Instabilities Driven Equilibration of the Quark-Gluon Plasma

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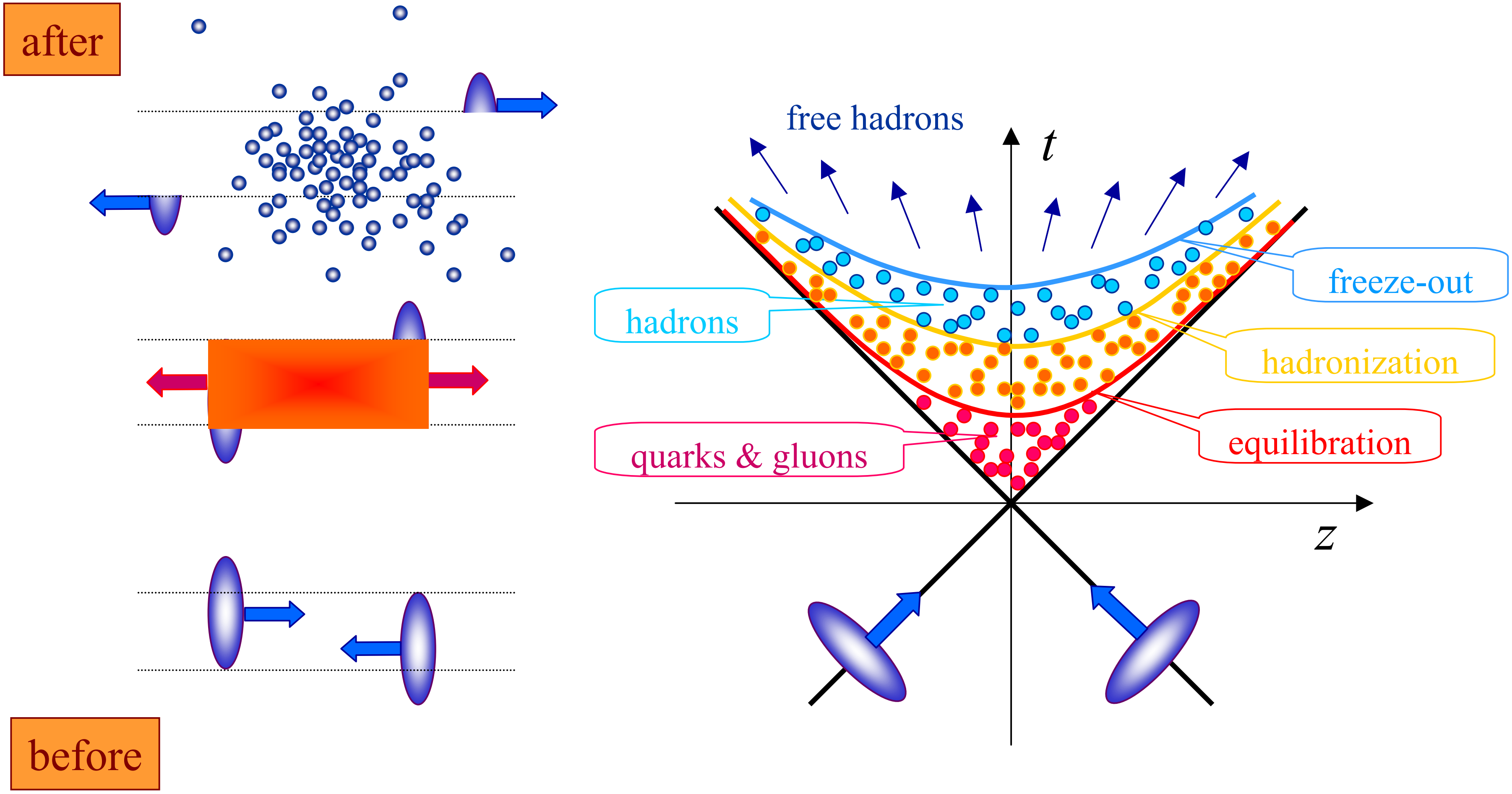
- why Mark’s favorite formula $e^{-E/T}$ is so good -

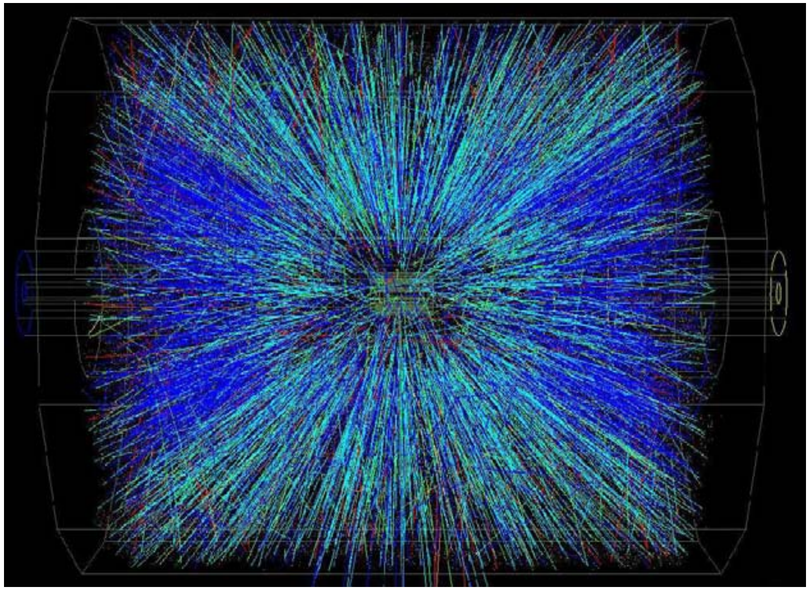
Happy Birthday Mark!

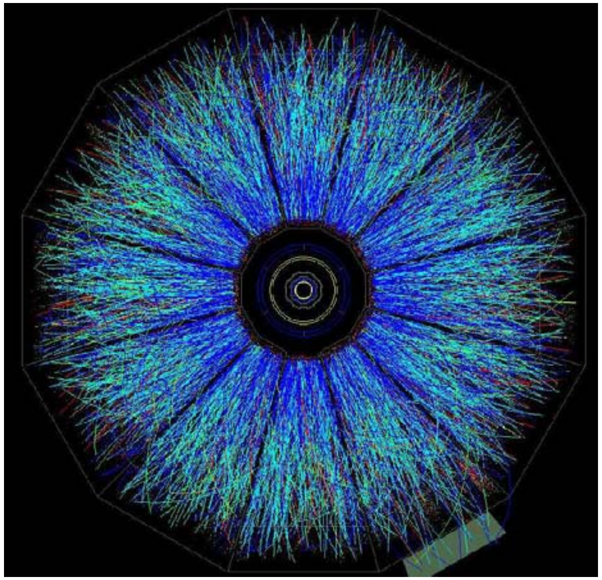
Relativistic heavy-ion collision – *Little Bang*



Relativistic heavy-ion collisions

$$\text{Au}-\text{Au collisions} \ @ \ \sqrt{s} = 100 + 100 \ \text{GeV/NN}$$

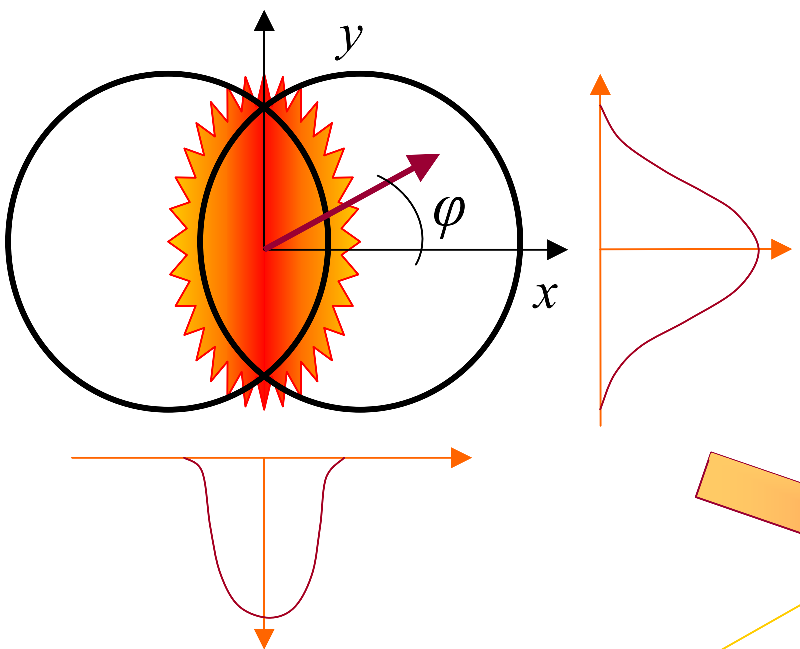




STAR experiment @ RHIC

Evidence of the early stage equilibration

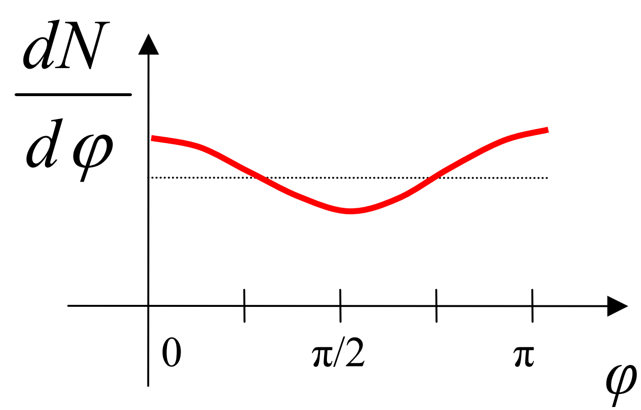
Success of hydrodynamic models in describing elliptic flow



Hydrodynamics

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)\mathbf{v}=-\frac{\nabla p}{\rho}$$

Hydrodynamics requires local thermodynamical equilibrium!



Equilibration is fast

$$v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle$$

Eccentricity decays due to the free streaming!

$\varepsilon \searrow \;\Rightarrow\; v_2 \searrow$ ➡ $t_{eq} \leq 1 \;\text{fm}/c$

time of equilibration

U. Heinz, AIP Conf. Proc. **739,** 163 (2004)

Collisions are too slow

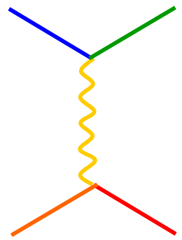
Time scale of hard parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

hard scattering ~ momentum transfer of order of T

either single hard scattering or multiple soft scatterings

dominated by



g – QCD coupling constant

$$\boxed{t_{\text{eq}} \approx t_{\text{hard}} \geq 2.6 \ \text{fm}/c}$$

R. Baier, A.H. Mueller, D. Schiff & D.T. Son, Phys. Lett. **B539**, 46 (2002)

Instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$

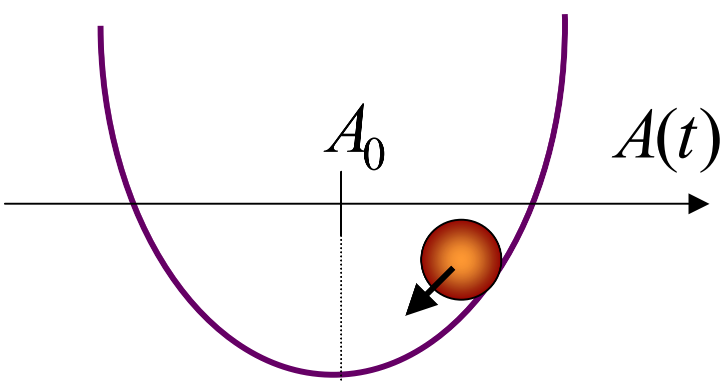
fluctuation

Instability

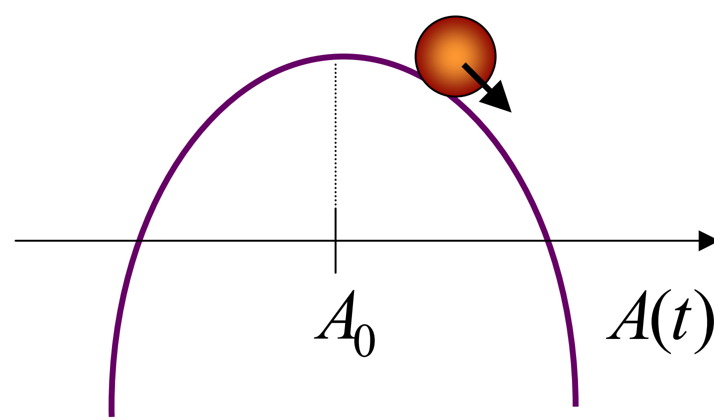
$$\delta A(t) \propto \mathrm{e}^{\gamma t}$$

$$\gamma > 0$$

stable configuration



unstable configuration



Terminology

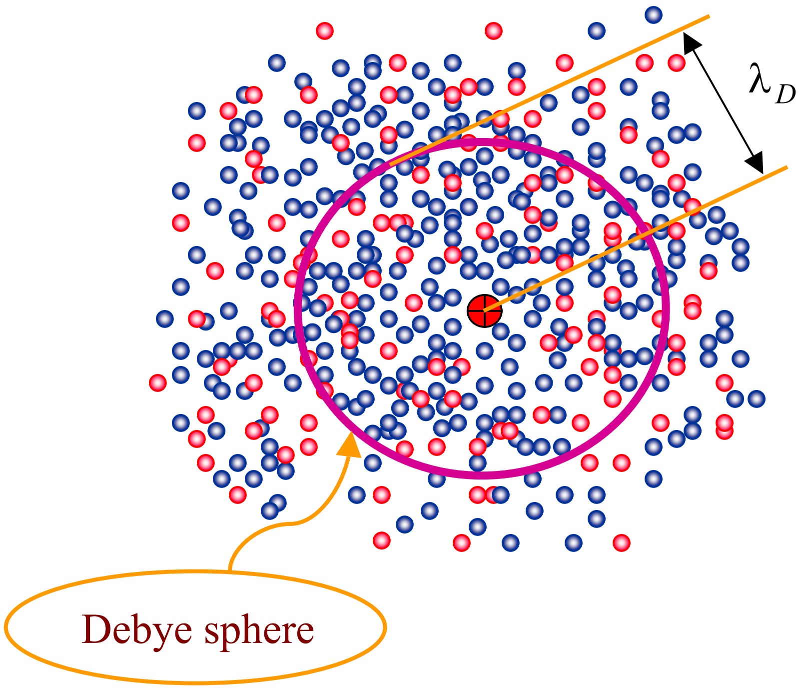
Plasma instabilities – interplay of particles and classical fields

Quantum Field Theory – no particles, no classical fields

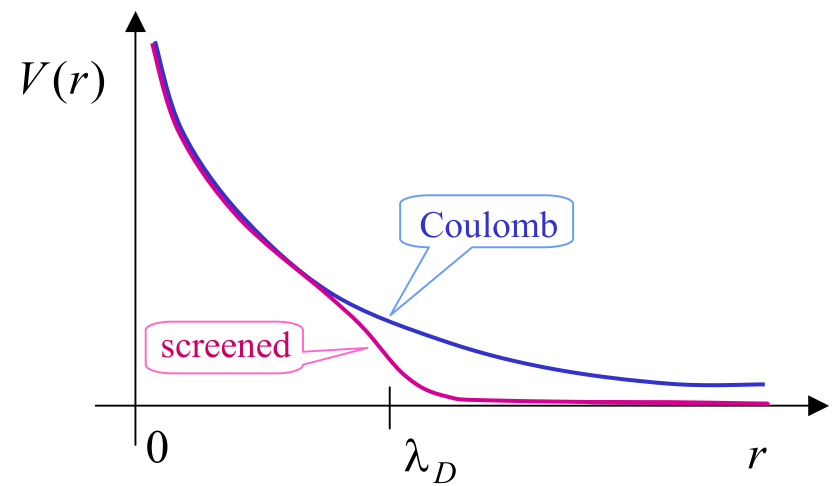
- particles – $\overbrace{\text{hard excitations, hard modes}}^{p_{\text{hard}} \sim T}$

- classical fields – $\underbrace{\text{highly populated}}_{\sim 1/g^2}$ $\underbrace{\text{soft excitations, soft modes}}_{p_{\text{soft}} \sim gT}$

Plasma manifests collective behavior



$$\lambda_D = \frac{1}{m_D} \sim \frac{1}{gT}$$ screening length $$V(r) \sim \frac{e^{-\frac{r}{\lambda_D}}}{r}$$



$$V_D = \frac{4}{3}\pi\lambda_D^3 \sim \frac{1}{g^3T^3}, \qquad n \sim T^3, \qquad n\,V_D \sim \frac{1}{g^3} >> 1 \quad \text{if} \quad g << 1$$

In a weakly coupled plasma, there are many particles in a Debye sphere!

Plasma instabilities

- **instabilities in configuration space – hydrodynamic instabilities**

- **instabilities in momentum space – kinetic instabilities**

instabilities due to non-equilibrium momentum distribution

$$f(\mathbf{p}) \text{ is not } \sim \exp\left(-\frac{E}{T}\right)$$

Kinetic instabilities

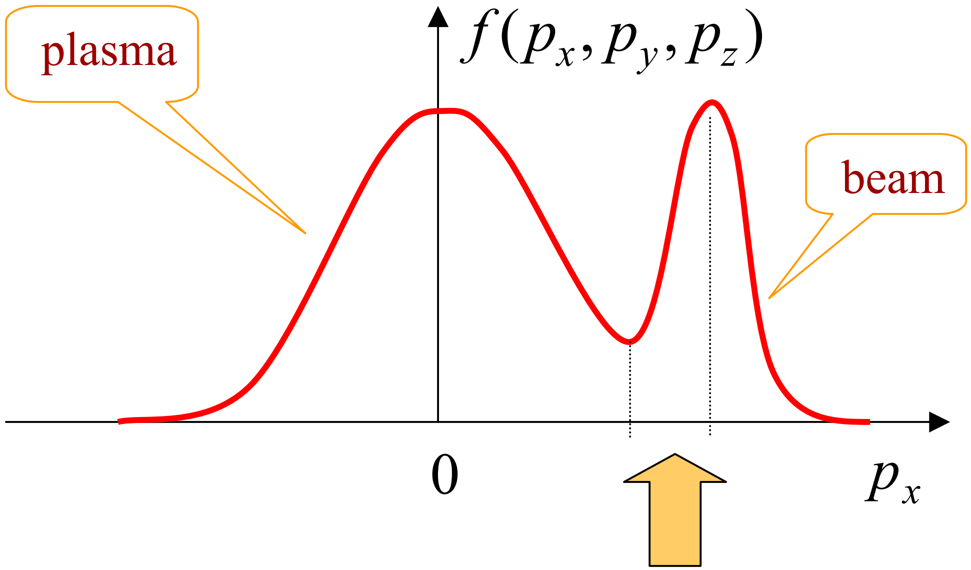
- **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}, \;\; \delta\rho \sim e^{-i(\omega t - \mathbf{kr})}$

- **transverse modes** – $\mathbf{k} \perp \mathbf{E}, \;\; \delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{kr})}$

E – electric field, **k** – wave vector, ρ – charge density, **j** - current

Logitudinal modes

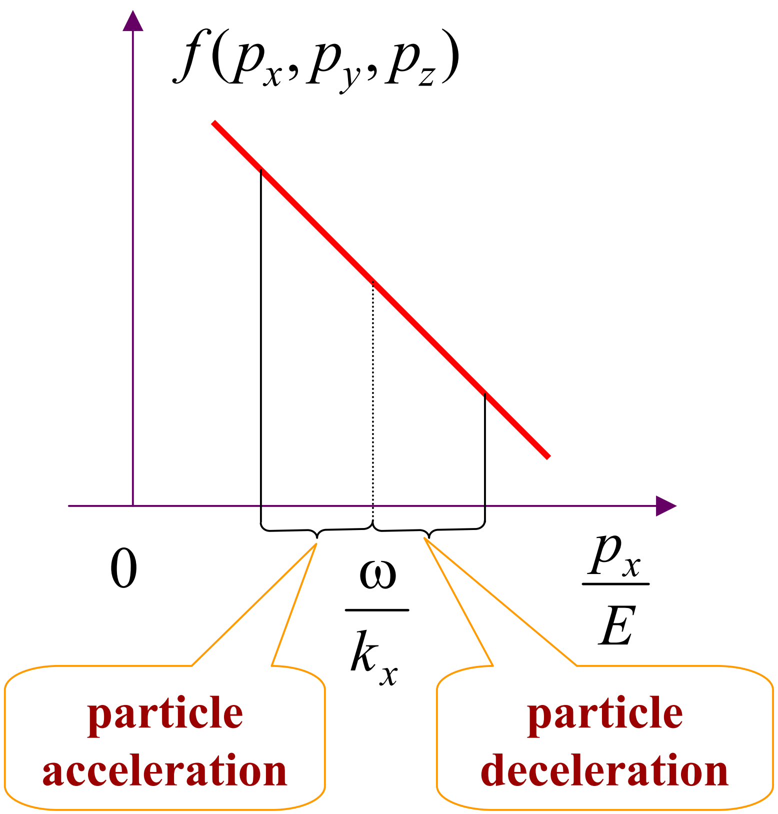
unstable configuration



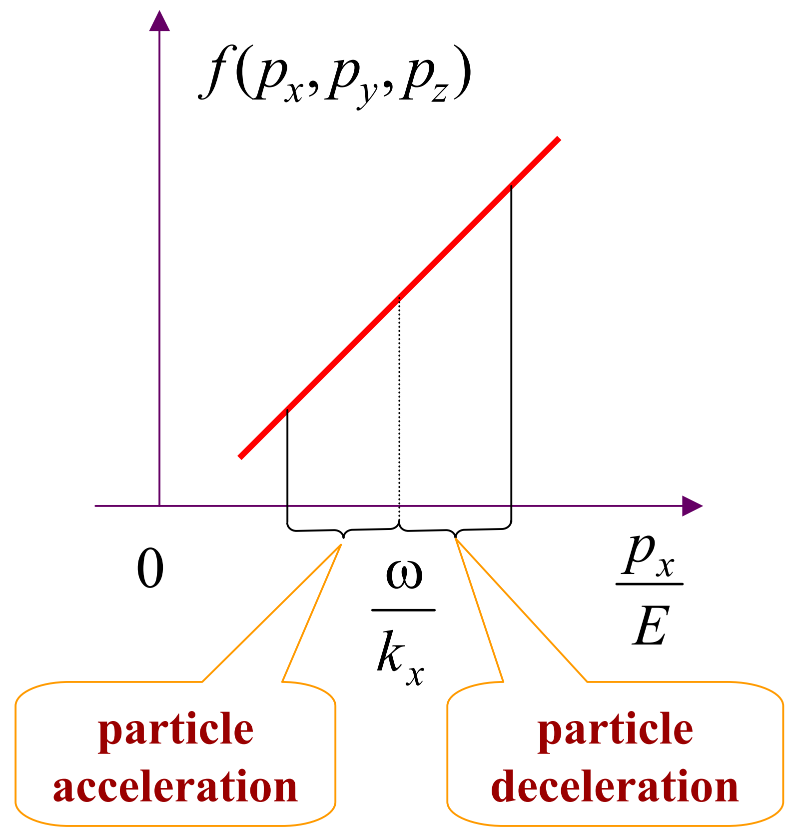
Energy is transferred from particles to fields

Logitudinal modes

Electric field decays - damping



Electric field grows - instability

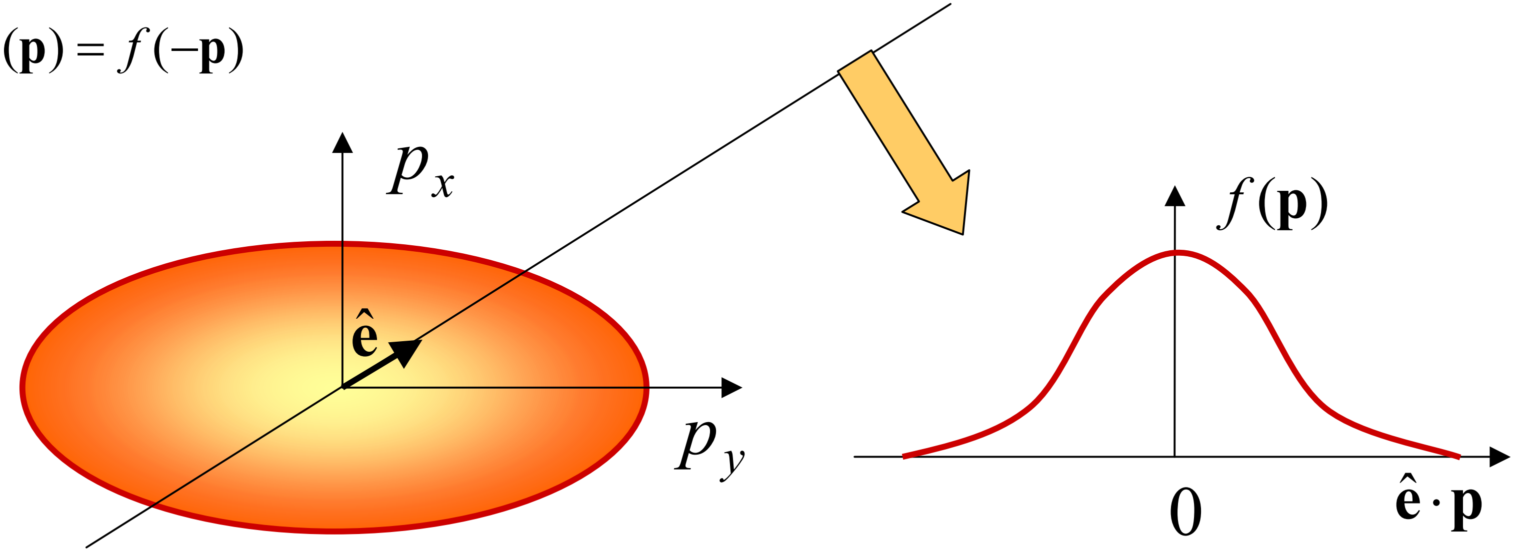


$\frac{\omega}{k_x}$ - phase velocity of the electric field wave, $\frac{p_x}{E}$ - particle's velocity

Transverse modes

Unstable modes occur due to anisotropy of the momentum distribution

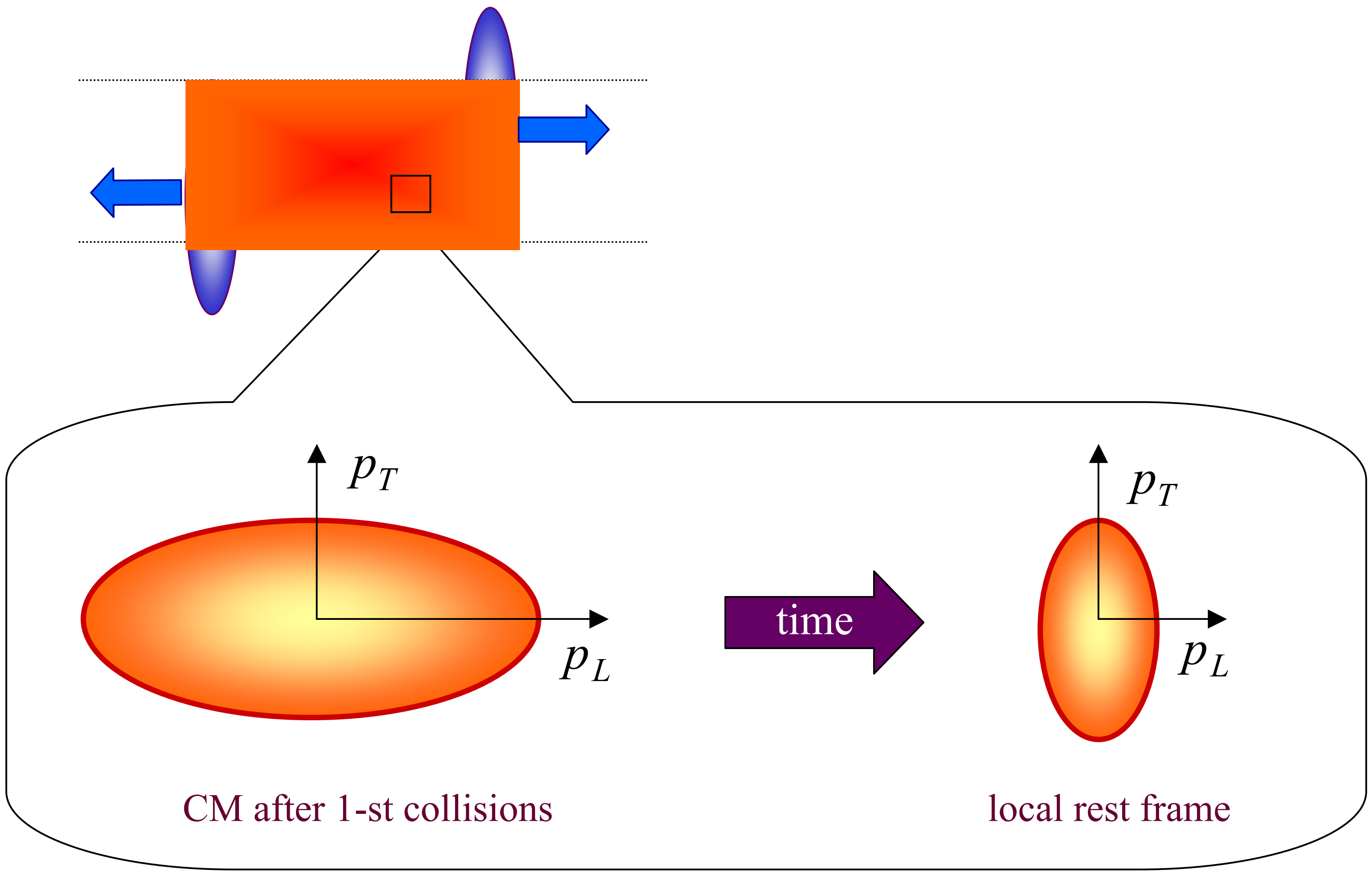
$f(\mathbf{p}) = f(-\mathbf{p})$



Momentum distribution distribution can monotonously decrease in every direction

Transverse modes are relevant for relativistic nuclear collisions!

Momentum Space Anisotropy in Nuclear Collisions

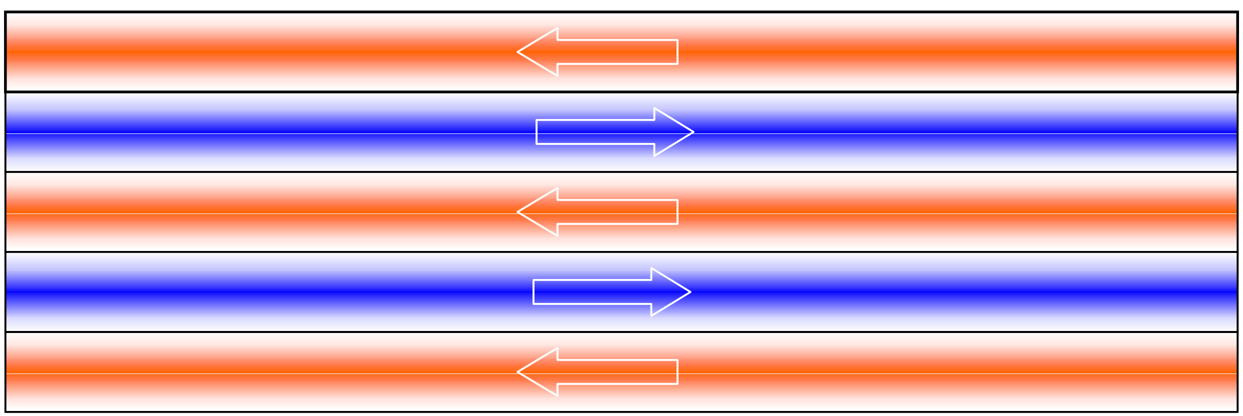


Seeds of instability

$\left\langle j_a^{\mu}(x) \right\rangle = 0$ **but current fluctuations are finite**

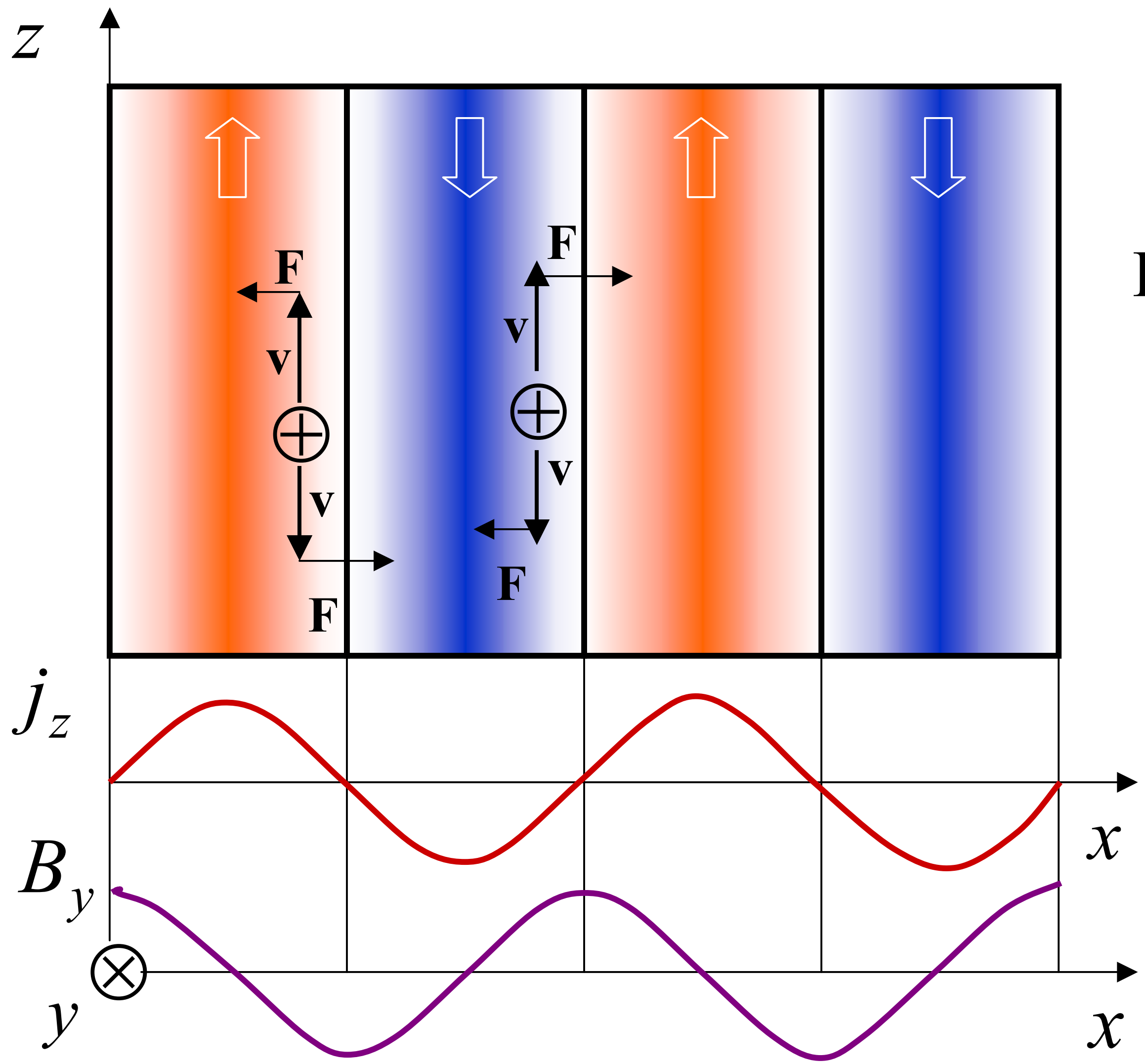
$$\boxed{\left\langle j_a^{\mu}(x_1)\, j_b^{\nu}(x_2) \right\rangle = \frac{1}{8}\delta^{ab} \int \frac{d^3p}{(2\pi)^3} \frac{p^{\mu} p^{\nu}}{E_p^2} f(\mathbf{p}) \delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0}$$

$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus

Mechanism of filamentation



Lorentz force

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

Ampere's law

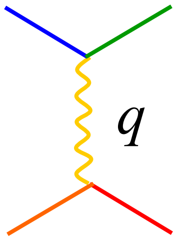
$$\nabla \times \mathbf{B} = \mathbf{j}$$

Instabilities vs. collisions

Time scale of parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

$$t_{\text{soft}} \sim \frac{1}{g^2 \ln(1/g)\, T}$$



hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Time scale of collective phenomena

$$t_{\text{collec}} \sim \frac{1}{g\,T}$$

$$g^2 << 1 \quad \Rightarrow \quad t_{\text{hard}} >> t_{\text{soft}} >> t_{\text{collec}}$$

The instabilities are fast!

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with Im$\omega > 0$ $\quad \Rightarrow \quad A^\mu(x) \sim \mathrm{e}^{\mathrm{Im}\,\omega t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. **How to get it?**

Transport theory

fundamental

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu Q\} = C$$ quarks

$$p_\mu D^\mu \overline{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu \overline{Q}\} = \overline{C}$$ antiquarks

adjoint

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{\mathcal{F}_{\mu\nu},(x)\partial_p^\nu G\} = C_g$$ gluons

free streaming

mean-field force

collisions

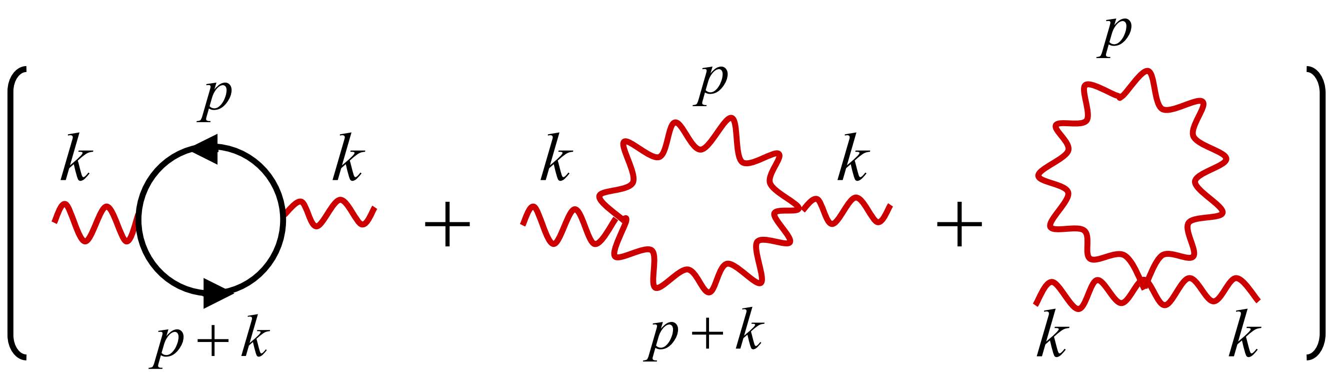
$$D^\mu \equiv \partial^\mu - ig[A^\mu, ...], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$\boxed{D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]}$$ mean-field generation

collisionless limit: $C = \overline{C} = C_g = 0$

Diagrammatic Hard Loop approach

$\Pi^{\mu\nu}(k) =$



Hard loop approximation: $k^\mu << p^\mu$

$$\boxed{\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\left[g^{\mu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\right]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

Chromo-hydrodynamic approach

$$D_{\mu} n^{\mu}(x) = 0$$

$$D_{\mu} T^{\mu\nu}(x) - \frac{g}{2}\left\{F^{\mu\nu}, n_{\mu}(x)\right\} = 0$$

Postulated form of $n^{\mu}(x)$ and $T^{\mu\nu}(x)$:

$$n^{\mu}(x) = n(x)\, u^{\mu}(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2}\left(\varepsilon(x) + p(x)\right)\left\{u^{\mu}(x), u^{\mu}(x)\right\} - p(x)\, g^{\mu\nu}$$

$n(x)$, $\varepsilon(x)$, $p(x)$, $u^{\mu}(x)$ matrices! $\quad u^{\mu}(x)\, u_{\mu}(x) = 1$

To close the system of equations: $\nabla p = 0$ or $\varepsilon = 3p \Leftarrow T^{\mu}_{\mu} = 0$

Dispersion equation

Dispersion equation

$$\boxed{\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0}$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$ **chromodielectric tensor**

$$k^\mu \equiv (\omega, \mathbf{k})$$

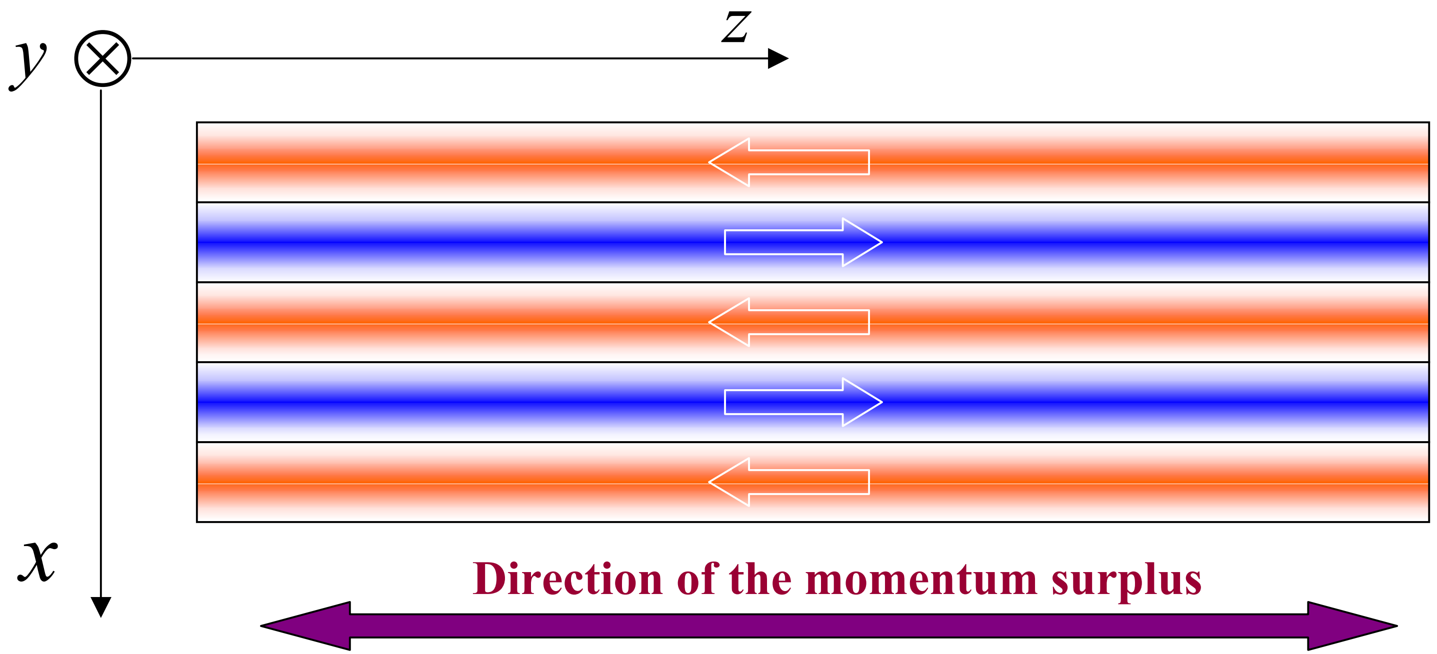
Dispersion equation

$$\boxed{\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0}$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int \frac{d^3 p}{(2\pi)^3} \frac{v^i}{\omega - \mathbf{kv} + i0^+} \frac{\partial f(\mathbf{p})}{\partial p^l}\left[\left(1 - \frac{\mathbf{kv}}{\omega}\right)\delta^{lj} + \frac{k^l v^j}{\omega}\right]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

Dispersion equation – configuration of interest



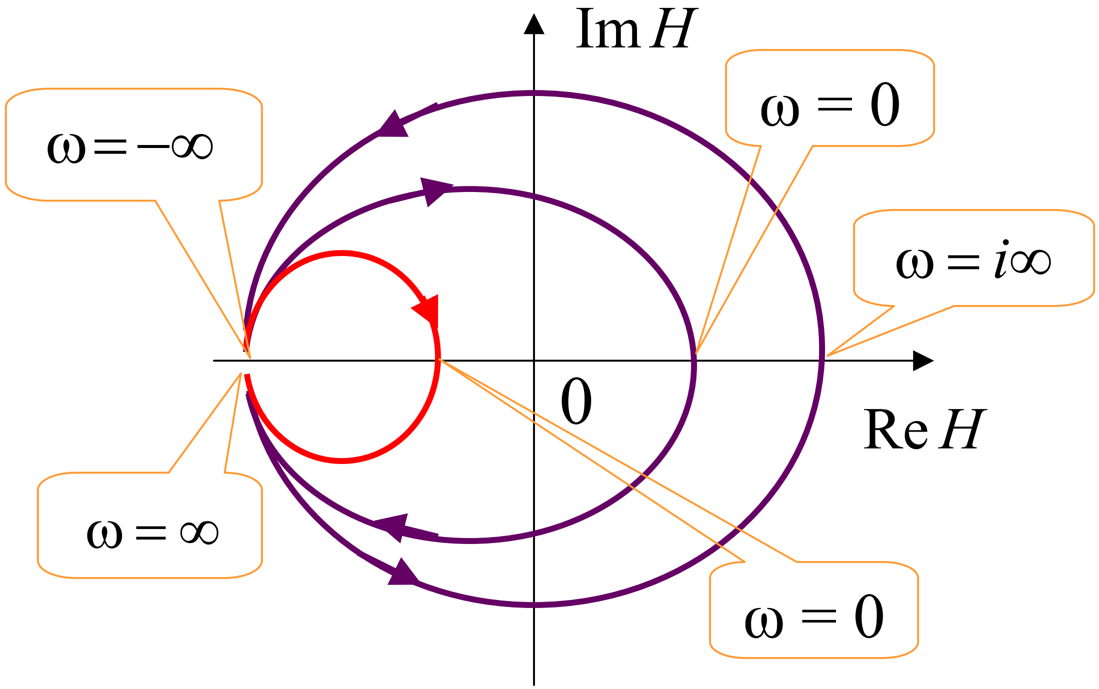
$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$

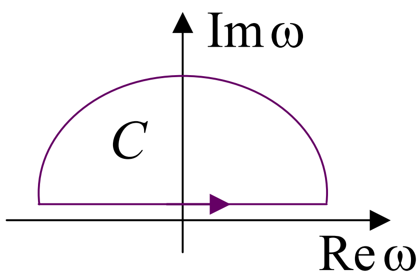
Dispersion equation $$\boxed{k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0}$$

Existence of unstable modes – Penrose criterion

$$\boxed{H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)}$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \dfrac{d\omega}{2\pi i} \dfrac{d \ln H(\omega)}{d\omega} = \ln H(\omega)\Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$





There are unstable modes if

$$\boxed{H(\omega = 0) < 0}$$

Anisotropy!

Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho\sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2 + \sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

$$\rho = 6\ \mathrm{fm}^{-3}$$

$$\alpha_s = g^2/4\pi = 0.3$$

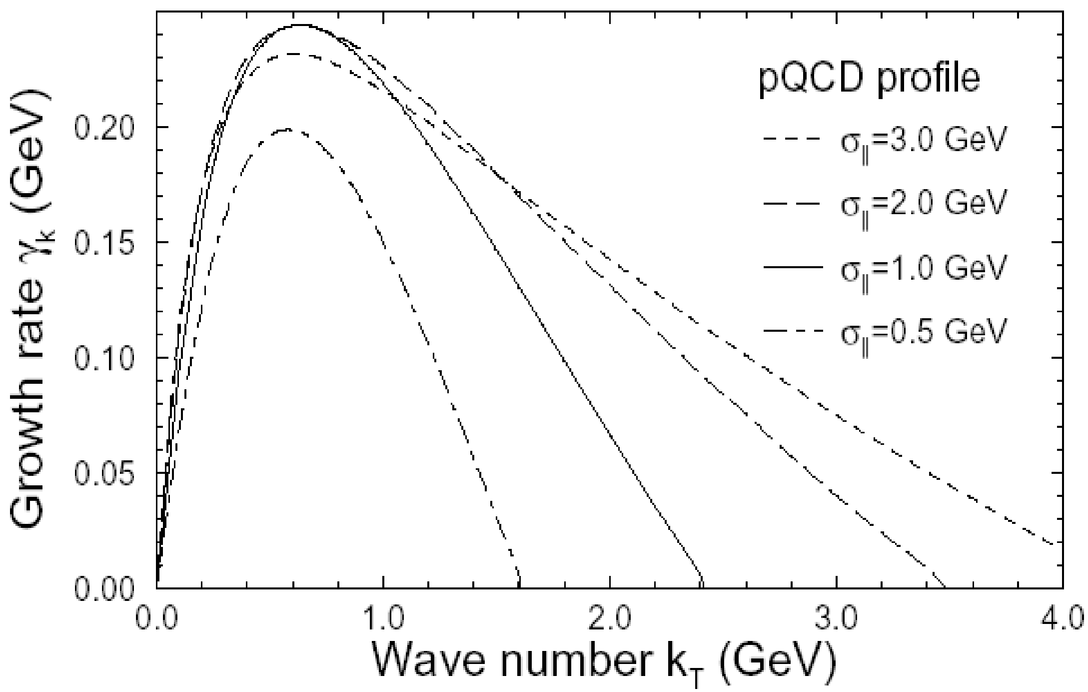
$$\sigma_\perp = 0.3\ \mathrm{GeV}$$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i\gamma_k$$

$$0 < \gamma_k \in \Re$$



J. Randrup & St. M., Phys. Rev. C **68**, 034909 (2003)

Hard-Loop dynamics

Soft fields in the passive background of hard particles

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\mathrm{eff}} = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\Big[f(\mathbf{p}) F^a_{\mu\nu}(x) \left(\frac{p^\nu p^\rho}{(p\cdot D)^2}\right)_{ab} F^{b\mu}_\rho(x)$$

$$+ i\,\frac{C_F}{3}\,\tilde{f}(\mathbf{p})\,\bar{\psi}(x)\,\frac{p\cdot\gamma}{p\cdot D}\,\psi(x)\Big]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\mu \Lambda^\mu(p,q,k) = \Sigma(p) + \Sigma(q)$$

St. M., A. Rebhan & M. Strickland, Phys. Rev. D **74**, 025004 (2004)

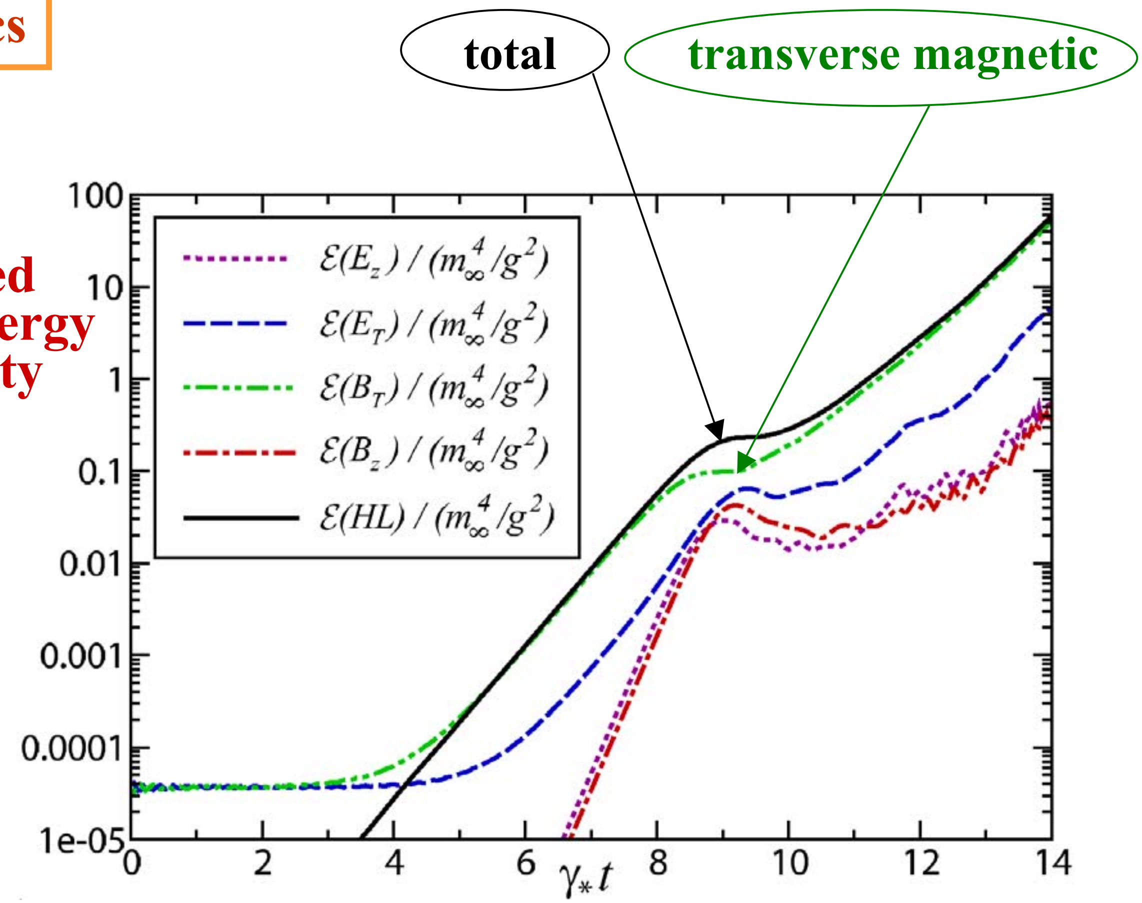
Growth of instabilities – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

1+1 dimensions

$$A_a^\mu = A_a^\mu(t, z)$$

Scaled field energy density



Anisotropic particle's momentum distribution

$$f(\mathbf{p}) = f_{\text{iso}}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi}\int_0^\infty dp\, p^2 \frac{df_{\text{iso}}(p)}{dp}$$

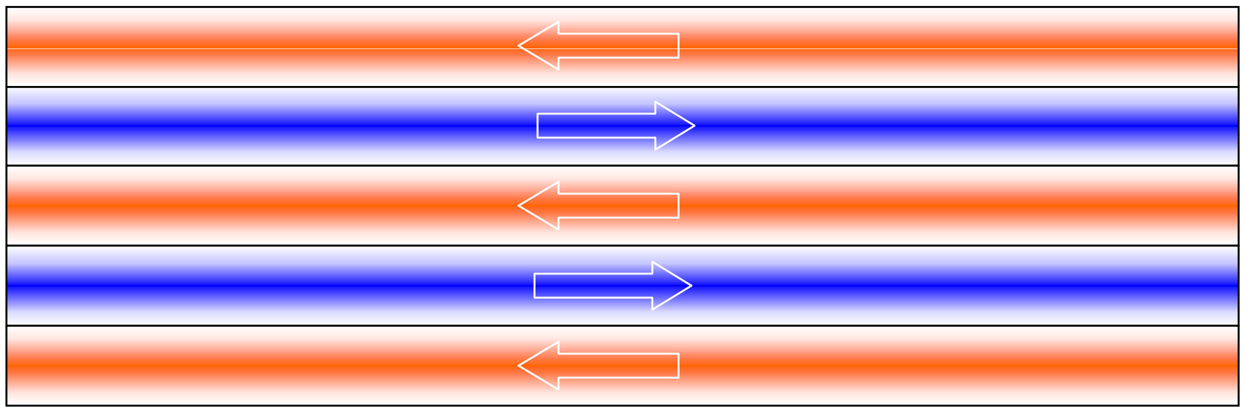
$$(m_D, \zeta)$$

Strong anisotropy $\zeta = 10$

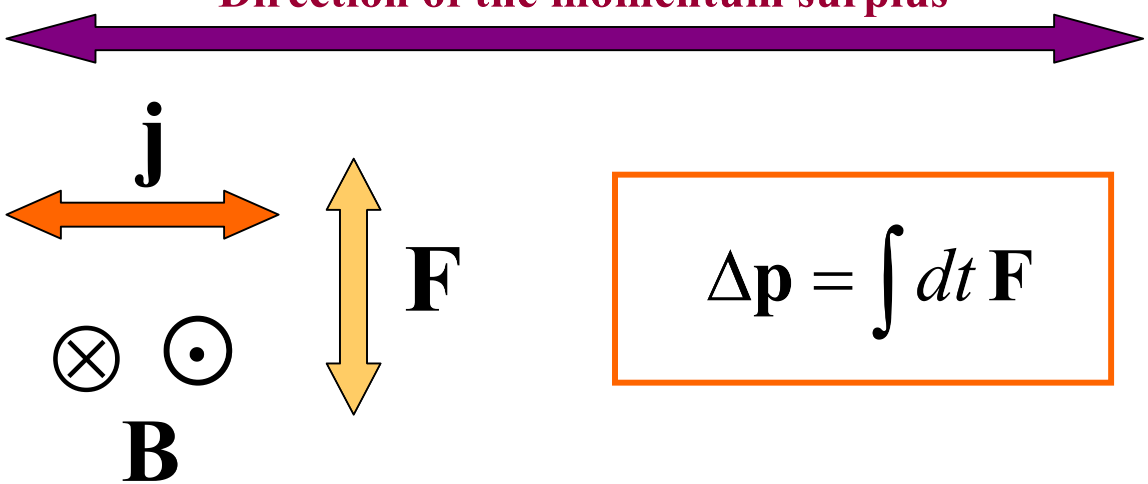
γ_* - maximal growth rate

A. Rebhan, P. Romatschke & M. Strickland, Phys. Rev. Lett. **94**, 102303 (2005)

Isotropization - particles

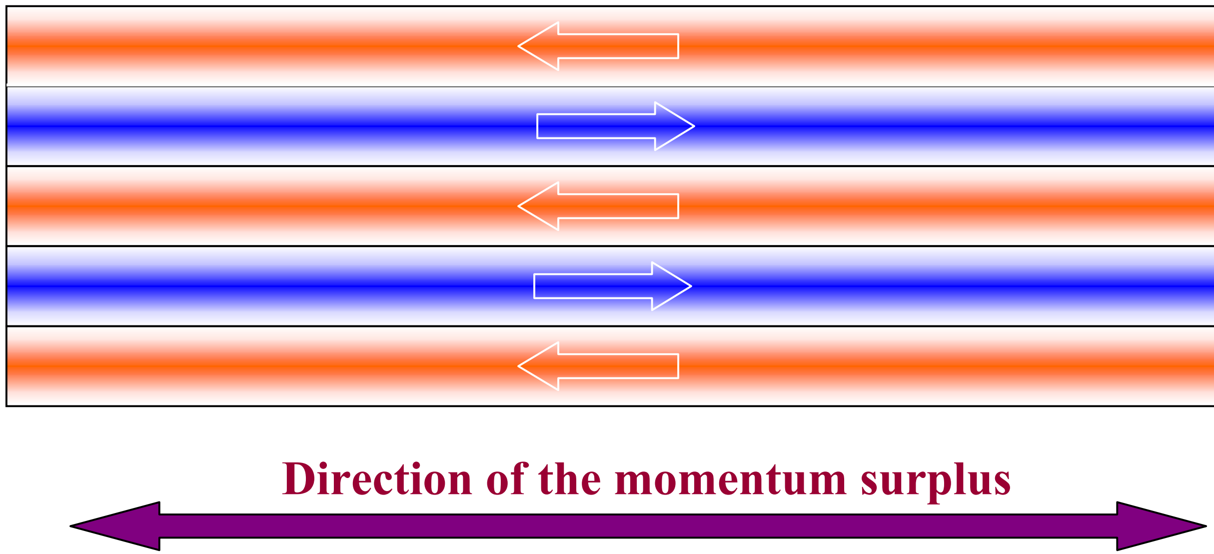


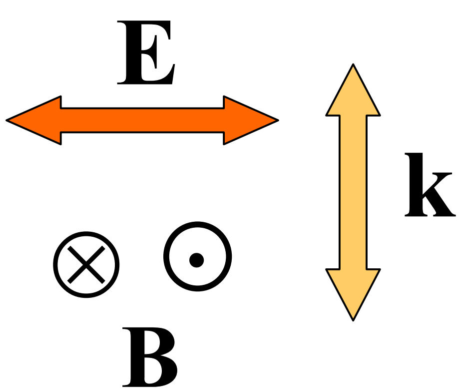
Direction of the momentum surplus



$$\boxed{\Delta \mathbf{p} = \int dt\, \mathbf{F}}$$

Isotropization - fields





$$\boxed{\mathbf{P}_{\text{fields}} \sim \mathbf{B}^{a} \times \mathbf{E}^{a} \sim \mathbf{k}}$$

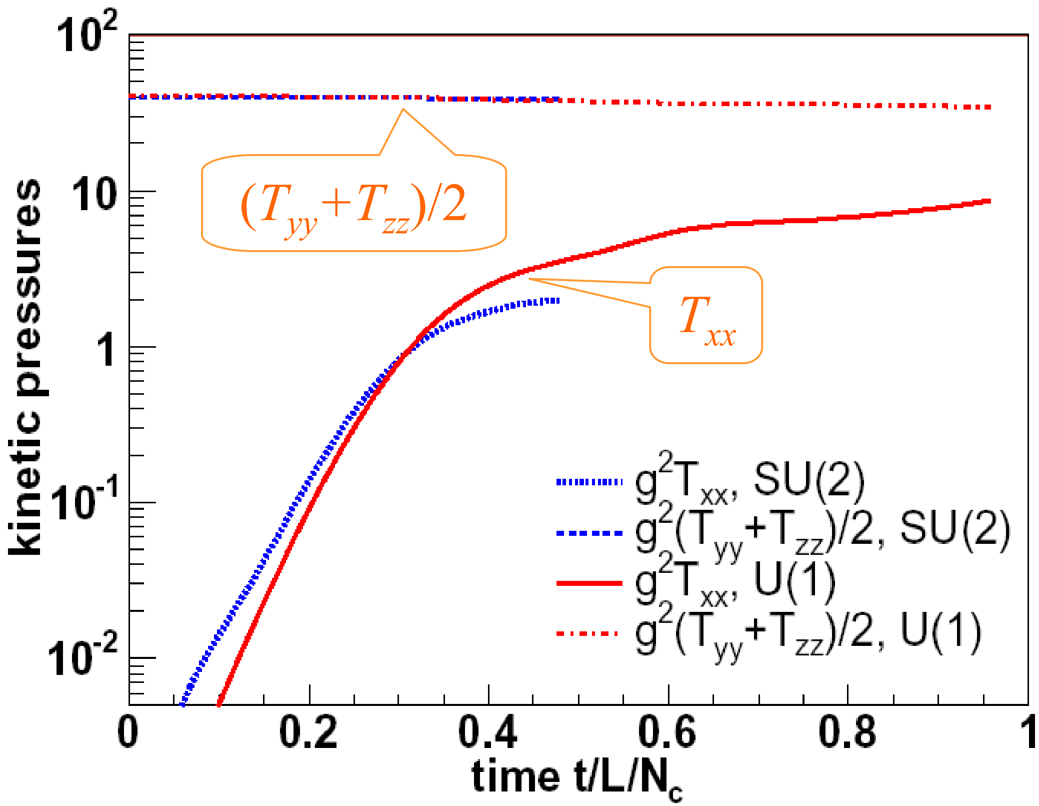
Isotropization – numerical simulation

Classical system of colored particles & fields

$$T_{ij} = \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

$$T_{xx} = (T_{yy} + T_{zz})/2$$



A. Dumitru & Y. Nara, Phys. Lett. **B621**, 89 (2005).

Conclusion

The scenario of instabilities driven equilibration provides a plausible solution of the fast equilibration problem