# Isotropization vs. Equilibration & Azimuthal Fluctuations

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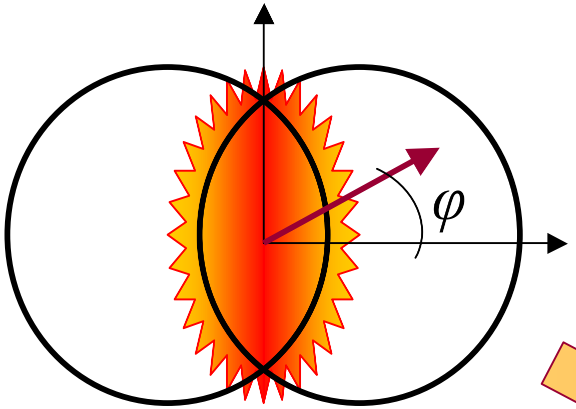
- elliptic flow & fast equilibration
- instabilities driven isotropization
- equilibration vs. isotropization
- azimuthal fluctuations & preequilibrium

# Evidence of equilibration at the early stage

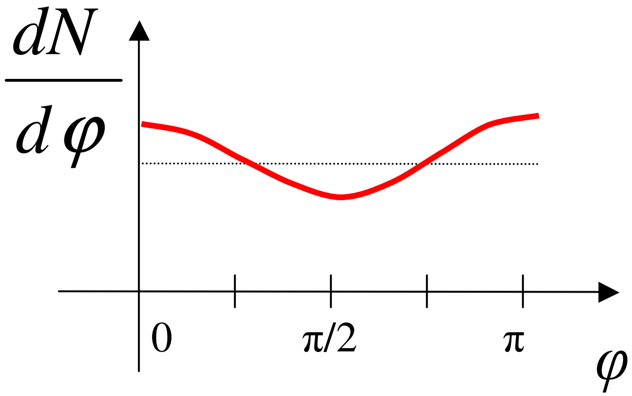
## Success of hydrodynamic models in describing elliptic flow



**Hydrodynamics**

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\nabla\right)\mathbf{v}=-\frac{\nabla p}{\rho}$$

**Hydrodynamic requires local thermodynamical equilibrium!**



# Equilibration is fast

$$\boxed{v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle}$$

**Eccentricity decays due to the free streaming!**

$$\boxed{\varepsilon \searrow \;\Rightarrow\; v_2 \searrow} \quad \Rightarrow \quad \boxed{t_{\mathrm{eq}} \leq 0.6 \;\; \mathrm{fm}/c}$$

time of equilibration

U. Heinz and P. Kolb, Nucl. Phys. **A702**, 269 (2002)

# Collisions are too slow

Time scale of hard parton-parton scattering

hard scattering ~ momentum transfer of order of $T$

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g)\, T}$$

either single hard scattering or multiple soft scatterings

$$\boxed{t_{\text{eq}} \approx t_{\text{hard}} \geq 2.6 \ \text{fm}/c}$$

R. Baier, A.H. Mueller, D. Schiff and D.T. Son, Phys. Lett. **B539**, 46 (2002)

# Scenarios of fast equilibration

- **Production mechanism of particles obeys equilibrium momentum distributions – instantaneous equilibration**

**Schwinger mechanism:**

$$\boxed{\frac{d^2 n}{dp_T^2} \sim e^{-\frac{2\pi m_T^2}{eE}}}$$

A. Białas, Phys.Lett. **B466**, 301 (1999)
W. Florkowski, Acta Phys. Pol. **B35**, 799 (2004)
D. Kharzeev & K. Tuchin, hep-ph/0501234

- **Equilibration is fast because quark-qluon plasma is strongly coupled**

**sQGP**

E.V. Shuryak, J. Phys.**G30**, S1221 (2004)
E.V. Shuryak & I. Zahed, Phys. Rev. **C70**, 021901 (2004)

- **Instabilities drive equilibration** **- as in the EM plasma**

# Instabilities driven equilibration

## The most important contributions

St. Mrówczyński,
*Color Collective Effects At The Early Stage Of Ultrarelativistic Heavy Ion Collisions,*
Phys. Rev. C **49**, 2191 (1994).

St. Mrówczyński,
*Color filamentation in ultrarelativistic heavy-ion collisions,*
Phys. Lett. B **393**, 26 (1997).

P. Romatschke and M. Strickland,
*Collective modes of an anisotropic quark gluon plasma,*
Phys. Rev. D **68**, 036004 (2003)

P. Arnold, J. Lenaghan and G.D. Moore,
*QCD plasma instabilities and bottom-up thermalization,*
JHEP **0308**, 002 (2003)

**Unstable Mode Analysis**

**Numerical Simulations**

A. Rebhan, P. Romatschke and M. Strickland,
*Hard-loop dynamics of non-Abelian plasma instabilities,*
Phys. Rev. Lett. **94**, 102303 (2005)

A. Dumitru and Y. Nara,
*QCD plasma instabilities and isotropization,*
arXiv:hep-ph/0503121.

# Instabilities driven equilibration

## The most important contributions cont.

St. Mrówczyński and M. Thoma,
*Hard Loop Approach to Anisotropic Systems,*
Phys. Rev. D **62**, 036011 (2000)

P. Arnold and J. Lenaghan,
*The abelianization of QCD plasma instabilities,*
Phys. Rev. D **70**, 114007 (2004)

St. Mrówczyński, A. Rebhan and M. Strickland,
*Hard-loop effective action for anisotropic plasmas,*
Phys. Rev. D **70**, 025004 (2004)

**Effective Action**

**Heavy-Ion Phenomenology**

J. Randrup and St. Mrówczyński,
*Chromodynamic Weibel instabilities in relativistic nuclear collisions,*
Phys. Rev. C **68**, 034909 (2003)

P. Arnold, J. Lenaghan, G.D. Moore and L.G. Yaffe,
*Apparent thermalization due to plasma instabilities in quark gluon plasma,*
Phys. Rev. Lett. **94**, 072302 (2005)

# Instabilities

**stationary state**

$$\boxed{A(t) = A_0 + \delta A(t)}$$

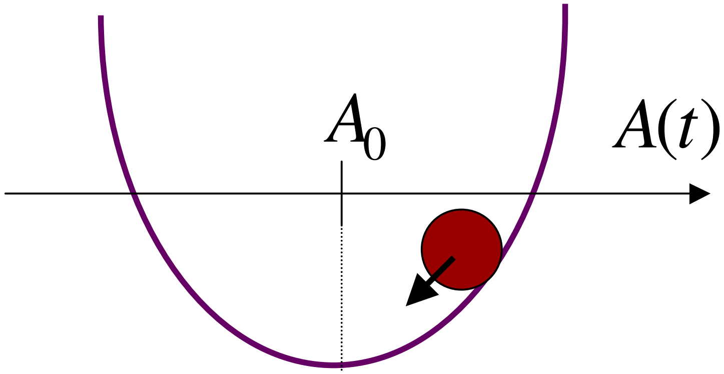
**fluctuation**

**Instability**

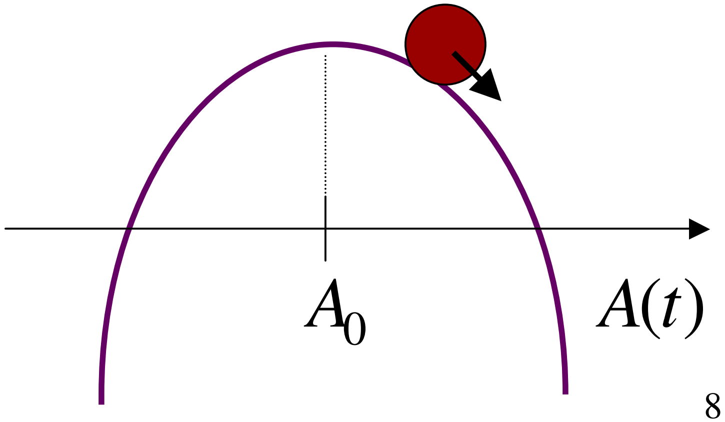
$$\boxed{\delta A(t) \propto e^{\gamma t}}$$

$$\gamma > 0$$

**stable configuration**



**unstable configuration**



# Plasma instabilities

- **instabilities in configuration space – hydrodynamic instabilities**
- **instabilities in momentum space – kinetic instabilities**

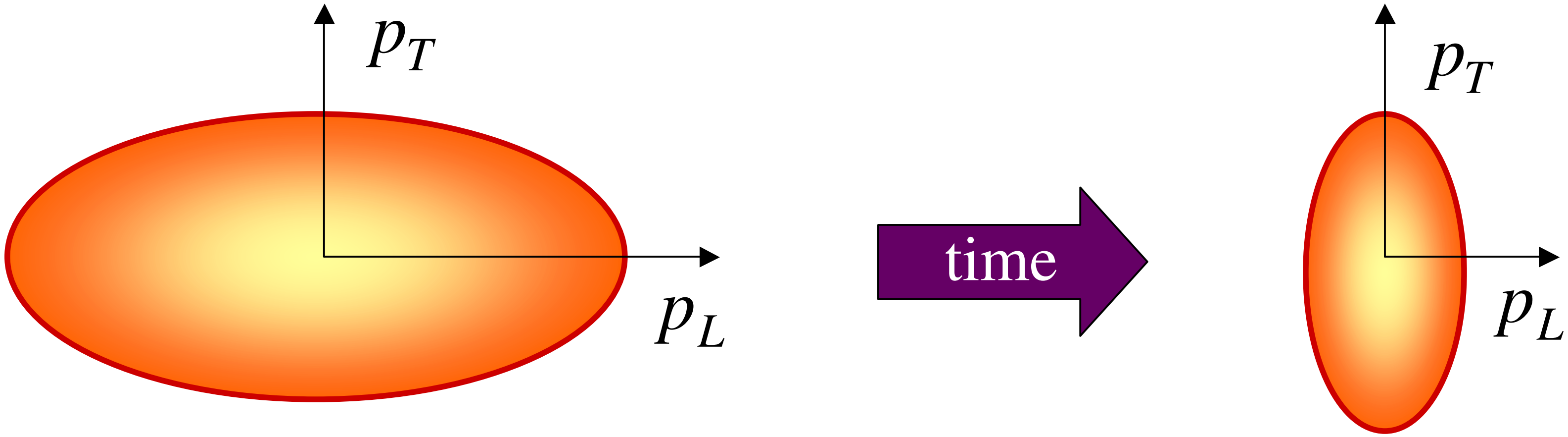
Instabilities due to non-equilibrium momentum distribution

- **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}, \ \delta\rho \sim e^{-i(\omega t - \mathbf{kr})}$
- **transverse modes** – $\mathbf{k} \perp \mathbf{E}, \ \delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{kr})}$

**E** – electric field, **k** – wave vector, $\rho$ – charge density, **j** - current

# Momentum Space Anisotropy in Nuclear Collisions

**Parton momentum distribution is initially strongly anisotropic**



CM after 1-st collisions

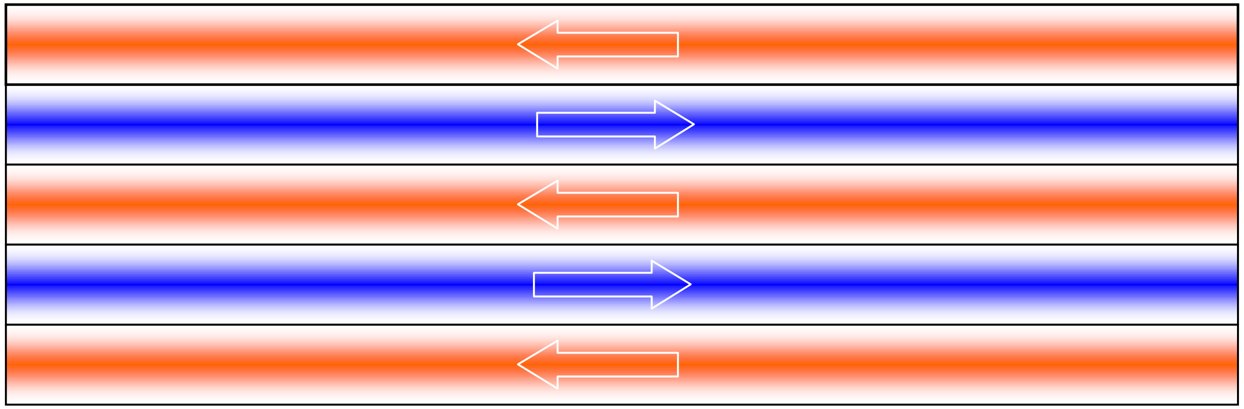
local rest frame

# Seeds of instability

**Current fluctuations**

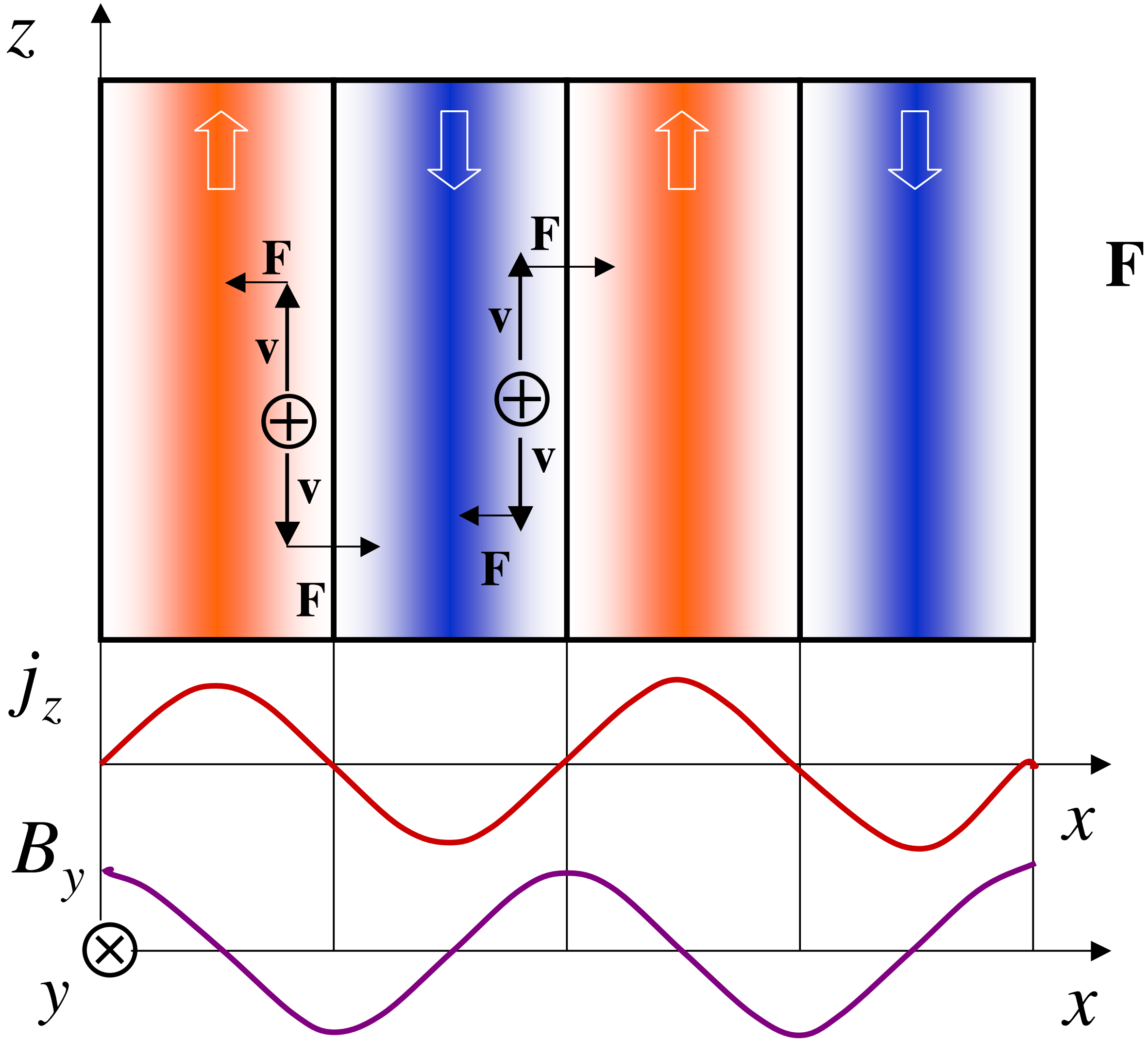
$$\left\langle j_a^\mu(x_1)\, j_b^\nu(x_2)\right\rangle = \frac{1}{2}\delta^{ab}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu p^\nu}{E_p^2} f(\mathbf{p})\delta^{(3)}(\mathbf{x}-\mathbf{v}t)$$

$$x_1 = (t_1, \mathbf{x}_1)\,, \quad x_2 = (t_2, \mathbf{x}_2)\,, \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus

# Mechanism of filamentation



$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}$$

$$\nabla \times \mathbf{B} = \mathbf{j}$$

# Dispersion equation

**Equation of motion of chromodynamic field $A^\mu$ in momentum space**

$$\boxed{[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)]A_\nu(k) = 0}$$

gluon self-energy

**Dispersion equation**

$$\boxed{\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0}$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

**Instabilities – solutions with Imω > 0** $\Rightarrow$ $A^\mu(x) \sim \mathrm{e}^{\mathrm{Im}\,\omega t}$

**Dynamical information is hidden in** $\Pi^{\mu\nu}(k)$. **How to get it?**

# Transport theory – transport equations

fundamental

$$\left(p_\mu D^\mu - g p^\mu F_{\mu\nu}(x)\partial_p^\nu\right) Q(p,x) = C$$

$$\left(p_\mu D^\mu + g p^\mu F_{\mu\nu}(x)\partial_p^\nu\right) \overline{Q}(p,x) = \overline{C}$$

adjoint

$$\left(p_\mu \mathcal{D}^\mu - g p^\mu \mathcal{F}_{\mu\nu}(x)\partial_p^\nu\right) G(p,x) = C_g$$

free streaming

mean-field force

collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, \ldots.], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$\boxed{D_\mu F^{\mu\nu} = j^\nu[Q, \overline{Q}, G]}$$

mean-field generation

collisionless limit: $C = \overline{C} = C_g = 0$

# Transport theory - linearization

fluctuation

$$\boxed{Q(p,x) = Q_0(p) + \delta Q(p,x)}$$

stationary colorless state $Q_0^{ij}(p) = \delta^{ij} n(p)$

$$|Q_0(p)| >> |\delta Q(p,x)|, \quad |\partial_p^\mu Q_0(p)| >> |\partial_p^\mu \delta Q(p,x)|$$

Linearized transport equations

$$\boxed{\begin{aligned} p_\mu D^\mu \delta Q(p,x) - g p^\mu F_{\mu\nu}(x) \partial_p^\nu Q_0(p) = 0 \\ p_\mu D^\mu \delta \overline{Q}(p,x) + g p^\mu F_{\mu\nu}(x) \partial_p^\nu \overline{Q}_0(p) = 0 \\ p_\mu \mathcal{D}^\mu \delta G(p,x) - g p^\mu \mathcal{F}_{\mu\nu}(x) \partial_p^\nu G_0(p) = 0 \end{aligned}}$$

# Transport theory – polarization tensor

$$\delta Q(p,x) = g\int d^4x'\,\Delta_p(x-x')\,p^\mu F_{\mu\nu}(x)\,\partial_p^\nu Q_0(p)$$

$$p_\mu D^\mu \Delta_p(x) = \delta^{(4)}(x)$$

$$\Downarrow$$

$$j^\mu[\delta Q, \delta\overline{Q}, \delta G]$$

$$\Updownarrow$$

$$j^\mu(k) = \Pi^{\mu\nu}(k)\,A_\nu(k)$$

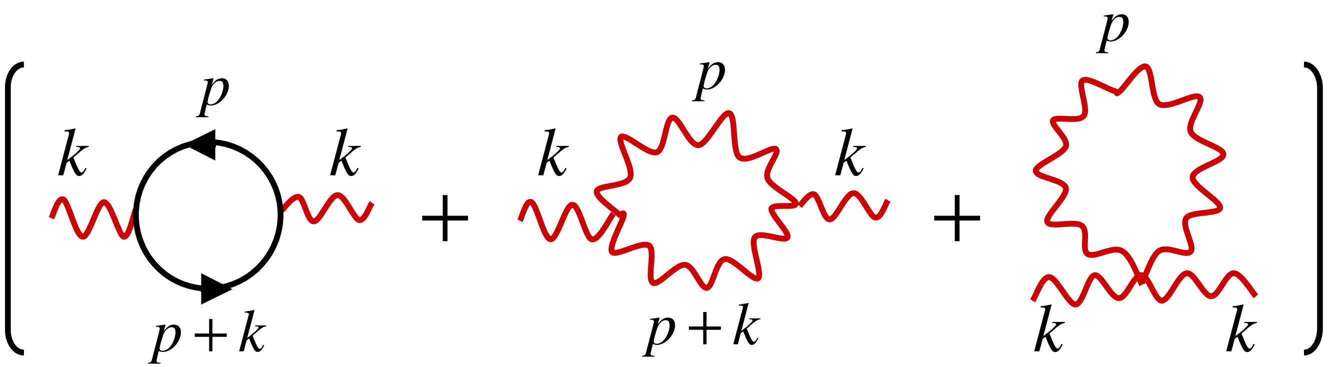
$$f(\mathbf{p}) \equiv n(\mathbf{p}) + \bar{n}(\mathbf{p}) + 2n_g(\mathbf{p})$$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\,\frac{p^\mu}{E}\Big[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\Big]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

# Diagrammatic Hard Loop approach

$\Pi^{\mu\nu}(k) =$



Hard loop approximation: $k^\mu << p^\mu$

$$\boxed{\Pi^{\mu\nu}(k) = \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E}\left[g^{\nu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+}\right]\frac{\partial f(\mathbf{p})}{\partial p^\lambda}}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

# Dispersion equation

**Dispersion equation**

$$\boxed{\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0}$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2}\Pi^{ij}(k)$$

**chromodielectric tensor**

$$k^\mu \equiv (\omega, \mathbf{k})$$

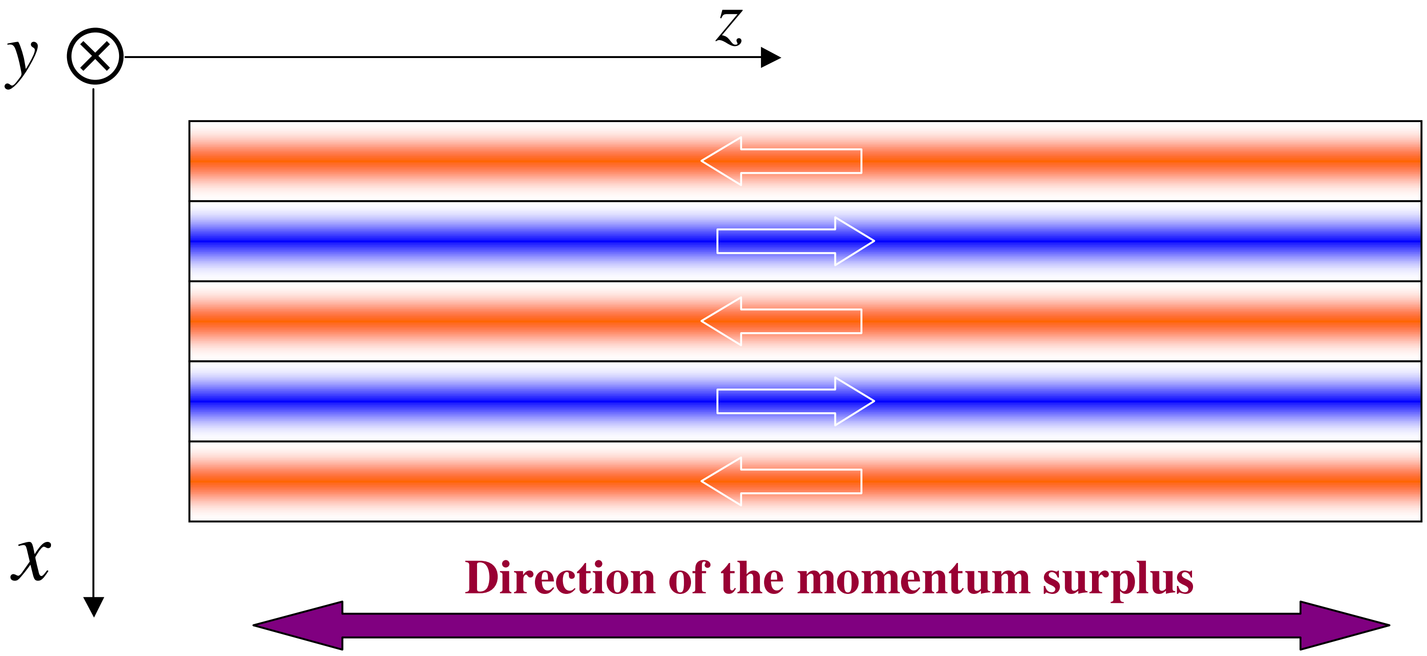
**Dispersion equation**

$$\boxed{\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0}$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega}\int \frac{d^3p}{(2\pi)^3} \frac{v^i}{\omega - \mathbf{k}\mathbf{v} + i0^+} \frac{\partial f(\mathbf{p})}{\partial p^l}\left[\left(1 - \frac{\mathbf{k}\mathbf{v}}{\omega}\right)\delta^{lj} + \frac{k^l v^j}{\omega}\right]$$

$$\mathbf{v} \equiv \mathbf{p}/E$$

# Dispersion equation – configuration of interest



$$\mathbf{j} = (0,0,j), \quad \mathbf{E} = (0,0,E), \quad \mathbf{k} = (k,0,0)$$

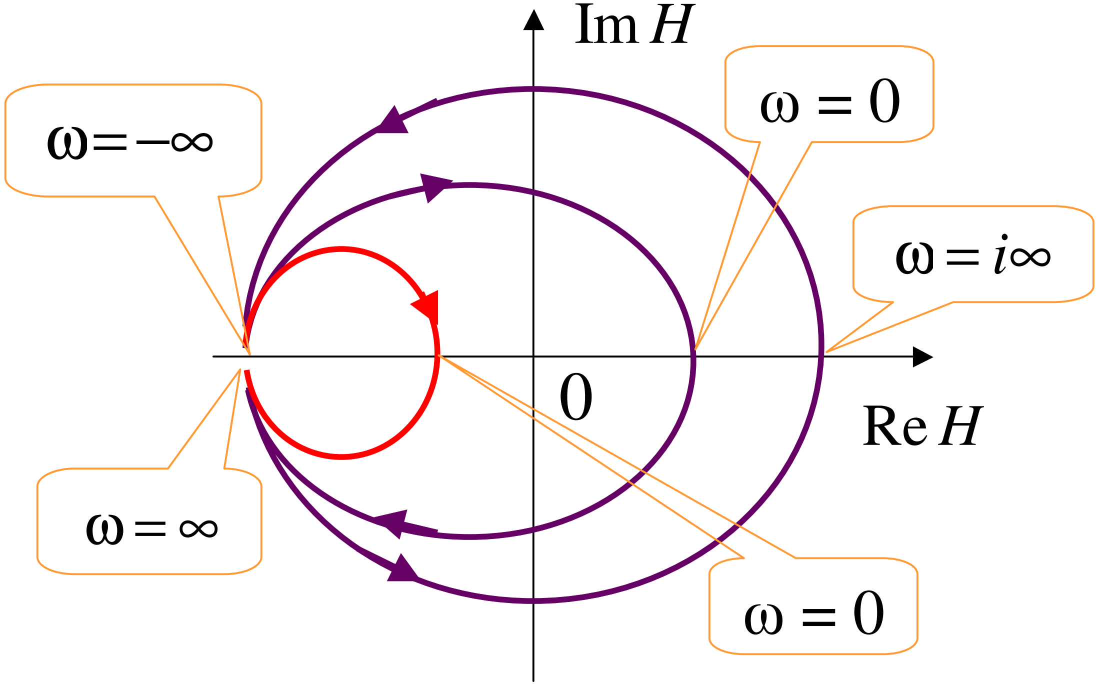
**Dispersion equation**

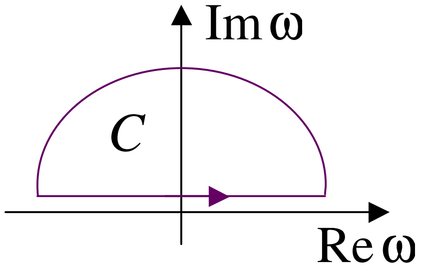
$$\boxed{k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0}$$

# Existence of unstable modes – Penrose criterion

$$\boxed{H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)}$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \displaystyle\oint_C \frac{d\omega}{2\pi i} \frac{d \ln H(\omega)}{d\omega} = \ln H(\omega)\Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$





**There are unstable modes if**

$$\boxed{H(\omega = 0) < 0}$$

# Unstable solution

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho\sigma_\perp^4}{\sigma_\parallel} \frac{1}{(p_\perp^2+\sigma_\perp^2)^3} e^{-\frac{p_\parallel^2}{2\sigma_\parallel^2}}$$

$\rho = 6\ \mathrm{fm}^{-3}$

$\alpha_s = g^2/4\pi = 0.3$

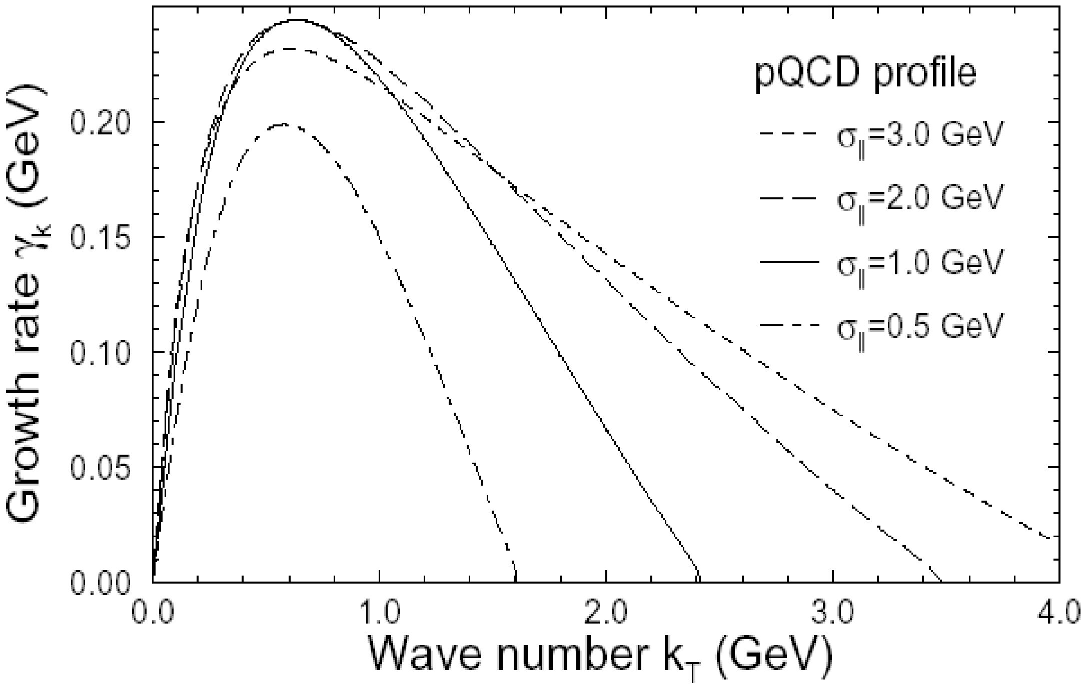
$\sigma_\perp = 0.3\ \mathrm{GeV}$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

**solution**

$$\omega(k) = \pm i\,\gamma_k$$

$$0 < \gamma_k \in \Re$$



J. Randrup and St. M., Phys. Rev. C **68**, 034909 (2003)

# Growth of instabilities – numerical simulation

**Classical system of colored particles & fields**

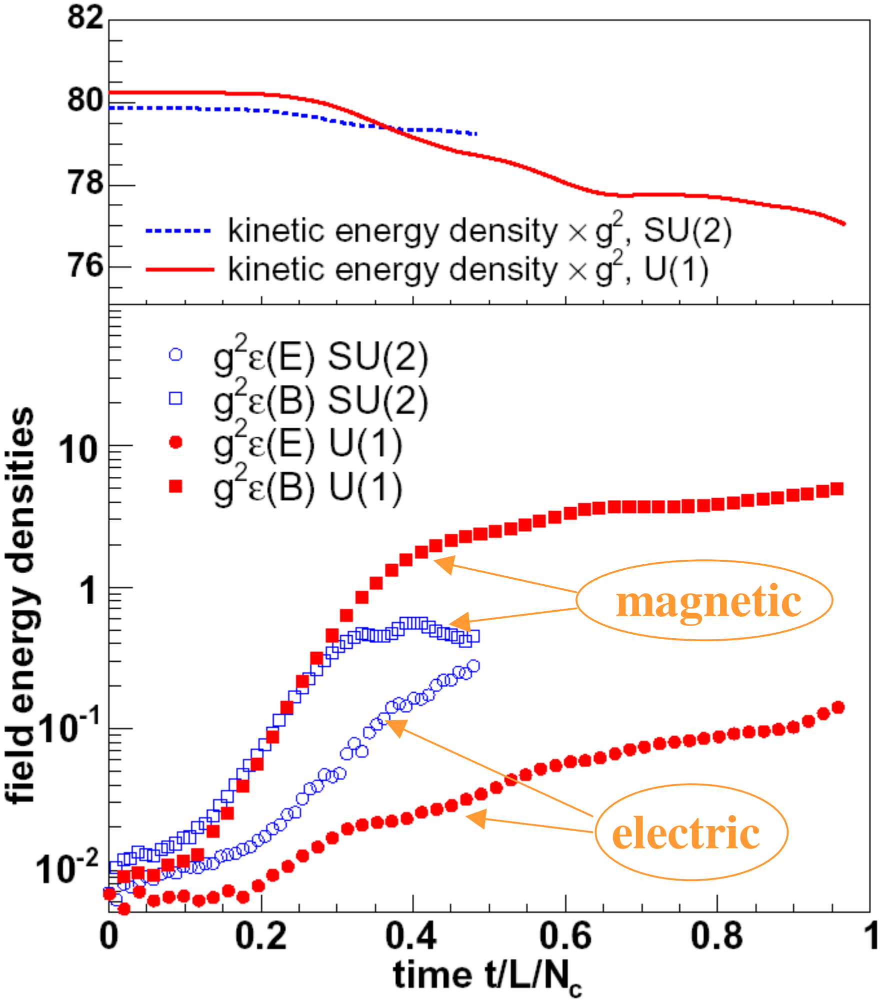
**initial fields: Gaussian noise as in Color Glass Condensate**

**initial particle distribution:**

$$f_0(\mathbf{p},\mathbf{x}) \sim \delta(p_x)\, e^{-\frac{\sqrt{p_y^2+p_z^2}}{p_{\mathrm{hard}}}}$$

$$p_{\mathrm{hard}} = 10\ \mathrm{GeV}$$

$$L = 40\ \mathrm{fm} \qquad \rho = 10\ \mathrm{fm}^{-3}$$



magnetic

electric

A. Dumitru and Y. Nara, arXiv:hep-ph/0503121

# Abelanization

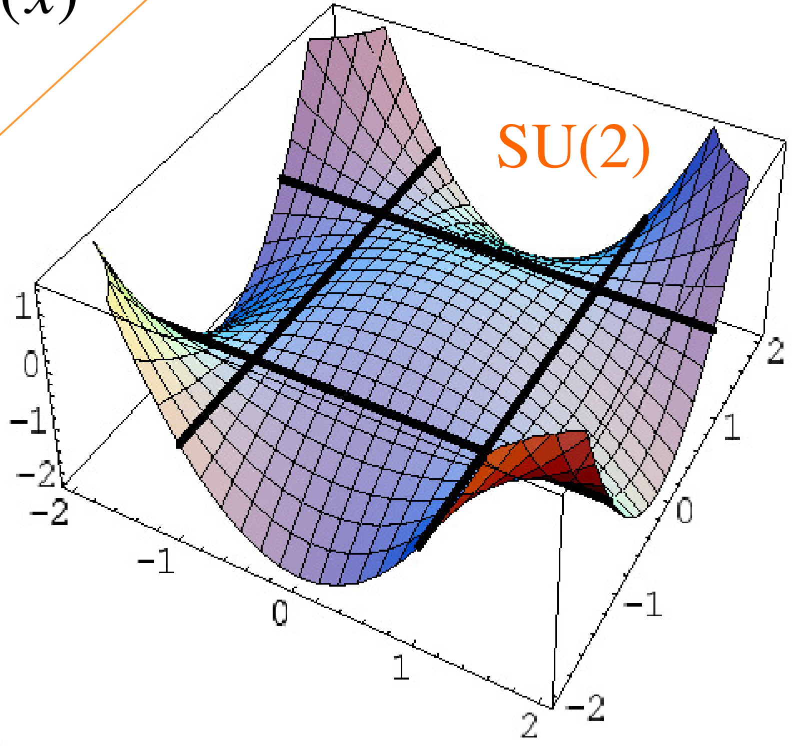
$$V_{\text{eff}}[\mathbf{A}^a] = -\mu^2 \mathbf{A}^a \cdot \mathbf{A}^a + \frac{1}{4} g^2 f_{abc} f_{ade} (\mathbf{A}^b \cdot \mathbf{A}^d)(\mathbf{A}^c \cdot \mathbf{A}^e)$$

**the gauge** $A_0^a = 0, \quad A_i^a(t, x, y, z) = A_i^a(x)$

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} = -\frac{1}{2} \mathbf{B}^a \mathbf{B}^a$$

$$= -\frac{1}{4} g^2 f_{abc} f_{ade} (\mathbf{A}^b \cdot \mathbf{A}^d)(\mathbf{A}^c \cdot \mathbf{A}^e)$$

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a + \overbrace{\frac{g}{2} f_{abc} \mathbf{A}^b \times \mathbf{A}^c}$$



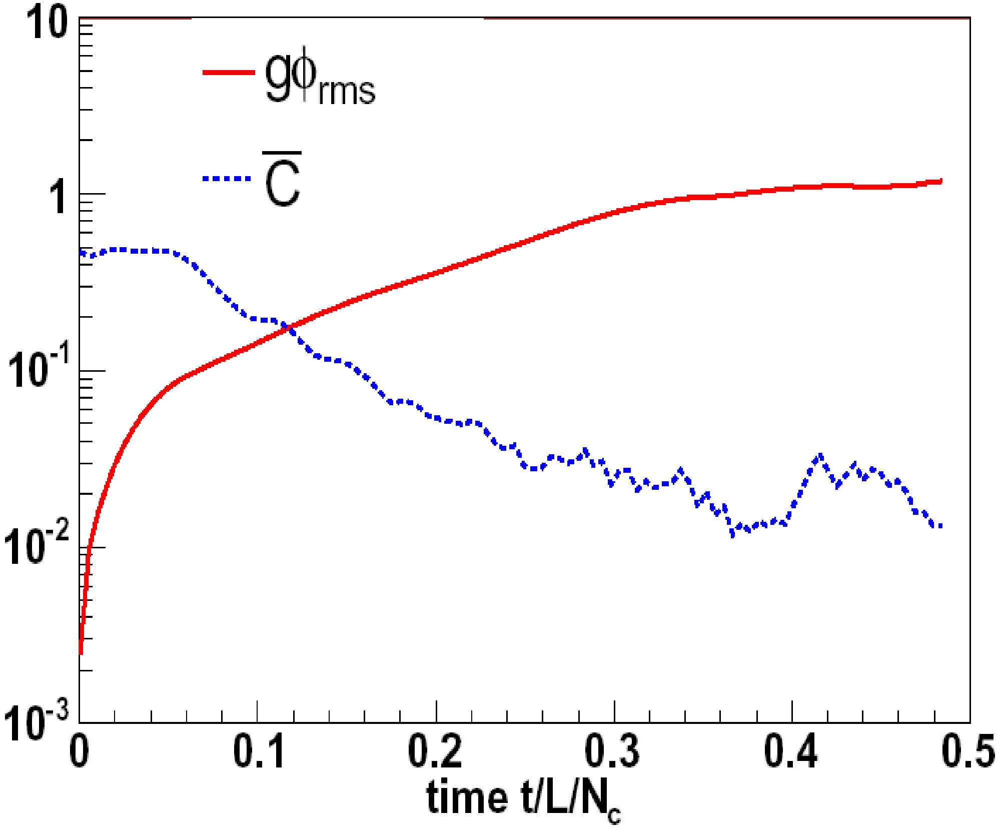
P. Arnold and J. Lenaghan, Phys. Rev. D **70**, 114007 (2004)

# Abelanization – numerical simulation

## Classical system of colored particles & fields

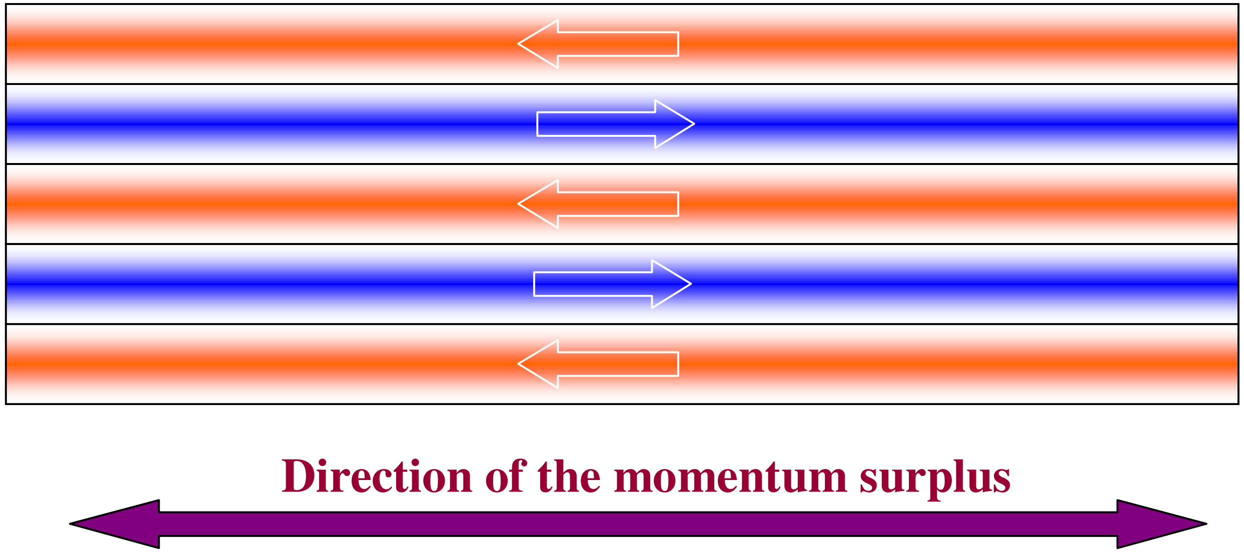
$$\phi_{\mathrm{rms}} = \sqrt{\int_0^L \frac{dx}{2L} \mathrm{Tr}[\mathbf{A}^2]}$$

$$\overline{C} = \int_0^L \frac{dx}{L} \frac{\sqrt{\mathrm{Tr}((i[A_y, A_z])^2)}}{\mathrm{Tr}[\mathbf{A}^2]}$$

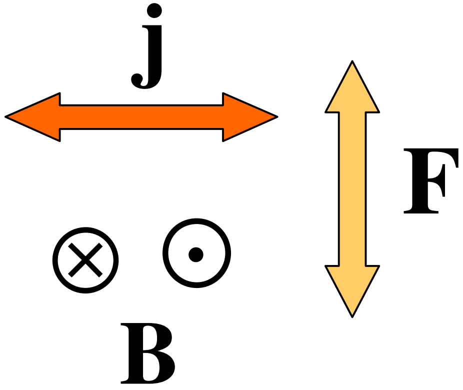


A. Dumitru and Y. Nara, arXiv:hep-ph/0503121

# Isotropization - particles

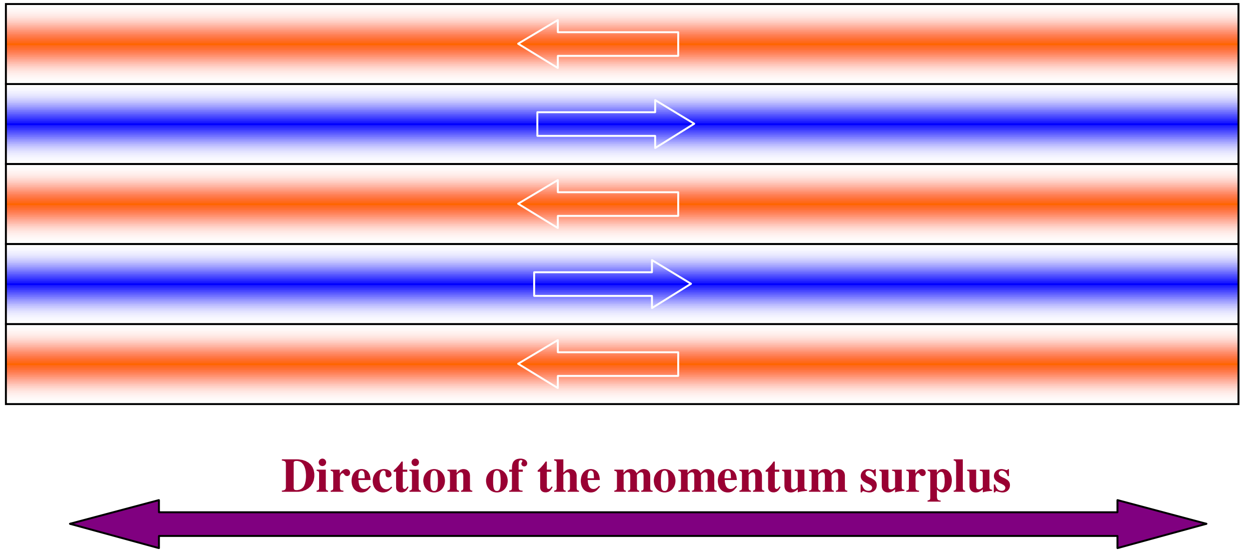


Direction of the momentum surplus

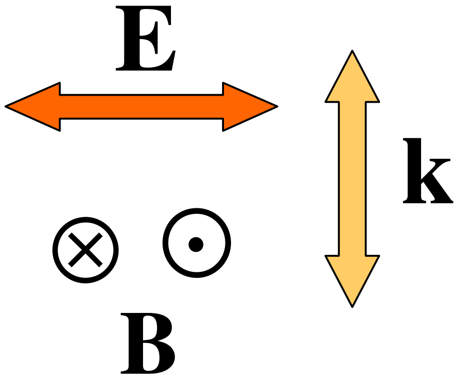


$$\boxed{\Delta \mathbf{p} = \int dt\, \mathbf{F}}$$

# Isotropization - fields



Direction of the momentum surplus



$$\boxed{\mathbf{P}_{\text{fields}} \sim \mathbf{B}^{a} \times \mathbf{E}^{a} \sim \mathbf{k}}$$

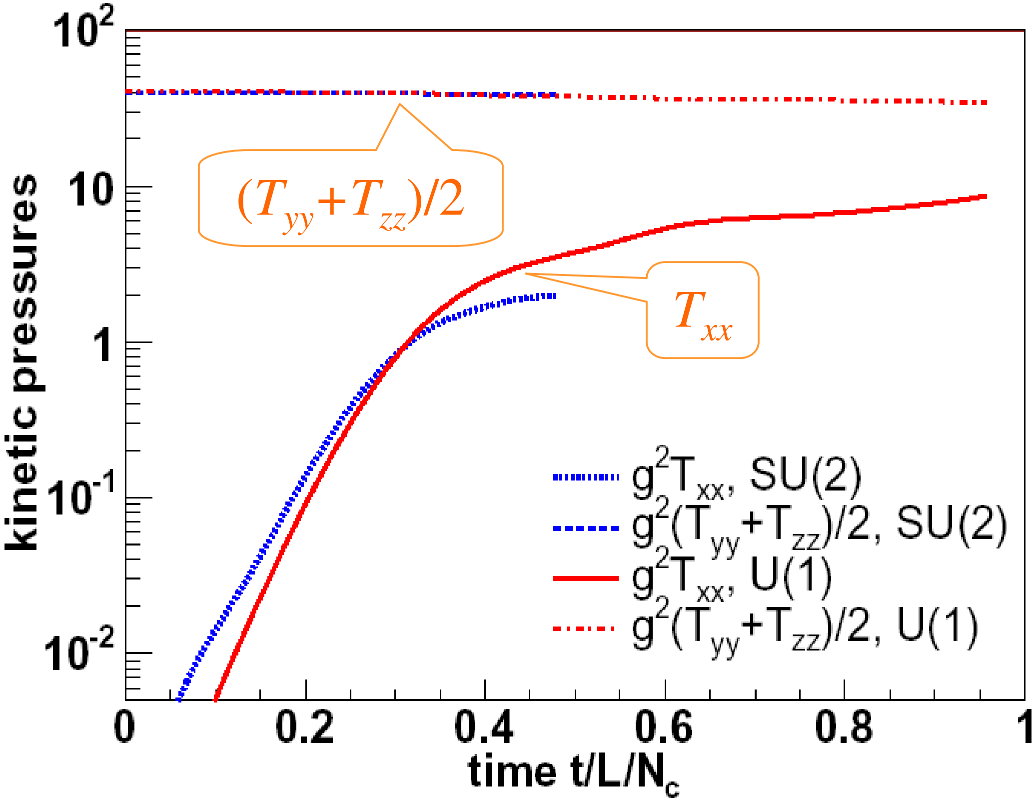
# Isotropization – numerical simulation

**Classical system of colored particles & fields**

$$T_{ij} = \int \frac{d^3p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

**Isotropy:**

$$\boxed{T_{xx} = (T_{yy} + T_{zz})/2}$$



A. Dumitru and Y. Nara, arXiv:hep-ph/0503121

# Isotropization vs. equilibration

Three comments:

Isotropization is a mean-field phenomenon which is not associated with the entropy production.

Collisions are needed for equilibration.

After the stage of instabilities, the system is in prequilibrium.

# Collective flow in preequilibrium

Two comments:

Elliptic flow starts in prequilibrium stage.

Approximate hydrodynamics requires not equilibrium
but merely isotropic momentum distribution.

Hydrodynamic equation

$$\partial_\mu T^{\mu\nu} = 0$$

Energy-momentum tensor of ideal fluid

$$T^{\mu\nu} = (\varepsilon + p) u^\mu u^\nu - p\, g^{\mu\nu}$$

**Equation of state?**

P. Arnold, J. Lenaghan, G.D. Moore and L.G. Yaffe, Phys. Rev. Lett. **94**, 072302 (2005)

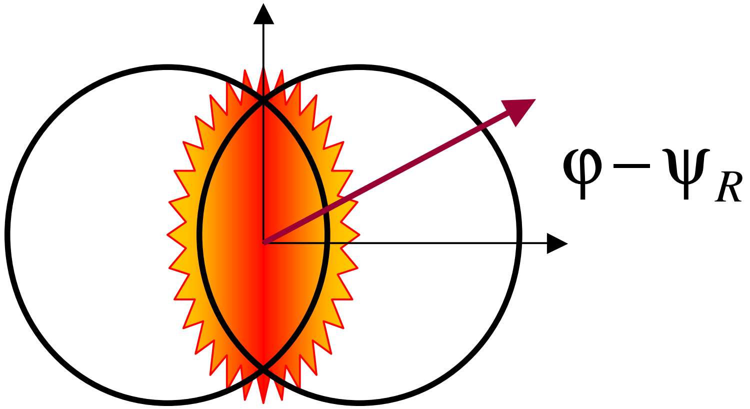
# Equilibrium vs. preequilibrium

Q: How to distinguish equilibrium form preequilibrium collective flow?

A: Look for flow fluctuations.

# Elliptic flow

$$P(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_n v_n \cos(n(\varphi - \psi_R))\right]$$



$$v_2 = \langle \cos(2(\varphi - \psi_R)) \rangle$$

# Elliptic flow fluctuations

$$R \equiv \cos(\psi_R - \psi_E)$$

$$\boxed{\langle v_2 \rangle = \frac{\left\langle \overline{\cos(2(\varphi - \psi_E))} \right\rangle}{\langle R \rangle}}$$

$\overline{\cdots}$ averaging over particles from a single event

$\langle \cdots \rangle$ averaging over events

$$\mathrm{Var}(v_2) \equiv \left\langle v_2^2 \right\rangle - \langle v_2 \rangle^2$$

$$\boxed{\mathrm{Var}(v_2) = \frac{1}{\langle R \rangle^2} \left( \left\langle \overline{\cos(2(\varphi - \psi_E))}^{\,2} \right\rangle - \left\langle \overline{\cos(2(\varphi - \psi_E))} \right\rangle^2 \right)}$$

St. M. and E.V. Shuryak, Acta Phys. Pol. **B34**, 4241 (2003)

# Statistical noise

Due to the finite particle multiplicity

$$\boxed{\mathrm{Var}(v_2) = \frac{1}{2\langle R\rangle^2\langle N\rangle} + \langle v_2\rangle^2 \frac{\mathrm{Var}(R)}{\langle R\rangle^2}}$$

$N$ number of particles used to compute $\cos(\varphi-\psi_E)$

$M$ number of particles used to determine $\psi_R$

$\psi_R - \psi_E \to 0$

$\langle(\psi_R-\psi_E)^2\rangle \sim \frac{1}{\langle M\rangle}$

$$\langle R\rangle^2 = \langle\cos(2(\psi_R-\psi_E))\rangle^2 \approx \left(1-2\langle(\psi_R-\psi_E)^2\rangle\right)^2 \approx \left(1-\frac{2\alpha}{\langle M\rangle}\right)^2 \approx \boxed{1-\frac{4\alpha}{\langle M\rangle}}$$

$$\langle R^2\rangle = \langle\cos^2(2(\psi_R-\psi_E))\rangle \approx 1-4\langle(\psi_R-\psi_E)^2\rangle \approx \boxed{1-\frac{4\alpha}{\langle M\rangle}}$$

$$\boxed{\mathrm{Var}(R) \sim \frac{1}{\langle M\rangle^2} << \frac{1}{\langle N\rangle}}$$

# Statistical noise

$$\boxed{\delta v_2 = \frac{1}{\langle R\rangle\sqrt{2\langle N\rangle}}} \qquad \delta v_2 \equiv \sqrt{\mathrm{Var}(v_2)}$$

$$\langle R\rangle \sim 1 \quad \& \quad \langle N\rangle \sim 10^3 \quad \Rightarrow \quad \boxed{\delta v_2 \sim 10^{-2}}$$

# Fluctuations due to b - variation

$$\boxed{\delta v_2 = \frac{d\langle v_2 \rangle}{db}\,\delta b}$$

$$b \rightarrow N_p \qquad N_p = 2Z\left(1 - \frac{b}{b_{\max}}\right)$$

$$\delta v_2 \approx 8 \times 10^{-4}\,\delta N_p$$

$$\delta N_p \sim 10 \quad \Rightarrow \quad \delta v_2 \sim 10^{-2}$$

# Elliptic flow fluctuations

Statistical noise & b variation are under control

Dynamical elliptic flow fluctuations seem to be measurable

# Integrated azimuthal fluctuations & Φ-measure

- single particle's variable $z = \varphi - \overline{\varphi}$ $\quad \overline{\ldots}$ inclusive averaging

- event's variable $Z = \sum_{i=1}^{N} (\varphi_i - \overline{\varphi})$ $\quad \langle \cdots \rangle$ averaging over events

$$\langle Z \rangle = 0$$

- Φ-measure

$$\boxed{\Phi = \sqrt{\frac{\langle Z^2 \rangle}{\langle N \rangle}} - \sqrt{\overline{z^2}}}$$

$\Phi = 0$ no correlations

M. Gaździcki and St. M., Z. Phys. **C54**, 127 (1992)

# Φ-measure of flow fluctuations

$$P_{\mathrm{ev}}(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_n v_n \cos(n(\varphi - \psi_n))\right]$$

$$\Phi \approx \frac{3}{2\pi^2}\langle N\rangle \begin{cases} \left\langle \sum_n \left(\dfrac{v_n}{n}\right)^2 \right\rangle & \text{for} \quad \langle\psi_n\psi_m\rangle = \langle\psi_n\rangle\langle\psi_m\rangle \\ & \qquad n \neq m \\ \left\langle \left(\sum_n \dfrac{v_n}{n}\right)^2 \right\rangle & \text{for} \quad n\psi_n + \alpha_n = \psi_R \end{cases}$$

**no flow fluctuations!**

$\langle N\rangle = 10^3, \quad v_1 = v_2 = 0.03, \quad v_n = 0,\, n \geq 3$

$$0.34 \leq \Phi \leq 0.62$$

St. M., Acta Phys. Pol. **B31**, 2065 (2000)

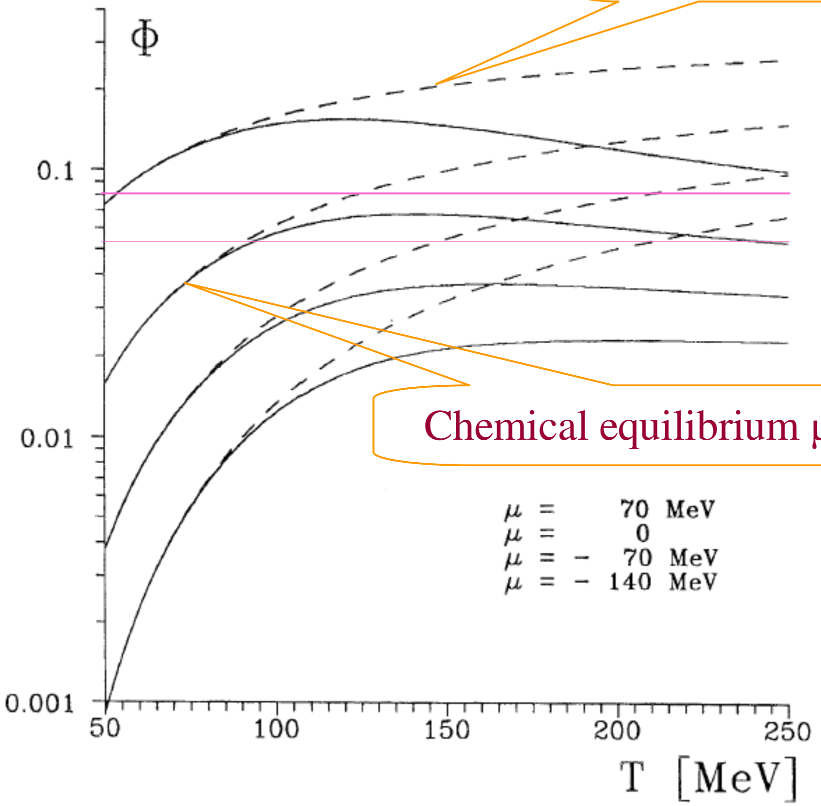
# Non-flow azimuthal fluctuations

**Bose-Einstein correlations**

$$\boxed{\Phi = \frac{\pi}{\sqrt{3}}\left(\sqrt{\frac{\tilde{\rho}}{\rho}} - 1\right)}$$

$$\rho = \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\beta(E-\mu)} - 1}$$

$$\tilde{\rho} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{\beta(E-\mu)}}{(e^{\beta(E-\mu)} - 1)^2}$$



no resonances

Chemical equilibrium μ = 0

St. M., Acta Phys. Pol. **B31**, 2065 (2000)

# Conclusions

Azimuthal fluctuations can tell us whether the elliptic flow is generated in the fully equilibrated sQGP or in the prequilibrium pQGP driven by instabilities