Feynman rules

It is not difficult to guess that the tedious method of calculating amplitudes and cross sections presented in Lecture IX can be changed into an algorithmic procedure with a set of mnemonic rules. Indeed, in 1948 Richard Feynman developed such useful rules, the key element of which are diagrams illustrating interaction processes. The easy-to-grasp intuitive meaning of diagrams, combined with their precise mathematical content, made them extremely popular. Physicists commonly think about collision processes by imagining Feynman diagrams, which have also become a symbol of the entire quantum field theory. It is hard to imagine any course of the theory where the Feynman diagrams are not included. So, simplified Feynman rules are presented and discussed in this lecture.

Cross section

- The cross section of a collisional process $1 + 2 \to 1' + 2' + 3' + \dots n'$ is written as

$$d\sigma = \frac{(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_1' - p_2' \dots - p_n')}{|\mathbf{v}_1 - \mathbf{v}_2|} \frac{1}{2E_1} \frac{1}{2E_2} |\mathcal{A}|^2 \frac{d^3p_1'}{(2\pi)^3 2E_1'} \frac{d^3p_2'}{(2\pi)^3 2E_2'} \cdots \frac{d^3p_n'}{(2\pi)^3 2E_n'} S, \tag{1}$$

  where $\mathcal{A}$ is the Lorentz invariant amplitude to be calculated according to the Feynman rules given below.

- $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of colliding initial-state particles and the velocities are assumed to be parallel to each other $\mathbf{v}_1 \parallel \mathbf{v}_2$.

- S is the combinatorial factor which depends on the number of identical particles in the final state. If there are k types of particles in the final state and there are n_i identical particles of i-th type with $i = 1, 2, \dots k$, the factor equals

$$S = \prod_{i=1}^{k} \frac{1}{n_i!}.$$

- All particles both in the final and initial states are assumed to be bosons in the formula (1). If a given particle is a fermion the factor $1/2E_i$ or $1/2E_j'$ is replaced by m/E_i or m/E_j', respectively. The different treatment of fermions and bosons is related to the different dimensionality of bosonic $\phi(x)$, $A^\mu(x)$ and fermionic $\psi(x)$ fields.

- In collisional processes initial-state particles, which have spin, are usually not polarized and spin of final-state particles is not measured. Then, the cross section (1) should summed over spin states of initial-state particles and averaged over spin states of final-state particles.

- The summation over spin states of electrons, positrons and photons is performed using the formulas

$$\sum_{\pm s} u_\alpha(\mathbf{p}, s)\, \bar{u}_\beta(\mathbf{p}, s) = \left(\frac{\gamma \cdot p + m}{2m}\right)_{\alpha\beta}, \tag{2}$$

$$\sum_{\pm s} v_\alpha(\mathbf{p}, s)\, \bar{v}_\beta(\mathbf{p}, s) = \left(\frac{\gamma \cdot p - m}{2m}\right)_{\alpha\beta}, \tag{3}$$

$$\sum_{\lambda=1}^{2} \epsilon_\mu(\mathbf{k}, \lambda)\, \epsilon_\nu(\mathbf{k}, \lambda) = -g_{\mu\nu}. \tag{4}$$

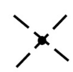
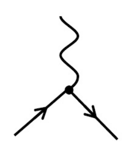
Amplitudes and diagrams

- One starts a computation of the amplitude $\mathcal{A}$ with drawing all possible diagrams of a given process.

- There are two types of lines in Feynman graphs: external lines which represent particles of initial or final states and internal lines which correspond to virtual particles being carriers of interaction.

- Assuming that the time flows from the left side of a diagram to its right side, the external lines in the left side of the diagram represent the initial-state particles and the external lines in the right side of the diagram the final-state particles.

Table 1: Feynman rules

	graphic element	mathematical counterpart
incoming boson line		1
outgoing boson line		1
boson propagator		$i\Delta(q) = \frac{i}{q^2 - m^2 + i0^+}$
incoming photon line		$\epsilon_\mu(\mathbf{k}, \lambda)$
outgoing photon line		$\epsilon_\mu(\mathbf{k}, \lambda)$
photon propagator		$iD^{\mu\nu}(q) = -i\frac{g^{\mu\nu}}{q^2 + i0^+}$
incoming electron line		$u(\mathbf{p}, s)$
outgoing electron line		$\bar{u}(\mathbf{p}, s)$
incoming positron line		$\bar{v}(\mathbf{p}, s)$
outgoing positron line		$v(\mathbf{p}, s)$
electron propagator		$iS(q) = i\frac{\gamma_\mu q^\mu + m}{q^2 - m^2 + i0^+}$
scalar-boson self-interaction		$i\frac{\lambda}{4!}$
electromagnetic interaction		$-ie\gamma^\mu$

- The lines of charge particles have arrows. If the arrow's orientation agrees with the direction of time flow, the line represents a particle and if the arrow's orientation is opposite to the time flow, the line represents an antiparticle.
- The lines of truly neutral particles have no arrows as particles cannot be distinguished from antiparticles.
- Three, four or even higher number of lines can be joined in a vertex which represent an interaction. The structure of vertices depends on dynamical theory under consideration.
- A set of Feynman rules, which is presented here, is limited to two theories: the self-interacting real scalar bosons and the quantum electrodynamics (QED).
- Real scalar bosons are represented by dashed lines with no arrows.
- Photons are represented by wavy lines with no arrows.
- Electrons are represented by solid lines with arrows oriented with the time flow.
- Positrons are represented by solid lines with arrows oriented against the time flow.
- A complete amplitude is a sum of partial amplitudes represented by a set of diagrams.

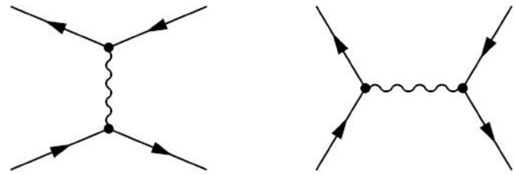


Figure 1: The lowest order diagrams of Bhabha scattering $e^-e^+ \to e^-e^+$

- A partial amplitude corresponding to a given diagram is computed combining mathematical expressions assigned to graphical elements as shown in Table 1.
- The spinor factors $u, \bar{u}, v, \bar{v}$ and matrices γ^μ are ordered along the continuous fermion lines formed by external and internal lines starting from the end indicated by the arrow. In other words, the spinor factors are ordered in the direction opposite to the arrow of fermion line.
- A diagram which is obtained from a given one by exchange of two identical fermions in the final or initial state gets an extra factor -1.
- A diagram which is obtained from a given one interchanging a fermion from the final (initial) state with a antifermion in the initial (final) state gets an extra factor -1.
- Since every vertex includes a coupling constant, the number of vertices determines a power of the coupling constant λ or e of the amplitude. Consequently, the diagrams of N vertices correspond to the $N-$th term of the operator $\hat{S}$ expansion.
- As the coupling constants are small, the diagrams of the smallest number of vertices provide the leading contribution to the amplitude a given collisional process. Such diagrams typically do not contain closed lines and are called tree diagrams.
- The diagrams with higher number of vertices, which include the closed lines – loops, provide corrections.
- The loop diagrams are a serious problem of quantum field theory as they correspond to infinite mathematical expressions and require the regularization and renormalization procedure.
- The Feynman rules given here are applicable only to tree diagrams.

Example – Bhabha scattering

- The Bhabha scattering is the elastic process $e^-e^+ \to e^-e^+$. Its cross section was first derived by Homi Bhabha in 1935.
- The two lowest order diagrams of the Bhabha scattering are shown in Fig. 1. The time flows from left to right and consequently the initial state particles are in the left side of the diagrams and the final state particles in the right side.
- The initial-state electron and positron have four-momenta $\mathbf{p}$, $\bar{\mathbf{p}}$ and spins s, $\bar{s}$, respectively. The four-momenta and spins of final-state electron and positron are $\mathbf{p}'$, $\bar{\mathbf{p}}'$ and s', $\bar{s}'$.
- The two amplitude corresponding to the two diagrams from Fig. 1 are

$$\mathcal{A}_1 = \Big[\bar{u}(\mathbf{p}', s')(-ie)\gamma_\mu u(\mathbf{p}, s)\Big] iD^{\mu\nu}(p - p')\Big[\bar{v}(\bar{\mathbf{p}}, \bar{s})(-ie)\gamma_\nu v(\bar{\mathbf{p}}', \bar{s}')\Big], \tag{5}$$

$$\mathcal{A}_2 = -\Big[\bar{u}(\mathbf{p}', s')(-ie)\gamma_\mu v(\bar{\mathbf{p}}', \bar{s}')\Big] iD^{\mu\nu}(p + \bar{p})\Big[\bar{v}(\bar{\mathbf{p}}, \bar{s})(-ie)\gamma_\nu u(\mathbf{p}, s)\Big]. \tag{6}$$

- Using the explicit form of the photon propagator the sum of the amplitudes equals

$$\mathcal{A} = ie^2\Big[\frac{\bar{u}(\mathbf{p}', s')\gamma_\mu u(\mathbf{p}, s)\ \bar{v}(\bar{\mathbf{p}}, \bar{s})\gamma^\mu v(\bar{\mathbf{p}}', \bar{s}')}{(p - p')^2} - \frac{\bar{u}(\mathbf{p}', s')\gamma_\nu v(\bar{\mathbf{p}}', \bar{s}')\ \bar{v}(\bar{\mathbf{p}}, \bar{s})\gamma^\nu u(\mathbf{p}, s)}{(p + \bar{p})^2}\Big]. \tag{7}$$

- The amplitude (7) substituted into Eq. (1) gives the cross section

$$d\sigma = \frac{(2\pi)^4\delta^{(4)}(p+\bar{p}-p'-\bar{p}')}{|\mathbf{v}-\bar{\mathbf{v}}|}\,\frac{m^4}{E\bar{E}E'\bar{E}'}\,|\mathcal{A}|^2\,\frac{d^3p'}{(2\pi)^3}\,\frac{d^3\bar{p}'}{(2\pi)^3}. \tag{8}$$

- The cross section (8) should be summed over initial spin states and averaged over final state spins according to the formula

$$d\bar{\sigma} = \frac{1}{2^2}\sum_{\pm s}\sum_{\pm\bar{s}}\sum_{\pm s'}\sum_{\pm\bar{s}'} d\sigma. \tag{9}$$

- The quantity, which is summed over spins, is $|W|^2$ equal to

$$\begin{aligned} |W|^2 \quad = \quad & \frac{\bar{u}(\mathbf{p}',s')\gamma_\mu u(\mathbf{p},s)\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu v(\bar{\mathbf{p}}',\bar{s}')}{(p-p')^2}\,\frac{\bar{u}(\mathbf{p},s)\gamma_\nu u(\mathbf{p}',s')\;\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma^\nu v(\bar{\mathbf{p}},\bar{s})}{(p-p')^2} \\ & -\frac{\bar{u}(\mathbf{p}',s')\gamma_\mu u(\mathbf{p},s)\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu v(\bar{\mathbf{p}}',\bar{s}')}{(p-p')^2}\,\frac{\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma_\nu u(\mathbf{p}',s')\;\bar{u}(\mathbf{p},s)\gamma^\nu v(\bar{\mathbf{p}},\bar{s})}{(p+\bar{p})^2} \\ & -\frac{\bar{u}(\mathbf{p}',s')\gamma_\mu v(\bar{\mathbf{p}}',\bar{s}')\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu u(\mathbf{p},s)}{(p+\bar{p})^2}\,\frac{\bar{u}(\mathbf{p},s)\gamma_\nu u(\mathbf{p}',s')\;\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma^\nu v(\bar{\mathbf{p}},\bar{s})}{(p-p')^2} \\ & +\frac{\bar{u}(\mathbf{p}',s')\gamma_\mu v(\bar{\mathbf{p}}',\bar{s}')\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu u(\mathbf{p},s)}{(p+\bar{p})^2}\,\frac{\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma_\nu u(\mathbf{p}',s')\;\bar{u}(\mathbf{p},s)\gamma^\nu v(\bar{\mathbf{p}},\bar{s})}{(p+\bar{p})^2}. \end{aligned} \tag{10}$$

- There are four terms which contribute to the cross section

$$S_1 \equiv \sum_{\pm s}\sum_{\pm\bar{s}}\sum_{\pm s'}\sum_{\pm\bar{s}'} \bar{u}(\mathbf{p}',s')\gamma_\mu u(\mathbf{p},s)\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu v(\bar{\mathbf{p}}',\bar{s}')\;\bar{u}(\mathbf{p},s)\gamma_\nu u(\mathbf{p}',s')\;\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma^\nu v(\bar{\mathbf{p}},\bar{s}), \tag{11}$$

$$S_2 \equiv \sum_{\pm s}\sum_{\pm\bar{s}}\sum_{\pm s'}\sum_{\pm\bar{s}'} \bar{u}(\mathbf{p}',s')\gamma_\mu u(\mathbf{p},s)\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu v(\bar{\mathbf{p}}',\bar{s}')\;\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma_\nu u(\mathbf{p}',s')\;\bar{u}(\mathbf{p},s)\gamma^\nu v(\bar{\mathbf{p}},\bar{s}), \tag{12}$$

$$S_3 \equiv \sum_{\pm s}\sum_{\pm\bar{s}}\sum_{\pm s'}\sum_{\pm\bar{s}'} \bar{u}(\mathbf{p}',s')\gamma_\mu v(\bar{\mathbf{p}}',\bar{s}')\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu u(\mathbf{p},s)\;\bar{u}(\mathbf{p},s)\gamma_\nu u(\mathbf{p}',s')\;\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma^\nu v(\bar{\mathbf{p}},\bar{s}), \tag{13}$$

$$S_4 \equiv \sum_{\pm s}\sum_{\pm\bar{s}}\sum_{\pm s'}\sum_{\pm\bar{s}'} \bar{u}(\mathbf{p}',s')\gamma_\mu v(\bar{\mathbf{p}}',\bar{s}')\;\bar{v}(\bar{\mathbf{p}},\bar{s})\gamma^\mu u(\mathbf{p},s)\;\bar{v}(\bar{\mathbf{p}}',\bar{s}')\gamma_\nu u(\mathbf{p}',s')\;\bar{u}(\mathbf{p},s)\gamma^\nu v(\bar{\mathbf{p}},\bar{s}), \tag{14}$$

such that

$$\sum_{\pm s}\sum_{\pm\bar{s}}\sum_{\pm s'}\sum_{\pm\bar{s}'} |W|^2 = \frac{S_1}{((p-p')^2)^2} - \frac{S_2+S_3}{(p-p')^2(p+\bar{p})^2} + \frac{S_4}{((p+\bar{p})^2)^2}. \tag{15}$$

- Using the relations (2, 3), one finds

$$\begin{aligned} 16m^4 S_1 &= \mathrm{Tr}[\gamma_\mu(\gamma\cdot p+m)\gamma_\nu(\gamma\cdot p'+m)]\;\mathrm{Tr}[\gamma^\mu(\gamma\cdot\bar{p}'-m)\gamma^\nu(\gamma\cdot\bar{p}-m)], & (16)\\ 16m^4 S_2 &= \mathrm{Tr}[\gamma_\mu(\gamma\cdot p+m)\gamma_\nu(\gamma\cdot\bar{p}-m)\gamma^\mu(\gamma\cdot\bar{p}'-m)\gamma^\nu(\gamma\cdot p'+m)], & (17)\\ 16m^4 S_3 &= \mathrm{Tr}[\gamma_\mu(\gamma\cdot\bar{p}'-m)\gamma_\nu(\gamma\cdot\bar{p}-m)\gamma^\mu(\gamma\cdot p+m)\gamma^\nu(\gamma\cdot p'+m)], & (18)\\ 16m^4 S_4 &= \mathrm{Tr}[\gamma_\mu(\gamma\cdot\bar{p}'-m)\gamma_\nu(\gamma\cdot p'+m)]\;\mathrm{Tr}[\gamma^\mu(\gamma\cdot p+m)\gamma^\nu(\gamma\cdot\bar{p}-m)]. & (19) \end{aligned}$$

- Computation of traces S_1, S_2, S_3, S_4 is not difficult but very tedious. One needs the following identities

$$\begin{aligned} \mathrm{Tr}[\gamma_\mu\gamma_\nu] &= 4g_{\mu\nu}, & (20)\\ \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] &= 4g_{\mu\sigma}\,g_{\nu\rho} - 4g_{\mu\rho}\,g_{\nu\sigma} + 4g_{\mu\nu}\,g_{\rho\sigma}, & (21)\\ \gamma_\mu\gamma^\nu\gamma^\mu &= -2\gamma^\nu, & (22)\\ \gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu &= 4g^{\nu\rho}. & (23) \end{aligned}$$

- The results are

$$m^4 S_1 = 2(p\cdot\bar{p})(p'\cdot\bar{p}') + 2(p\cdot\bar{p}')(p'\cdot\bar{p}) - 2m^2(p\cdot p') - 2m^2(\bar{p}'\cdot\bar{p}) + 4m^4, \tag{24}$$

$$m^4 S_2 = m^4 S_3 = -2(p'\cdot\bar{p})(p\cdot\bar{p}') - m^2(p'\cdot\bar{p}') - m^2(p'\cdot\bar{p}) + m^2(\bar{p}\cdot\bar{p}') + m^2(p\cdot p') - m^2(p\cdot\bar{p}') - m^2(p\cdot\bar{p}) - 2m^4, \tag{25}$$

$$m^4 S_4 = 2(p\cdot\bar{p}')(\bar{p}\cdot p') + 2(\bar{p}\cdot\bar{p}')(p\cdot p') + 2m^2(p'\cdot\bar{p}') + 2m^2(p\cdot\bar{p}) + 4m^4. \tag{26}$$

Exercise: Compute the traces (16, 17, 18, 19) which give the formulas (24, 25, 26).

- Using the Mandelstam variables

$$s \equiv (p+\bar{p})^2 = (p'+\bar{p}')^2 = 2(p\cdot\bar{p}) + 2m^2 = 2(p'\cdot\bar{p}') + 2m^2, \tag{27}$$

$$t \equiv (p-p')^2 = (\bar{p}'-\bar{p})^2 = -2(p\cdot p') + 2m^2 = -2(\bar{p}'\cdot\bar{p}) + 2m^2, \tag{28}$$

$$u \equiv (p-\bar{p}')^2 = (p'-\bar{p})^2 = -2(p\cdot\bar{p}') + 2m^2 = -2(p'\cdot\bar{p}) + 2m^2, \tag{29}$$

which obey the condition

$$s + t + u = 4m^2, \tag{30}$$

the formulas (24, 25, 26) can be written as

$$m^4 S_1 = \frac{s^2+u^2}{2} + 4m^2(t-m^2), \tag{31}$$

$$m^4 S_2 = m^4 S_3 = -\frac{u^2}{2} + 4m^2 u - 6m^4, \tag{32}$$

$$m^4 S_4 = \frac{t^2+u^2}{2} + 4m^2(s-m^2). \tag{33}$$

Exercise: Derive the formulas (31, 32, 33) starting with (24, 25, 26).

- The spin averaged Bhabha cross section (9) is

$$\frac{d\bar{\sigma}}{dt} = -\frac{e^4}{16\pi}\frac{m^4}{(p\cdot\bar{p})^2 - m^4}\left[\frac{S_1}{((p-p')^2)^2} - \frac{S_2+S_3}{(p-p')^2(p+\bar{p})^2} + \frac{S_4}{((p+\bar{p})^2)^2}\right], \tag{34}$$

which using the results (31, 32, 33) gives the final formula

$$\frac{d\bar{\sigma}}{dt} = -\frac{2\pi\alpha^2}{s(s-4m^2)}\left[\frac{s^2+u^2+8m^2(t-m^2)}{t^2} + 2\frac{u^2-8m^2u+12m^4}{st} + \frac{t^2+u^2+8m^2(s-m^2)}{s^2}\right], \tag{35}$$

where $\alpha \equiv \frac{e^2}{4\pi}$.

- The first and third terms of the formula (35) correspond the squares of the amplitudes (5, 6) while the second term is the interference term.
- Due to the first term the Bhabha cross section has a strong maximum (is actually divergent) as $t \to 0$, similarly to the Mott and Rutherford cross sections.
- In the ultrarelativistic approximation when $m = 0$, the cross section (35) simplifies to

$$\frac{d\bar{\sigma}}{dt} = -\frac{2\pi\alpha^2}{s^2}\left[\frac{s^2+u^2}{t^2} + \frac{2u^2}{st} + \frac{t^2+u^2}{s^2}\right]. \tag{36}$$

- In nonrelativistic approximation the cross section (35) becomes the Rutherford cross section. It happens because the amplitude (5) dominates while the amplitude (6) and the interference term are suppressed.

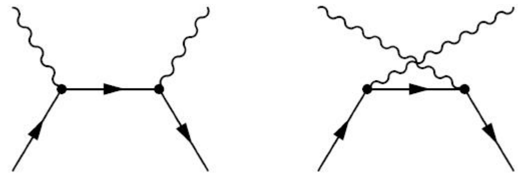


Figure 2: The lowest order diagrams of Compton scattering $\gamma e^- \to \gamma e^-$

Exercise: Prove that the Bhabha cross section (35) becomes the Rutherford cross section in nonrelativistic approximation. Take into account that in the center-of-mass frame the invariants s, t, u can be expressed as

$$s = 4m^2 + 4\mathbf{p}^2, \qquad t = -4\mathbf{p}^2 \sin^2\frac{\theta}{2}, \qquad u = -4\mathbf{p}^2 \cos^2\frac{\theta}{2}. \tag{37}$$

Example – Compton scattering

- The Compton scattering $\gamma e^- \to \gamma e^-$ was historically very important as it revealed a corpuscular nature of photons.
- If the photon wavelength is much bigger that the inverse electron mass, we deal with the classical Thomson scattering where photon wavelength remains unchanged.
- In 1923 Arthur Compton observed that the wavelength of photons of energy 20 keV scattered on electrons becomes significantly longer. Specifically, in the electron rest frame the wavelength change is

$$\Delta\lambda = 2\pi\Big(\frac{1}{\omega} - \frac{1}{\omega'}\Big) = \frac{2\pi}{m}(1-\cos\theta), \tag{38}$$

  where ω and ω' are initial and final photon energies and $\cos\theta$ is the photon scattering angle.
- The wavelength shift is independent of the initial wavelength. For this reason it is visible for photons of a sufficiently short wavelength, such that $\Delta\lambda$ is comparable to λ.
- In 1928 Oskar Klein and Yoshio Nishina gave a theoretical description of the phenomenon deriving the scattering cross section known as the Klein-Nishina formula.

Exercise: Derive the formula (38).

- The energy, momentum and spin of initial- and final-state electron are E, $\mathbf{p}$, s and E', $\mathbf{p}'$ and s'. The same quantities which characterize the photons are ω, $\mathbf{k}$, λ and ω', $\mathbf{k}'$, λ'.
- The two amplitudes corresponding to the two lowest order diagrams shown in Fig. 2 are

$$\mathcal{A}_1 = \epsilon_\mu(\mathbf{k}',\lambda')\Big[\bar{u}(\mathbf{p}',s')(-ie)\gamma^\mu iS(q_+)(-ie)\gamma^\nu u(\mathbf{p},s)\Big]\epsilon_\nu(\mathbf{k},\lambda), \tag{39}$$

$$\mathcal{A}_2 = \epsilon_\mu(\mathbf{k}',\lambda')\Big[\bar{u}(\mathbf{p}',s')(-ie)\gamma^\nu iS(q_-)(-ie)\gamma^\mu u(\mathbf{p},s)\Big]\epsilon_\nu(\mathbf{k},\lambda), \tag{40}$$

  where $q_+ \equiv p + k$ and $q_- \equiv p - k'$.
- The sum of the amplitudes $\mathcal{A}_1$ and $\mathcal{A}_2$ is

$$\mathcal{A} = -ie^2\epsilon_\mu(\mathbf{k}',\lambda')\bar{u}(\mathbf{p}',s')\Big(\gamma^\mu S(q_+)\gamma^\nu + \gamma^\nu S(q_-)\gamma^\mu\Big)u(\mathbf{p},s)\epsilon_\nu(\mathbf{k},\lambda). \tag{41}$$

- The cross section (1) equals

$$d\sigma = \frac{(2\pi)^4\delta^{(4)}(k+p-k'-p')}{\sqrt{(k\cdot p)^2}}\,\frac{m^2}{4\omega' E'}\,|\mathcal{A}|^2\,\frac{d^3k'}{(2\pi)^3}\,\frac{d^3p'}{(2\pi)^3}, \tag{42}$$

where we have taken into account that there is one boson and one fermion both in the initial and final states. We have also taken into account that when $\mathbf{v_p}||\mathbf{v_k}$

$$E\omega|\mathbf{v_p}-\mathbf{v_k}| = \sqrt{(p\cdot k)^2}. \tag{43}$$

- The cross section (42) can be written as

$$\frac{d\sigma}{dt} = -\frac{m^2}{4\pi(s-m^2)^2}|\mathcal{A}|^2. \tag{44}$$

- The amplitude square summed over the initial and final spin states is

$$\sum_{s,s',\lambda,\lambda'} |\mathcal{A}|^2 = e^4 \sum_{s,s',\lambda,\lambda'} \epsilon_\mu(\mathbf{k}',\lambda')\bar{u}(\mathbf{p}',s')\Big(\gamma^\mu S(q_+)\gamma^\nu + \gamma^\nu S(q_-)\gamma^\mu\Big)u(\mathbf{p},s)\epsilon_\nu(\mathbf{k},\lambda) \tag{45}$$
$$\times\, \epsilon_\rho(\mathbf{k},\lambda)\bar{u}(\mathbf{p},s)\Big(\gamma^\rho S(q_+)\gamma^\sigma + \gamma^\sigma S(q_-)\gamma^\rho\Big)u(\mathbf{p}',s')\epsilon_\sigma(\mathbf{k}',\lambda').$$

- Using the relations (2, 4) and the explicit form of the electron propagator one finds

$$\sum_{s,s',\lambda,\lambda'} |\mathcal{A}|^2 = \frac{e^4}{4m^2}\Big\{\frac{1}{(q_+^2-m^2)^2}\mathrm{Tr}\Big[(\gamma\cdot p'+m)\gamma^\mu(\gamma\cdot q_+ + m)\gamma^\nu(\gamma\cdot p+m)\gamma_\nu(\gamma\cdot q_+ + m)\gamma_\mu\Big] \tag{46}$$
$$+\frac{1}{(q_+^2-m^2)(q_-^2-m^2)}\mathrm{Tr}\Big[(\gamma\cdot p'+m)\gamma^\mu(\gamma\cdot q_+ + m)\gamma^\nu(\gamma\cdot p+m)\gamma_\mu(\gamma\cdot q_- + m)\gamma_\nu\Big]$$
$$+\frac{1}{(q_-^2-m^2)(q_+^2-m^2)}\mathrm{Tr}\Big[(\gamma\cdot p'+m)\gamma^\nu(\gamma\cdot q_- + m)\gamma^\mu(\gamma\cdot p+m)\gamma_\nu(\gamma\cdot q_+ + m)\gamma_\mu\Big]$$
$$+\frac{1}{(q_-^2-m^2)^2}\mathrm{Tr}\Big[(\gamma\cdot p'+m)\gamma^\nu(\gamma\cdot q_- + m)\gamma^\mu(\gamma\cdot p+m)\gamma_\mu(\gamma\cdot q_- + m)\gamma_\nu\Big]\Big\}.$$

- After long and tedious calculations one obtains

$$\sum_{s,s',\lambda,\lambda'} |\mathcal{A}|^2 = \frac{2e^4}{m^2}\Big[\frac{-su+3sm^2+um^2+m^4}{(s-m^2)^2} + \frac{2(s+u)m^2+4m^4}{(s-m^2)(u-m^2)} + \frac{-su+3um^2+sm^2+m^4}{(u-m^2)^2}\Big], \tag{47}$$

where

$$s \equiv (p+k)^2 = (p'+k')^2, \tag{48}$$
$$t \equiv (p-p')^2 = (k'-k)^2, \tag{49}$$
$$u \equiv (p-k')^2 = (p'-k)^2. \tag{50}$$

- Substituting the result (47) into the cross section formula (44) we obtain the famous Klein-Nishina cross section

$$\frac{d\bar{\sigma}}{dt} = -\frac{2\pi\alpha^2}{(s-m^2)^2}\Big[\frac{-su+3sm^2+um^2+m^4}{(s-m^2)^2} + \frac{2(s+u)m^2+4m^4}{(s-m^2)(u-m^2)} + \frac{-su+3um^2+sm^2+m^4}{(u-m^2)^2}\Big], \tag{51}$$

where the factor 1/4 takes into account four initial spin states.

- In the limit of massless electrons the formula simplifies to

$$\frac{d\bar{\sigma}}{dt} = \frac{2\pi\alpha^2}{s^2}\Big[\frac{u}{s} + \frac{s}{u}\Big]. \tag{52}$$

- In contrast to the Rutherford, Mott and Bhabha scattering the total cross section of Compton scattering is finite and it equals

$$\bar{\sigma} = \frac{8\pi}{3}\frac{\alpha^2}{m^2}. \tag{53}$$

Crossed processes

- Except the Bhabha and Compton scattering there are three other electromagnetic binary processes with the cross sections of the order α^2 which are: the Møller scattering $e^{\mp}e^{\mp} \to e^{\mp}e^{\mp}$, the two-photon annihilation of electron and positron $e^-e^+ \to \gamma\gamma$ and the creation of electron-positron pair $\gamma\gamma \to e^-e^+$. The amplitude of theses processes can be obtained from the Bhabha or Compton scattering by means of the crossing symmetry.

- Let us consider the reaction
$$1 + 2 + \dots n \to 1' + 2' + \dots n' \tag{54}$$
of the amplitude $\mathcal{A}(p_1, p_2, \dots p_n | p_{1'}, p_{2'}, \dots p_{n'})$. The example of the crossed process is
$$1 + \bar{2}' + \dots n \to 1' + \bar{2} + \dots n', \tag{55}$$
where the bar over 2 denotes the antiparticle, and its amplitude is $\mathcal{A}_{\rm cross}(p_1, p_{2'}, \dots p_n | p_{1'}, p_2, \dots p_{n'})$.

- According to the principle of crossing symmetry
$$\mathcal{A}_{\rm cross}(p_1, p_{2'}, \dots p_n | p_{1'}, p_2, \dots p_{n'}) = \mathcal{A}(p_1, -p_{2'}, \dots p_n | p_{1'}, -p_2, \dots p_{n'}). \tag{56}$$

- Due to the sign change of four-momenta of interchanged particles, the amplitudes (56) should be understood as analytically continued to negative energies. The continuation causes some difficulties. However, we are going to apply the crossing symmetry to the amplitudes squares where the difficulties are absent.

Electron-positron annihilation

- The process $e^+e^- \to \gamma\gamma$ is described by two diagrams shown in Fig. 2 if the time flows from the bottom to the top.

- The electron and positron four-momenta are $p = (E, \mathbf{p})$ and $\bar{p} = (\bar{E}, \bar{\mathbf{p}})$ and those of photons are $k_1 = (\omega_1, \mathbf{k}_1)$ and $k_2 = (\omega_2, \mathbf{k}_2)$.

- To convert the amplitude square of the Compton scattering into that of the e^+e^- annihilation we rearrange the four-momenta as
$$k \to -k_1, \quad k' \to k_2, \quad p \to p, \quad p' \to -\bar{p},$$
which leads to the following changes of the Mandelstam invariants
$$\begin{aligned} s = (p+k)^2 &\quad \to \quad (p-k_1)^2 = t, \\ t = (p-p')^2 &\quad \to \quad (p+\bar{p})^2 = s, \\ u = (p-k')^2 &\quad \to \quad (p-k_2)^2 = u. \end{aligned}$$

- The amplitude square summed over spins, which is obtained from Eq. (47), is
$$\sum_{s,s',\,\lambda,\lambda'} |\mathcal{A}|^2 = \frac{2e^4}{m^2}\Big[\frac{-tu + 3tm^2 + um^2 + m^4}{(t-m^2)^2} + \frac{2(t+u)m^2 + 4m^4}{(t-m^2)(u-m^2)} + \frac{-tu + 3um^2 + tm^2 + m^4}{(u-m^2)^2}\Big]. \tag{57}$$

- Using the formula
$$E\bar{E}|\mathbf{v} - \bar{\mathbf{v}}| = \sqrt{(p\cdot\bar{p})^2 - m^2} = \sqrt{\frac{s-4m^2}{2}}, \tag{58}$$
the annihilation cross section equals
$$\frac{d\bar{\sigma}}{dt} = -\frac{\pi\alpha^2}{(s-4m^2)^2}\Big[\frac{-tu + 3tm^2 + um^2 + m^4}{(t-m^2)^2} + \frac{2(t+u)m^2 + 4m^4}{(t-m^2)(u-m^2)} + \frac{-tu + 3um^2 + tm^2 + m^4}{(u-m^2)^2}\Big]. \tag{59}$$
where we have included the combinatorial factor 1/2 due to the two identical photons in the final state.

- When $E, \bar{E} \gg m$ the formula simplifies to
$$\frac{d\bar{\sigma}}{dt} = \frac{\pi\alpha^2}{s^2}\Big[\frac{u}{t} + \frac{t}{u}\Big]. \tag{60}$$

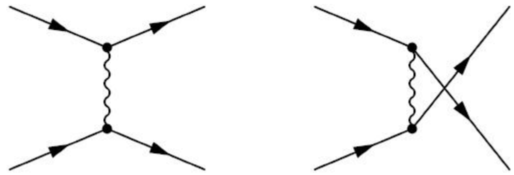


Figure 3: The lowest order diagrams of Møller scattering $e^-e^- \to e^-e^-$.

Creation of electron-positron pair

- The cross section of the process $\gamma\gamma \to e^-e^+$ can be also easily found from the Compton cross section.
- We denote the photon four-momenta as $k_1 = (\omega_1, \mathbf{k}_1)$ i $k_2 = (\omega_2, \mathbf{k}_2)$, and those of electron and positron as $p = (E, \mathbf{p})$ and $\bar{p} = (\bar{E}, \bar{\mathbf{p}})$.
- To go from the Compton scattering to the e^+e^- pair creation we change the four-momenta as

$$k \to k_1, \quad k' \to -k_2, \quad p \to -p, \quad p' \to \bar{p},$$

which leads to

$$\begin{aligned} s = (p+k)^2 \quad &\to \quad (p-k_1)^2 = t, \\ t = (p-p')^2 \quad &\to \quad (p+\bar{p})^2 = s, \\ u = (p-k')^2 \quad &\to \quad (p-k_2)^2 = u. \end{aligned}$$

- Modifying the flux factor because of the vanishing photon masses, the cross section is

$$\frac{d\bar{\sigma}}{dt} = -\frac{2\pi\alpha^2}{s}\Big[\frac{-tu + 3tm^2 + um^2 + m^4}{(t-m^2)^2} + \frac{2(t+u)m^2 + 4m^4}{(t-m^2)(u-m^2)} + \frac{-tu + 3um^2 + tm^2 + m^4}{(u-m^2)^2}\Big]. \tag{61}$$

Møller scattering

- The elastic process $e^-e^- \to e^-e^-$ or $e^+e^+ \to e^+e^+$ is called the Møller scattering as it was theoretically studied by Christian Møller.
- The two lowest order diagrams of the Møller scattering are shown in Fig. 3.
- The cross section of the Møller scattering can be obtained from that of Bhabha scattering (35), changing the sign of positron four-momenta in the initial and final states. It leads to the replacement $s \leftrightarrow u$. Consequently, the cross section equals

$$\frac{d\bar{\sigma}}{dt} = -\frac{\pi\alpha^2}{s(s-4m^2)}\Big[\frac{s^2 + u^2 + 8m^2(t-m^2)}{t^2} + \frac{t^2 + s^2 + 8m^2(u-m^2)}{u^2} + 2\frac{s^2 - 8m^2 s + 12m^4}{tu}\Big], \tag{62}$$

where the combinatorial factor 1/2 is included.

- In contrast to the Bhabha cross section, the Mott cross section is invariant under the replacement $t \leftrightarrow u$. For this reason the Mott cross section has two maxima at $t = 0$ and $u = 0$ which correspond to the scattering angles $\theta = 0$ and $\theta = \pi$. Since the cross section is infinite at $t = 0$ and $u = 0$, the total cross section is ill defined.
- In the unltrarelativistic limit the cross section (62) simplifies to

$$\frac{d\bar{\sigma}}{dt} = -\frac{\pi\alpha^2}{s^2}\Big[\frac{s^2 + u^2}{t^2} + \frac{t^2 + s^2}{u^2} + \frac{2s^2}{tu}\Big]. \tag{63}$$

The End