left by $\overline{\psi}(x)$ and Eq. (30) from the right by $\psi(x)$, and summing up the resulting equations, we get

$$\overline{\psi}(x)\gamma_\mu \partial^\mu \psi(x) + \overline{\psi}(x)\gamma_\mu \overleftarrow{\partial^\mu} \psi(x) = \partial^\mu \Big(\overline{\psi}(x)\gamma_\mu \psi(x) \Big) = 0, \tag{34}$$

which means that the current

$$j_\mu(x) \equiv \overline{\psi}(x)\gamma_\mu \psi(x), \tag{35}$$

which transforms as a four-vector, is conserved ($\partial^\mu j_\mu = 0$).

Hamilton formalism of spinor field

- The canonical momentum conjugate to $\psi(x)$ is

$$\pi_\alpha(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial \dot{\psi}_\alpha(x)} = i\Big(\overline{\psi}(x)\gamma^0\Big)_\alpha = i\psi^\dagger_\alpha(x), \tag{36}$$

where the spinor index $\alpha = 1, 2, 3, 4$ is explicitly written.

- The Poisson bracket, which is defined as

$$\{A(t,\mathbf{x}), B(t,\mathbf{x}')\}_{\rm PB} \equiv \int d^3x'' \bigg(\frac{\delta A(t,\mathbf{x})}{\delta \psi_\alpha(t,\mathbf{x}'')} \frac{\delta B(t,\mathbf{x}')}{\delta \pi_\alpha(t,\mathbf{x}'')} - \frac{\delta A(t,\mathbf{x})}{\delta \pi_\alpha(t,\mathbf{x}'')} \frac{\delta B(t,\mathbf{x}')}{\delta \psi_\alpha(t,\mathbf{x}'')} \bigg), \tag{37}$$

equals for the pair of canonical variables

$$\{\psi_\alpha(t,\mathbf{x}), \pi_\beta(t,\mathbf{x}')\}_{\rm PB} = \delta_{\alpha\beta}\, \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \tag{38}$$

and

$$\{\psi_\alpha(t,\mathbf{x}), \psi_\beta(t,\mathbf{x}')\}_{\rm PB} = \{\pi_\alpha(t,\mathbf{x}), \pi_\beta(t,\mathbf{x}')\}_{\rm PB} = 0. \tag{39}$$

- Using the Legendre transformation, we define the Hamiltonian density

$$\mathcal{H}(x) \equiv \pi(x)\,\dot{\psi}(x) - \mathcal{L}(x) = \pi(x)\Big(- \boldsymbol{\alpha}\cdot\nabla - i\beta m\Big)\,\psi(x) = \overline{\psi}(x)\Big(- i\boldsymbol{\gamma}\cdot\nabla + m\Big)\,\psi(x) \quad (40)$$

and the Hamiltonian equals $H = \int d^3x\,\mathcal{H}$.

- The canonical equations of motion are

$$\dot{\psi}(x) = \frac{\delta H}{\delta\pi(x)} = \pi(x), \quad (41)$$

$$\dot{\pi}(x) = -\frac{\delta H}{\delta\psi(x)} = -\pi(x)\Big(\boldsymbol{\alpha}\cdot\overleftarrow{\nabla} - i\beta m\Big). \quad (42)$$

The first equation relates the canonical momentum to the canonical variable and the second equation combined with the first one gives the equation equivalent to the Dirac equation. It should be remembered that the time t is merely a parameter in Eqs. (41, 42).

- Canonical equations of motion can be also written as

$$\dot{\psi}(t,\mathbf{x}) = \{\psi(t,\mathbf{x}), H\}_{\rm PB} = \pi(t,\mathbf{x}), \quad (43)$$

$$\dot{\pi}(t,\mathbf{x}) = \{\pi(t,\mathbf{x}), H\}_{\rm PB} = -\pi(t,\mathbf{x})\Big(\boldsymbol{\alpha}\cdot\overleftarrow{\nabla} - i\beta m\Big). \quad (44)$$

Solutions of free Dirac equation

- We already know from the Lecture I the solutions of Dirac equation with zero momentum $\mathbf{p} = 0$. The solutions are

$$\psi_\alpha(x) = e^{\mp imt} w_\alpha. \quad (45)$$

- The solutions of non-zero $\mathbf{p}$ can be obtained performing the Lorentz transformation of (45). Then, mt changes into px. Since the mass m transforms into $E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2}$, and $-m$ into $-E_{\mathbf{p}}$, the Lorentz transformation does not mix up the positive and negative energy solutions.

- The solutions of momentum $\mathbf{p}$ with positive and negative energies can be written as

$$\psi^{(+)}_{\mathbf{p},s}(x) = e^{-ipx}u(\mathbf{p},s), \qquad \psi^{(-)}_{\mathbf{p},s}(x) = e^{ipx}v(\mathbf{p},s), \quad (46)$$

where $p^\mu = (E_{\mathbf{p}}, \mathbf{p})$ and the spinors $u(\mathbf{p},s)$, $v(\mathbf{p},s)$ satisfies the algebraic equations

$$[\gamma_\mu p^\mu - m]u(\mathbf{p},s) = 0, \qquad [\gamma_\mu p^\mu + m]v(\mathbf{p},s) = 0. \quad (47)$$

The variable s, which takes two values, determines a spin state.

- The spinors $u(\mathbf{p},s)$, $v(\mathbf{p},s)$ can be found as solutions of the equations (47) or transforming the spinor w which enters the formula (45). However, an explicit form of $u(\mathbf{p},s)$, $v(\mathbf{p},s)$ is usually not needed. It is usually sufficient to know that the spinors satisfy the conditions of orthogonality and completeness.

- Orthogonality:

$$\bar{u}(\mathbf{p},s)\,u(\mathbf{p},s') = \delta^{ss'} = -\bar{v}(\mathbf{p},s)\,v(\mathbf{p},s'), \tag{48}$$

$$\bar{u}(\mathbf{p},s)\,v(\mathbf{p},s') = 0 \quad = \bar{v}(\mathbf{p},s)\,u(\mathbf{p},s'), \tag{49}$$

where $\bar{u} \equiv u^\dagger\gamma^0$ and $\bar{v} \equiv v^\dagger\gamma^0$. The relation (48) shows that the spinors u and v are dimensionless. The product $\bar{v}(\mathbf{p},s)\,v(\mathbf{p},s)$ is negative, even so the product $v^\dagger(\mathbf{p},s)\,v(\mathbf{p},s)$ is positive. The matrix γ^0 changes the sign which is evident in the Pauli-Dirac representation (9).

- Completeness:

$$\sum_{\pm s} u_\alpha(\mathbf{p},s)\,\bar{u}_\beta(\mathbf{p},s) = \left(\frac{\gamma\cdot p + m}{2m}\right)_{\alpha\beta}, \tag{50}$$

$$\sum_{\pm s} v_\alpha(\mathbf{p},s)\,\bar{v}_\beta(\mathbf{p},s) = \left(\frac{\gamma\cdot p - m}{2m}\right)_{\alpha\beta}, \tag{51}$$

where $p^\mu \equiv (E_\mathbf{p},\mathbf{p})$.

- If we subtract the equations (50, 51) from each other we get the standard completeness relation

$$\sum_{\pm s}\Big(u_\alpha(\mathbf{p},s)\,\bar{u}_\beta(\mathbf{p},s) - v_\alpha(\mathbf{p},s)\,\bar{v}_\beta(\mathbf{p},s)\Big) = \delta_{\alpha\beta}, \tag{52}$$

which tell us that at a given $\mathbf{p}$ the spinors $u(\mathbf{p},s)$, $u(\mathbf{p},-s)$, $v(\mathbf{p},s)$ and $v(\mathbf{p},-s)$ constitute a basis in the four-dimensional vector space.

- Eqs. (50, 51) provide the completeness relations in the two subspaces corresponding to positive and negative energies.The matrices

$$P_\pm \equiv \frac{\pm\gamma\cdot p + m}{2m} \tag{53}$$

are the projection operators which allows one to split any spinor into its positive and negative energy parts. (A projection operators P obeys by definition the equation $P^2 = P$.) When P_+ or P_- acts on the sum of u and v, it cancles v or u, respectively. The operators $P_\pm$ are mutually orthogonal $P_+P_- = 0$ because $p^2 = m^2$ and complementary $P_+ + P_- = 1$.

Excercise: Prove that $P_\pm$ given by Eq. (53) obey $P_\pm^2 = P_\pm$ and $P_+P_- = 0$.

- We write down the general solutions of the Dirac equation as

$$\psi(x) = \sum_{\pm s}\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{m}{E_\mathbf{p}}}\,\Big[e^{-ipx}a(\mathbf{p},s)\,u(\mathbf{p},s) + e^{ipx}b^*(\mathbf{p},s)\,v(\mathbf{p},s)\Big], \tag{54}$$

$$\bar{\psi}(x) = \sum_{\pm s}\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{m}{E_\mathbf{p}}}\,\Big[e^{-ipx}b(\mathbf{p},s)\,\bar{v}(\mathbf{p},s) + e^{ipx}a^*(\mathbf{p},s)\,\bar{u}(\mathbf{p},s)\Big], \tag{55}$$

where $a(\mathbf{p},s)$ and $b(\mathbf{p},s)$ are arbitrary functions of the dimension $m^{-3/2}$.

- Substituting the expressions (54, 55) into Eq. (40), the Hamiltonian equals

$$H = \int d^3x\, \mathcal{H}(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \Big[a^*(\mathbf{p},s)\, a(\mathbf{p},s) - b(\mathbf{p},s)\, b^*(\mathbf{p},s) \Big], \tag{56}$$

  which is not positive definite. However, as we will see, it gives the quantum counterpart which is positive definite.

Excercise: Derive the formula (56).

Hint: Use the relations

$$\begin{aligned} \bar{u}(\mathbf{p},s)\, \mathbf{p}\cdot\boldsymbol{\gamma}\, u(\mathbf{p},s') &= \bar{v}(\mathbf{p},s)\, \mathbf{p}\cdot\boldsymbol{\gamma}\, v(\mathbf{p},s') = \frac{\mathbf{p}^2}{m}\delta^{ss'}, & (57)\\ \bar{u}(\mathbf{p},s)\, \mathbf{p}\cdot\boldsymbol{\gamma}\, v(\mathbf{p},s') &= \bar{v}(\mathbf{p},s)\, \mathbf{p}\cdot\boldsymbol{\gamma}\, u(-\mathbf{p},s') = 0, & (58) \end{aligned}$$

which can be obtained from Eqs. (48, 49) together with Eqs. (47).

- Defining the charge $Q \equiv \int d^3x\, j^0(x)$, where j^0 is the time component of the conserved current (35), and using the solutions (54, 55), the charge equals

$$Q = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3} \Big[a^*(\mathbf{p},s)\, a(\mathbf{p},s) + b(\mathbf{p},s)\, b^*(\mathbf{p},s) \Big]. \tag{59}$$

  In contrast to the Hamiltonian the charge Q is positive definite. In the quantum theory there will be the opposite situation.

Canonical quantization of spinor filed

- The procedure of canonical quantization of the spinor filed is similar to that of the scalar filed but commutators are replaced with anticommutators. It will be shown later on that the quantization of scalar field with anticommutators or spinor field with commutators leads to physically unacceptable results.

Field operators

- The quantum spinor field is represented by the operators $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$ which satisfy the Dirac equations

$$[i\gamma_\mu \partial^\mu - m]\hat{\psi}(x) = 0, \qquad \hat{\bar{\psi}}(x)[i\gamma_\mu \overleftarrow{\partial^\mu} + m] = 0. \tag{60}$$

- Replacing the Poisson brackets with the anticommutators $\{...,...\}_{\rm PB} \to -i\{...,...\}$, Eqs. (38, 39) provide the relations

$$\{\hat{\psi}_\alpha(t,\mathbf{x}), \hat{\pi}_\beta(t,\mathbf{x}')\} = i\delta_{\alpha\beta}\, \delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{61}$$

$$\{\hat{\psi}_\alpha(t,\mathbf{x}), \hat{\psi}_\beta(t,\mathbf{x}')\} = \{\hat{\pi}_\alpha(t,\mathbf{x}), \hat{\pi}_\beta(t,\mathbf{x}')\} = 0, \tag{62}$$

  where the canonical momentum $\hat{\pi}(x)$ conjugate to $\psi(x)$ equals $\hat{\pi}(x) = i\hat{\psi}^\dagger(x) = i\hat{\bar{\psi}}(x)\gamma^0$.

- The field operators $\hat{\psi}(x)$ and $\hat{\overline{\psi}}(x)$ are expressed through the creation and annihilation operators $\hat{a}^\dagger(\mathbf{p},s)$, $\hat{a}(\mathbf{p},s)$, $\hat{b}^\dagger(\mathbf{p},s)$ and $\hat{b}(\mathbf{p},s)$ in the same was as the classical fields $\psi(x)$ and $\overline{\psi}(x)$ through the functions $a^*(\mathbf{p},s)$, $a(\mathbf{p},s)$, $b^*(\mathbf{p},s)$, $b(\mathbf{p},s)$. Therefore, using Eqs. (54, 55), we have

$$\hat{\psi}(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{m}{E_{\mathbf{p}}}} \Big[e^{-ipx} \hat{a}(\mathbf{p},s)\, u(\mathbf{p},s) + e^{ipx} \hat{b}^\dagger(\mathbf{p},s)\, v(\mathbf{p},s) \Big], \tag{63}$$

$$\hat{\overline{\psi}}(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{m}{E_{\mathbf{p}}}} \Big[e^{-ipx} \hat{b}(\mathbf{p},s)\, \bar{v}(\mathbf{p},s) + e^{ipx} \hat{a}^\dagger(\mathbf{p},s)\, \bar{u}(\mathbf{p},s) \Big]. \tag{64}$$

- One easily checks that the relations (61, 62) can be obtained assuming that the creation and annihilation operators satisfy the relations

$$\{\hat{a}(\mathbf{p},s), \hat{a}^\dagger(\mathbf{p}',s')\} = \delta^{ss'} \delta^{(3)}(\mathbf{p}-\mathbf{p}') = \{\hat{b}(\mathbf{p},s), \hat{b}^\dagger(\mathbf{p}',s')\}, \tag{65}$$

  with the vanishing anticommutators of remaining pairs of the operators $\hat{a}^\dagger(\mathbf{p},s)$, $\hat{a}(\mathbf{p},s)$, $\hat{b}^\dagger(\mathbf{p},s)$, $\hat{b}(\mathbf{p},s)$.

- We note that due to the vanishing anticommutator like $\{\hat{a}(\mathbf{p},s), \hat{a}(\mathbf{p}',s')\}$, the squares and higher powers of $\hat{a}(\mathbf{p},s)$ vanish. So, we have

$$\hat{a}^2(\mathbf{p},s) = \Big(\hat{a}^\dagger(\mathbf{p},s)\Big)^2 = \hat{b}^2(\mathbf{p},s) = \Big(\hat{b}^\dagger(\mathbf{p},s)\Big)^2 = 0. \tag{66}$$

- The Hamiltonian is expressed through the creation and annihilation operators as

$$\hat{H} = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \Big[\hat{a}^\dagger(\mathbf{p},s)\, \hat{a}(\mathbf{p},s) - \hat{b}(\mathbf{p},s)\, \hat{b}^\dagger(\mathbf{p},s) \Big]. \tag{67}$$

- Interchanging $\hat{b}^\dagger(\mathbf{p},s)$ and $\hat{b}(\mathbf{p},s)$ in Eq. (67), we get

$$\hat{H} = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \Big[\hat{a}^\dagger(\mathbf{p},s)\, \hat{a}(\mathbf{p},s) + \hat{b}^\dagger(\mathbf{p},s)\, \hat{b}(\mathbf{p},s) - \delta^{(3)}(\mathbf{p}=0) \Big], \tag{68}$$

  where the pathological negative term has appeared. Ignoring the term of equivalently applying the rule of normal ordering, which requires that annihilation operators are on the right side of creation operators, the Hamiltonian is

$$\hat{H} = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3}\, E_{\mathbf{p}} \Big[\hat{a}^\dagger(\mathbf{p},s)\, \hat{a}(\mathbf{p},s) + \hat{b}^\dagger(\mathbf{p},s)\, \hat{b}(\mathbf{p},s) \Big]. \tag{69}$$

- Proceeding analogously the quantum counterpart of the charge (59) is

$$\hat{Q} = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^3} \Big[\hat{a}^\dagger(\mathbf{p},s)\, \hat{a}(\mathbf{p},s) - \hat{b}^\dagger(\mathbf{p},s)\, \hat{b}(\mathbf{p},s) \Big]. \tag{70}$$

Space of states

- To simplify the construction of the space of states, we discretize the momentum space as in the case of scalar field. The Hamiltonian (69) then reads

$$\hat{H} = \sum_{\pm s} \sum_i E_i \Big[\hat{a}_i^\dagger(s)\, \hat{a}_i(s) + \hat{b}_i^\dagger(s)\, \hat{b}_i(s)\Big], \tag{71}$$

where $E_i \equiv \sqrt{m^2 + \mathbf{p}_i^2}$ and the dimensionless creation and annihilation operators are defined as

$$\hat{a}_i(s) \equiv \frac{1}{\sqrt{L^3}}\, \hat{a}(\mathbf{p}_i, s), \qquad \hat{a}_i^\dagger(s) \equiv \frac{1}{\sqrt{L^3}}\, \hat{a}^\dagger(\mathbf{p}_i, s), \tag{72}$$

$$\hat{b}_i(s) \equiv \frac{1}{\sqrt{L^3}}\, \hat{b}(\mathbf{p}_i, s), \qquad \hat{b}_i^\dagger(s) \equiv \frac{1}{\sqrt{L^3}}\, \hat{b}^\dagger(\mathbf{p}_i, s). \tag{73}$$

- The charge operator (70) is

$$\hat{Q} = \sum_{\pm s} \sum_i \Big[\hat{a}_i^\dagger(s)\, \hat{a}_i(s) - \hat{b}_i^\dagger(s)\, \hat{b}_i(s)\Big]. \tag{74}$$

- The annihilation and creation operators satisfy the relations

$$\{\hat{a}_i(s), \hat{a}_j^\dagger(s')\} = \delta^{ij}\delta^{ss'} = \{\hat{b}_i(s), \hat{b}_j^\dagger(s')\}, \tag{75}$$

obtained from Eq. (65). The anticommutators of remaining pairs of the operators vanish.

- Since the Hamiltonian (71) is positive definite, we postulate that the lowest energy vacuum state $|0\rangle$ exists.

- The annihilation operators annihilate the vacuum state that is

$$\hat{a}_i(s)|0\rangle = 0 = \hat{b}_i(s)|0\rangle. \tag{76}$$

- The vacuum expectation value of the Hamiltonian (71) is zero *i.e.* $\langle 0|\hat{H}|0\rangle = 0$.

- Assuming that the spin variable s equals ± 1, which correspond to the projection of the spin on a given axis $\hbar/2$ or $-\hbar/2$, we have

$$\hat{a}_i^\dagger(\pm 1)|0\rangle = |1_i^\pm\rangle, \qquad \hat{b}_i^\dagger(\pm 1)|0\rangle = |\bar{1}_i^\pm\rangle. \tag{77}$$

So, the operator $\hat{a}_i^\dagger(\pm 1)$ which acts on the vacuum state produces the one-particle state of two possible spin orientations. The operator $\hat{b}_i^\dagger(\pm 1)$ produces one antiparticle state. Eq. (74) shows that antiparticles carry opposite charge to particles.

- Since squares and higher powers of $\hat{a}_i^\dagger(s)$ and $\hat{b}_i^\dagger(s)$ vanish we have

$$\Big(\hat{a}_i^\dagger(\pm 1)\Big)^n|0\rangle = 0, \qquad \Big(\hat{b}_i^\dagger(\pm 1)\Big)^n|0\rangle = 0, \qquad \text{for} \quad n > 1. \tag{78}$$

There are only states with single particles of a given momentum and spin. Therefore, the spinor field quantized with the anticommutators describe a system of fermions – particles obeying the Fermi-Dirac statistics.

- If the annihilation operator $\hat{a}_i(\pm 1)$ or $\hat{b}_i(\pm 1)$ acts on $|1_i^{\pm}\rangle$ or $|\bar{1}_i^{\pm}\rangle$ the vacuum state is generated
$$\hat{a}_i(\pm 1)|1_i^{\pm}\rangle = |0\rangle, \qquad \hat{b}_i(\pm 1)|\bar{1}_i^{\pm}\rangle = |0\rangle. \tag{79}$$

- The states $|n_1^+, n_1^-, \bar{n}_1^+, \bar{n}_1^-; n_2^+, n_2^-, \bar{n}_2^+, \bar{n}_2^-; \ldots\rangle$, where $n_i^{\pm} = 0,\ 1$ and $\bar{n}_i^{\pm} = 0,\ 1$ are numbers of particles and antiparticles, respectively, with momentum $\mathbf{p}_i$ and spin $\pm\hbar/2$, form the orthonormal basis of the Fock space.

- The basis states are eigenstates of the energy and charge operators (71) and (74). The energy eigenvalues are nonnegative while the charge eigenvalues can be both positive and negative.