Wigner functional of fermion fields

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Motivation – Wigner function

1932 – Eugene Wigner

$$\varphi(t,x) \quad \rightarrow \quad W(t,x,p)$$

wave function — Wigner function

$$W(t,x,p) \equiv \int du\, e^{-ipu} \varphi^*(t, x+u/2)\, \varphi(t, x-u/2)$$

- $W(t,x,p) \in \mathbf{R}$
- $\int dx\, |\varphi(t,x)|^2 = 1 \quad \Rightarrow \quad \int dx \int \frac{dp}{2\pi} W(t,x,p) = 1$
- $W(t,x,p) \gtrless 0$

Motivation – bosonic Wigner functional

1994 – St. Mrówczyński & B. Müller

$$\Phi(t,x) \quad \rightarrow \quad W[t,\Phi(x),\Pi(x)]$$

scalar quantum field — Wigner functional

$$W(t,x,p) \equiv \int du\, e^{-ipu} \langle x+u/2 | \hat{\rho}(t) | x-u/2 \rangle$$

$|x\rangle$ – position eigenstate

$$W[t,\Phi(x),\Pi(x)] \equiv \int D\varphi(x)\, e^{-i\int dx \Pi(x)\varphi(x)} \langle \Phi(x)+\varphi(x)/2 | \hat{\rho}(t) | \Phi(x)-\varphi(x)/2 \rangle$$

$\Phi(x)$ – time-independent scalar field in Schrödinger picture

Motivation – bosonic Wigner functional

$$\left\langle O(\hat{\Phi},\hat{\Pi})\right\rangle = \left\langle O(\Phi,\Pi)\right\rangle$$

- $$\left\langle O\left(\hat{\Phi},\hat{\Pi}\right)\right\rangle \equiv \frac{1}{\mathrm{Tr}[\hat{\rho}]}\mathrm{Tr}\left[\hat{\rho}\, O\left(\hat{\Phi},\hat{\Pi}\right)\right]$$

- $$\left\langle O\left(\Phi,\Pi\right)\right\rangle \equiv \frac{1}{Z}\int D\Phi\int\frac{D\Pi}{2\pi}\, O\left(\Phi,\Pi\right)W[t,\Phi,\Pi]$$

$$Z \equiv \int D\Phi\int\frac{D\Pi}{2\pi}\; W[t,\Phi,\Pi]$$

$W[t,\Phi,\Pi]$ – density in phase-space spanned by $\Phi \,\&\, \Pi$

Motivation – bosonic Wigner functional

$$i\frac{\partial}{\partial t}\hat{\rho}(t) = \left[\hat{H}, \hat{\rho}(t)\right]$$

Semiclassical equation of motion

$$\left[\frac{\partial}{\partial t} + \int dx \left(\frac{\delta H}{\delta \Pi(x)} \frac{\delta}{\delta \Phi(x)} - \frac{\delta H}{\delta \Phi(x)} \frac{\delta}{\delta \Pi(x)} \right)\right] W[t, \Phi, \Pi] = 0$$

Liouville's equation

Motivation – bosonic Wigner functional

Free field in thermal equilibrium

$$\hat{\rho} = e^{-\beta \hat{H}}$$

$$\Downarrow$$

classical limit

$$W[\Phi,\Pi] \sim \exp\left[-\frac{\beta}{2} \int \frac{dp}{2\pi} \Delta(p) \left(\Pi^*(p)\Pi(p) + (p^2 + m^2)\Phi^*(p)\Phi(p) \right) \right] \approx \exp[-\beta H]$$

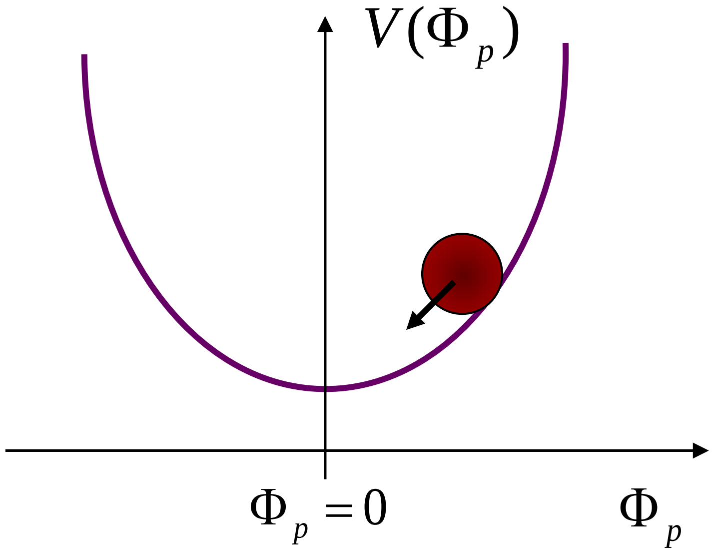
$$\Delta(p) \equiv \frac{2T}{E_p} \tanh\left(\frac{E_p}{2T} \right), \qquad E_p \equiv \sqrt{p^2 + m^2}, \qquad \beta \equiv \frac{1}{T}$$

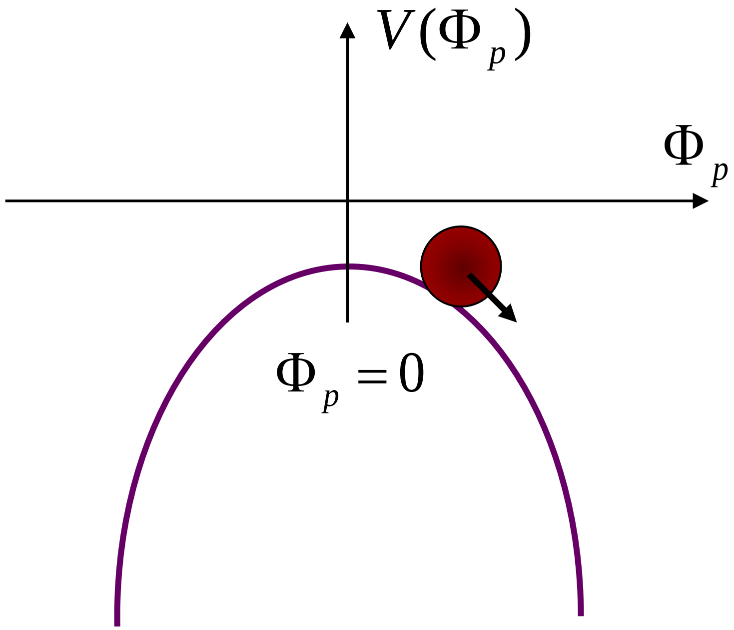
Motivation – bosonic Wigner functional

Simple application – *rollover* phase transition

$$V(\Phi) = -\frac{1}{2} m^2(t)\, \Phi^2(x), \qquad m^2(t) = \begin{cases} m^2 & t < 0 \\ -\mu^2 & t \geq 0 \end{cases}$$

$p^2 < \mu^2$





Definition of fermionic Wigner functional

$$W[t,\Psi(x),\Pi(x)] \equiv \int D\varphi(x)\, e^{-i\int dx\,\Pi(x)\varphi(x)} \left\langle \Psi(x)+\varphi(x)/2 \middle| \hat{\rho}(t) \middle| \Psi(x)-\varphi(x)/2 \right\rangle$$

Does it make any sense?

$\Psi(x),\ \Pi(x),\ \varphi(x)$ - Grassmann (anticommuting) fields

$$\Psi(x) \quad \rightarrow \quad \{\Psi_1, \Psi_2, \Psi_3, \ldots\} \qquad \Psi_i \equiv \Psi(x_i)$$

- $\Psi_i\Psi_j + \Psi_j\Psi_i = 0$ $\qquad \Psi_i^2 = 0$
- Berezin integrals: $\int d\Psi_i = 0, \qquad \int d\Psi_i \Psi_j = \delta^{ij}$

Simple example

Harmonic oscillator in a ground state

$$\langle x_1|\hat{\rho}(t)|x_2\rangle \sim \exp\left[-\frac{1}{2}\left(a\,x_1^2 + a\,x_2^2\right)\right] \Rightarrow W(t,x,p) \sim \exp\left[-a\,x^2 - \frac{p^2}{a}\right]$$

$$\langle \Psi_1(x)|\hat{\rho}(t)|\Psi_2(x)\rangle = \exp\left[-\frac{1}{2}\int dx\,\left(\Psi_1(x)\mathbf{A}\,\Psi_1(x) + \Psi_2(x)\mathbf{A}\,\Psi_2(x)\right)\right]$$

$\Psi_i = \begin{pmatrix} \chi_i \\ \psi_i \end{pmatrix}$ - Weyl spinor

$\mathbf{A}$ - antisymmetric 2×2 matrix

$$\mathbf{A}^T = -\mathbf{A}$$

$$W[t,\Psi(x),\Pi(x)] = \exp\left[-\int dx\,\left(\Psi(x)\mathbf{A}\,\Psi(x) - \Pi(x)\mathbf{A}^{-1}\,\Pi(x)\right)\right]$$

Fermionic Wigner functional

$$\left\langle O(\hat{\Psi},\hat{\Pi})\right\rangle = \left\langle O(\Psi,\Pi)\right\rangle$$

- $$\left\langle O\left(\hat{\Psi},\hat{\Pi}\right)\right\rangle \equiv \frac{1}{Z}\mathrm{Tr}\left[\hat{\rho}\, O\left(\hat{\Psi},\hat{\Pi}\right)\right]$$

- $$\left\langle O\left(\Psi,\Pi\right)\right\rangle \equiv \frac{1}{Z}\int D\Psi \int \frac{D\Pi}{2\pi} O\left(\Psi,\Pi\right) W[t,\Psi,\Pi]$$

$$Z \equiv \mathrm{Tr}[\hat{\rho}] = \int D\Psi \int \frac{D\Pi}{2\pi} W[t,\Psi,\Pi]$$

$W[t,\Psi,\Pi]$ - density in phase-space spanned by Ψ & Π

Example of computation

$$Z = \int D\Psi \int \frac{D\Pi}{2\pi} W[t,\Psi,\Pi] = \int D\Psi \int \frac{D\Pi}{2\pi} \int D\varphi(x)\, e^{-i\int dx\,\Pi(x)\varphi(x)} \left\langle \Psi(x)+\varphi(x)/2 \right| \hat{\rho}(t) \left| \Psi(x)-\varphi(x)/2 \right\rangle$$

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx\,\Pi(x)\varphi(x)} = \quad ?$$

Discretization

$$\begin{cases} \varphi(x) \to \{\varphi_1,\varphi_2,\varphi_3,\ldots\varphi_n\}, & \varphi_i \equiv \varphi(x_i) \\ \Pi(x) \to \{\Pi_1,\Pi_2,\Pi_3,\ldots\Pi_n\}, & \Pi_i \equiv \Pi(x_i) \end{cases}$$

Berezin integrals: $\int d\Pi_i = 0, \quad \int d\Pi_i \Pi_j = \delta^{ij}$

$$\int \frac{D\Pi_n}{2\pi} \frac{D\Pi_{n-1}}{2\pi} \cdots \frac{D\Pi_1}{2\pi} e^{-i\Delta x \sum_{j=1}^{n} \Pi_j \varphi_j} = (-i\Delta x)^n \int \frac{D\Pi_n}{2\pi} \frac{D\Pi_{n-1}}{2\pi} \cdots \frac{D\Pi_1}{2\pi} \left(\sum_{j=1}^{n} \Pi_j \varphi_j \right)^n$$

Example of computation cont.

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx\,\Pi(x)\varphi(x)} = (-i\Delta x)^n \int \frac{D\Pi_n}{2\pi}\frac{D\Pi_{n-1}}{2\pi}\dots\frac{D\Pi_1}{2\pi}\left(\sum_{j=1}^{n}\Pi_j\varphi_j\right)^n = -\left(-\frac{i\Delta x}{2\pi}\right)^n \varphi_1\varphi_2\dots\varphi_n$$

$$\int d\varphi_n d\varphi_{n-1}\dots d\varphi_1 \overbrace{\varphi_1\varphi_2\dots\varphi_n}^{\delta(\varphi_1,\varphi_2,\dots\varphi_n)} f(\varphi_1,\varphi_2,\dots\varphi_n) = f_0$$

function in n-dimesional Grassmann algebra spanned by $\{1,\varphi_i,\varphi_i\varphi_j,\dots\underbrace{\varphi_i\varphi_j\dots\varphi_k}_{n}\}$

$$f(\varphi_1,\varphi_2,\dots\varphi_n) = f_0 + \sum_{i=1}^{n} f_1(i)\varphi_i + \sum_{i,j=1}^{n} f_2(i,j)\varphi_i\varphi_j + \dots$$

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx\,\Pi(x)\varphi(x)} \propto \delta[\varphi(x)]$$

Example of computation cont.

$$Z = \int D\Psi \int \frac{D\Pi}{2\pi} \int D\varphi(x)\, e^{-i\int dx\,\Pi(x)\varphi(x)} \left\langle \Psi(x)+\varphi(x)/2 \middle| \hat{\rho}(t) \middle| \Psi(x)-\varphi(x)/2 \right\rangle$$

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx\,\Pi(x)\varphi(x)} \propto \delta[\varphi(x)]$$

$$Z = \int D\Psi \left\langle \Psi(x) \middle| \hat{\rho}(t) \middle| \Psi(x) \right\rangle = \mathrm{Tr}[\hat{\rho}]$$

$$\int D\Psi \int \frac{D\Pi}{2\pi} W[t,\Psi,\Pi] = \mathrm{Tr}[\hat{\rho}]$$

Fermionic Wigner functional

$$i\frac{\partial}{\partial t}\hat{\rho}(t)=\left[\hat{H},\hat{\rho}(t)\right]$$

$$\Downarrow$$

Equation of motion

$$\left[\frac{\partial}{\partial t}+\int dx\left(\frac{\delta H}{\delta\Pi(x)}\frac{\delta}{\delta\Psi(x)}-\frac{\delta H}{\delta\Psi(x)}\frac{\delta}{\delta\Pi(x)}\right)\right]W[t,\Psi,\Pi]=0$$

Exact equation of motion is of the Liouville form

Conclusions

Although bosonic and fermionic fields are very different,
the Wigner functionals have very similar properties.

For more: St. Mrówczyński, Physical Review D **87**, 065026 (2013)