Sum rule of the correlation function

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(Received 12 October 2004; published 27 April 2005)

We discuss a sum rule satisfied by the correlation function of two particles with small relative momenta. The sum rule, which results from the completeness condition of the quantum states of two particles, is derived and checked to see how it works in practice. The sum rule is shown to be trivially satisfied by free particle pairs. We then analyze three different systems of interacting particles: neutron and proton pairs in the s-wave approximation, the so-called hard spheres with phase shifts taken into account up to $l = 4$, and finally, the Coulomb system of two charged particles.

DOI: 10.1103/PhysRevC.71.044905 PACS number(s): 25.75.Gz

I. INTRODUCTION

The correlation functions of two identical or nonidentical particles with "small" relative momenta have been extensively studied in nuclear collisions for bombarding energies from tens of MeV [1] to hundreds of GeV [2]. These functions provide information about space-time characteristics of particle sources in the collisions. As shown by one of us [3], the correlation function integrated over particle relative momentum satisfies a simple and exact relation because of the completeness of the particle quantum states. The relation can be used to get a particle phase-space density, following the method [4,5], with no need to extract the Coulomb interaction. The sum rule offers a constraint for the procedure of imaging [6,7] which inverts the correlation function and provides the source function. The sum rule is also helpful in handling the correlation functions of exotic systems such as $\bar{p}\Lambda$ [8] when the interparticle interaction is poorly known.

The aim of this paper is to discuss the sum rule in detail and to prove or disprove its usefulness in experimental studies. Therefore, we first derive the sum rule and show that it is trivially satisfied by free particles. Then, we analyze the correlations in the neutron-proton (n-p) system, where there are both the two-particle scattering states and a bound state, i.e., a deuteron. We prove that in spite of the attractive interaction, the n-p correlation can be negative because of the deuteron formation. Although some qualitative features dictated by the sum rule are certainly seen, the calculated correlation function does not satisfy the sum rule. This is not surprising as not only the momenta of order $1/R$, where R is the source size, but also larger momenta contribute to the sum-rule integral. Consequently, the s-wave approximation, which is used to compute the n-p correlation function, is insufficient.

To assess the importance of the higher l phase shifts, we discuss a system of the so-called hard spheres and compute the correlation function of identical and nonidentical particles, taking into account phase shifts up to $l = 4$. Paradoxically, the higher l contributions do not improve the situation but make it even worse.

Finally, we discuss the correlations caused by the Coulomb interaction. This case is of particular interest as one usually measures the correlation functions of charge particles that experience the Coulomb interaction. Also, the integral, which is controlled by the sum rule, is used to determine the particle phase-space density [4,5]. In the case of Coulomb interactions, the exact wave functions are known, and consequently, the exact correlation functions can be computed. Unfortunately, as we discuss in detail, the integral of interest appears to be divergent, and the sum rule does not hold for this most important case.

II. PRELIMINARIES

To avoid unnecessary complications, our considerations are nonrelativistic, and we start with the formula repeatedly discussed in the literature which expresses the correlation function $R(\mathbf{q})$ of two particles with the relative momentum $\mathbf{q}$ as

$$R(\mathbf{q}) = \int d^3r\, \mathcal{D}_r(\mathbf{r})\, |\phi_{\mathbf{q}}(\mathbf{r})|^2, \tag{1}$$

where $\phi_{\mathbf{q}}(\mathbf{r})$ is the wave function of relative motion of the two particles. $\mathcal{D}_r(\mathbf{r})$ is the *effective* source function defined through the probability density $D_r(\mathbf{r}, t)$ to emit the two particles at the relative distance $\mathbf{r}$ and the time difference t as

$$\mathcal{D}_r(\mathbf{r}) = \int dt\, D_r(\mathbf{r} - \mathbf{v}t, t), \tag{2}$$

with $\mathbf{v}$ being the particle pair velocity with respect to the source; the relative source function $D_r(\mathbf{r}, t)$ is given by the single-particle source function $D(\mathbf{r}, t)$, which describes the space-time emission points of a single particle, in the following

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way:

$$D_r(\mathbf{r},t) = \int d^3R\, dT\; D\left(\mathbf{R} - \frac{1}{2}\mathbf{r}, T - \frac{1}{2}t\right) \times D\left(\mathbf{R} + \frac{1}{2}\mathbf{r}, T + \frac{1}{2}t\right).$$

We note that $\mathcal{D}_r(\mathbf{r})$, $D_r(\mathbf{r},t)$, and $D(\mathbf{r},t)$ are all normalized as

$$\int d^3r\; \mathcal{D}_r(\mathbf{r}) = \int d^3r\; dt\; D_r(\mathbf{r},t) = \int d^3r\; dt\; D_r(\mathbf{r},t) = 1. \tag{3}$$

We also observe that the spherically symmetric single-particle source function $D(\mathbf{r},t)$ provides, in general, the effective $\mathcal{D}_r(\mathbf{r})$, which is elongated along the velocity $\mathbf{v}$. To simplify calculations, however, we assume here that the source function $\mathcal{D}_r(\mathbf{r})$ is spherically symmetric. Such an assumption makes sense when the single-particle source function $D(\mathbf{r},t)$ is spherically symmetric and particles are emitted instantaneously, i.e., $D(\mathbf{r},t) = D(|\mathbf{r}|)\,\delta(t-t_0)$. Then,

$$\mathcal{D}_r(r) = \int d^3R D\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) D\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right).$$

The single-particle source function is often chosen in the Gaussian form

$$D(\mathbf{r}) = \frac{1}{\left(2\pi r_0^2\right)^{3/2}} \exp\left(-\frac{\mathbf{r}^2}{2r_0^2}\right).$$

It gives the mean radius squared of a source-equal to $\langle \mathbf{r}^2 \rangle = 3r_0^2$, and it leads to the effective relative source function as

$$\mathcal{D}_r(\mathbf{r}) = \frac{1}{\left(4\pi r_0^2\right)^{3/2}} \exp\left(-\frac{\mathbf{r}^2}{4r_0^2}\right). \tag{4}$$

III. SUM RULE

Let us consider the correlation function integrated over the relative momentum. Since $R(\mathbf{q}) \to 1$ when $\mathbf{q} \to \infty$, we rather discuss the integral of $R(\mathbf{q}) - 1$. Using Eq. (1) and taking into account the normalization condition of $\mathcal{D}_r(\mathbf{r})$ (3), one finds, after changing the order of the $\mathbf{r}$ and $\mathbf{q}$ integrations, the expression

$$\int \frac{d^3q}{(2\pi)^3}\; (R(\mathbf{q}) - 1) = \int d^3r\; \mathcal{D}_r(\mathbf{r}) \int \frac{d^3q}{(2\pi)^3} (|\phi_{\mathbf{q}}(\mathbf{r})|^2 - 1). \tag{5}$$

It appears that the integral over $\mathbf{q}$ in the right-hand side (rhs) of Eq. (5) is determined by the quantum-mechanical completeness condition. Indeed, the wave functions satisfy the well-known closure relation

$$\int \frac{d^3q}{(2\pi)^3}\; \phi_{\mathbf{q}}(\mathbf{r})\phi^*_{\mathbf{q}}(\mathbf{r}') + \sum_\alpha \phi_\alpha(\mathbf{r})\phi^*_\alpha(\mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \pm \delta^{(3)}(\mathbf{r} + \mathbf{r}'), \tag{6}$$

where ϕ_α represents a possible bound state of the two particles of interest. When the particles are not identical, the second term in the rhs of Eq. (6) is not present. This term guarantees the right symmetry for both sides of the equation for the case of identical particles. The upper sign is for bosons, while the lower one is for fermions. The wave function of identical bosons (fermions) $\phi_{\mathbf{q}}(\mathbf{r})$ is (anti-)symmetric when $\mathbf{r} \to -\mathbf{r}$, and the rhs of Eq. (6) is indeed (anti-)symmetric when $\mathbf{r} \to -\mathbf{r}$ or $\mathbf{r}' \to -\mathbf{r}'$. If the particles of interest carry spin, the summation over spin degrees of freedom in the left-hand side (lhs) of Eq. (6) is implied. When the integral representation of $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ is used, the relation (5) can be rewritten as

$$\int \frac{d^3q}{(2\pi)^3} [\phi_{\mathbf{q}}(\mathbf{r})\phi^*_{\mathbf{q}}(\mathbf{r}') - e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')}] + \sum_\alpha \phi_\alpha(\mathbf{r})\phi^*_\alpha(\mathbf{r}') = \pm\delta^{(3)}(\mathbf{r} + \mathbf{r}').$$

Now, we take the limit $\mathbf{r}' \to \mathbf{r}$ and get the relation

$$\int \frac{d^3q}{(2\pi)^3}\; [|\phi_{\mathbf{q}}(\mathbf{r})|^2 - 1] = \pm\, \delta^{(3)}(2\mathbf{r}) - \sum_\alpha |\phi_\alpha(\mathbf{r})|^2 . \tag{7}$$

When Eq. (7) is substituted into Eq. (5), one finds the desired sum rule

$$\int d^3q\;\; [R(\mathbf{q}) - 1] = \pm\pi^3\, \mathcal{D}_r(0) - \sum_\alpha A_\alpha, \tag{8}$$

where A_α is the formation rate of a bound state α

$$A_\alpha = (2\pi)^3 \int d^3r\; \mathcal{D}_r(\mathbf{r})|\phi_\alpha(\mathbf{r})|^2. \tag{9}$$

A_α relates the cross section to produce the bound state α with momentum $\mathbf{P}$ to the cross section to produce two particles with momenta $\mathbf{P}/2$ as

$$\frac{d\sigma^\alpha}{d\mathbf{P}} = A_\alpha \frac{d\widetilde{\sigma}}{d(\mathbf{P}/2)d(\mathbf{P}/2)}.$$

The tilde means that the short-range correlations are removed from the two-particle cross section, which is usually taken as a product of the single-particle cross sections.

The completeness condition is, obviously, valid for any interparticle interaction. It is also valid when the pair of particles interact with the time-independent external field, e.g., the Coulomb field generated by the particle source. Thus, the sum rule (8) holds under very general conditions as long as the basic formula (1) is justified; in particular, as long as the source function $\mathcal{D}_r(\mathbf{r})$ is $\mathbf{q}$ independent and spin independent. The validity of these assumptions can only be tested within a microscopic model of nucleus-nucleus collision which properly describes the quantum particle correlations and bound state formation.

There are several potential applications of relation (8). As already mentioned, the integral of the correlation function in the lhs of Eq. (8) is used to determine a particle phase-space density. The method [4,5] assumes that the correlation function represents noninteracting particles. Consequently, before computing the integral, the correlation function is corrected in such a way that the Coulomb interaction is removed. However, the sum rule (8) shows that the integral of the correlation function is independent of the interparticle interaction. Therefore, there is no need to extract the Coulomb interaction. We note that this procedure is rather model dependent.

To obtain information about the source function $D(\mathbf{r}, t)$, one usually parametrizes the function, computes the correlation function, and compares it with experimental data. The method of imaging [6,7] provides the source function directly, inverting the functional $R[\mathcal{D}_r]$ (1) with the experimental correlation function as an input. As seen, relation (8), which gives $\mathcal{D}_r(0)$, can be treated as a useful constraint of the imaging method.

The sum rule is also helpful in understanding the correlation functions. In particular, the sum rule shows that the correlation function can be negative in spite of an attractive interparticle interaction. This happens when the particles form bound states represented by the second term in the rhs of Eq. (8) or the interaction is strongly absorptive as in the case of an antiproton-lambda system which annihilates into mesons. Then, the scattering states of the $\bar{p}\Lambda$ do not give a complete set of quantum states of the system, and the integral from the lhs of Eq. (8) has to be negative. The recently measured correlation function of $\bar{p}$ and Λ [8] is indeed negative at small relative momenta.

IV. FREE PARTICLES

In the case of noninteracting particles, the correlation function differs from unity only for identical particles. Then, the wave function, which enters the correlation function (1), is an (anti-)symmetrized plane wave

$$\phi_{\mathbf{q}}(\mathbf{r}) = \frac{1}{\sqrt{2}}(e^{i\mathbf{qr}} \pm e^{-i\mathbf{qr}}),$$

with the upper sign for bosons and lower for fermions. Then, the integration over $\mathbf{q}$ in Eq. (5) can be explicitly performed without reference to the completeness condition (7), and one finds

$$\int d^3q \;\; [R(\mathbf{q}) - 1] = \pm\pi^3 \, \mathcal{D}_r(0). \tag{10}$$

The sum rule (10) was found this way in [9]; see also [4].

Although the sum rule (8) assumes integration up to the infinite momentum, one expects that the integral in Eq. (8) saturates at sufficiently large q. To discuss the problem quantitatively, we define the function

$$S(q_{\max}) = 4\pi \int_0^{q_{\max}} dq \; q^2[R(q) - 1]. \tag{11}$$

As already mentioned, the source function is assumed to be spherically symmetric; consequently, the correlation function depends only on $q \equiv |\mathbf{q}|$. Thus, the angular integration is trivially performed.

For the Gaussian effective source (4), when $\pi^3\mathcal{D}_r(0) = (\sqrt{\pi}/2r_0)^3$, the free functions $R(q)$ and $S(q_{\max})$ equal

$$R(q) = 1 \pm e^{-4r_0^2q^2},$$

$$S(q_{\max}) = \pm\left(\frac{\sqrt{\pi}}{2r_0}\right)^3 E_{2/3}[(2r_0q_{\max})^3],$$

where

$$E_n(x) \equiv \frac{1}{\Gamma(1+1/n)} \int_0^x dt e^{-t^n},$$

and $\Gamma(z)$ is the Euler gamma function. Since for large x, the function $E_n(x)$ can be expressed as (see, e.g., [10]),

$$E_n(x) = 1 - \frac{1}{\Gamma(1+1/n)} \frac{e^{-x^n}}{nx^{n-1}}\left[1 + \mathcal{O}\left(\frac{1}{x^n}\right)\right],$$

we have the approximation

$$S(q_{\max}) \approx \pm\left(\frac{\sqrt{\pi}}{2r_0}\right)^3 \left(1 - \frac{4r_0q_{\max}}{\sqrt{\pi}} e^{-4r_0^2q_{\max}^2}\right),$$

for $(2r_0\, q_{\max})^3 \gg 1$. As seen, the sum-rule integral (10) is saturated for $q_{\max}$ not much exceeding $1/2r_0$, and obviously, the sum rule is satisfied.

V. NEUTRON-PROTON SYSTEM

In this section, we discuss the interacting neutron-proton pair in either the spin singlet or triplet state. The nucleons, produced in high-energy nuclear collisions, are usually assumed to be unpolarized, and one considers the spin-averaged correlation function R which is a sum of the singlet and triplet correlation functions $R^{s,t}$ with the weight coefficients 1/4 and 3/4, respectively. Here, we consider, however, the singlet and the triplet correlation functions separately. Then, the sum rule (8) reads

$$\int d^3q \;(R^s(\mathbf{q}) - 1) = 0, \tag{12}$$

$$\int d^3q(R^t(\mathbf{q}) - 1) = -A_d \, . \tag{13}$$

Following [11], we calculate the correlation functions $R^{s,t}$ assuming that the source radius is significantly larger than the n-p interaction range. Then, the wave function of the n-p pair (in a scattering state) can be approximated by its asymptotic form

$$\phi_{np}^{s,t}(\mathbf{r}) = e^{i\mathbf{qr}} + f^{s,t}(q)\frac{e^{iqr}}{r}, \tag{14}$$

where $f^{s,t}(q)$ is the scattering amplitude. It is chosen as

$$f^{s,t}(q) = \frac{-a^{s,t}}{1 - \frac{1}{2}d^{s,t}a^{s,t}q^2 + iqa^{s,t}}, \tag{15}$$

where $a^{s,t}$ ($d^{s,t}$) is the scattering length (effective range) of the n-p scattering; $a^s = -23.7$ fm, $d^s = 2.7$ fm, and $a^t = 5.4$ fm, $d^t = 1.7$ fm [12]. The amplitude (15) takes into account only the s-wave scattering. This is justified as long as only small relative momenta are considered.

Substituting the wave function (14) into formula (1) with the source function (4), we get

$$\begin{aligned} R^{s,t}(q) = 1 &+ \mathrm{Re}[f^{s,t}(q)]\frac{1}{2r_0^2q}e^{-4r_0^2q^2}\mathrm{erfi}(2r_0q) \\ &- \mathrm{Im}[f^{s,t}(q)]\,\frac{1}{2r_0^2q}(1 - e^{-4r_0^2q^2}) + |f^{s,t}(q)|^2\frac{1}{2r_0^2}, \end{aligned} \tag{16}$$

where

$$\mathrm{erfi}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x dt e^{t^2}.$$

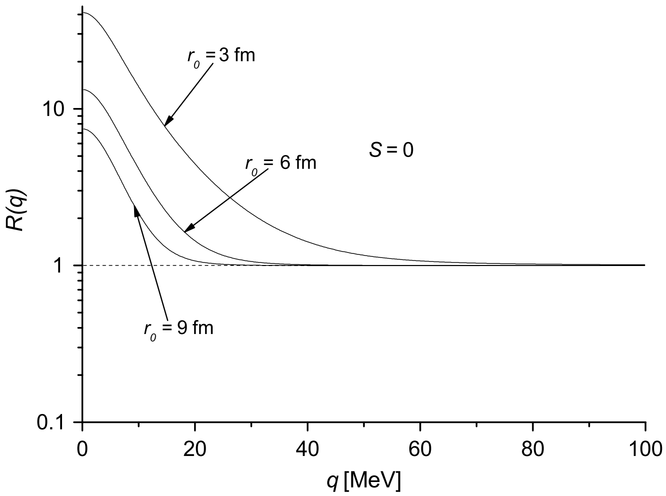

FIG. 1. Singlet correlation function of neutron and proton computed for three values of source size parameter r_0.

Because the source described by formula (4) is spherically symmetric, the correlation function (16) does not depend on $\mathbf{q}$ but on q only.

In Figs. 1 and 2, we show, respectively, the singlet and triplet correlation functions computed for three values of the source size parameter r_0. As seen, the triplet correlation is negative in spite of the attractive neutron-proton interaction. This happens, in accordance with the sum rule (13), because the neutron and proton, which are close to each other in the phase space, tend to exist in a bound not a scattering state. Moreover, the n-p pairs that form a deuteron deplete the sample of n-p pairs and produce a dip of the correlation function at small relative momenta.

The deuteron formation rate, which enters the sum rule (13), is computed with the deuteron wave function in the Hulthén form

$$\phi_d(\mathbf{r}) = \left(\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}\right)^{1/2} \frac{e^{-\alpha r} - e^{-\beta r}}{r}, \tag{17}$$

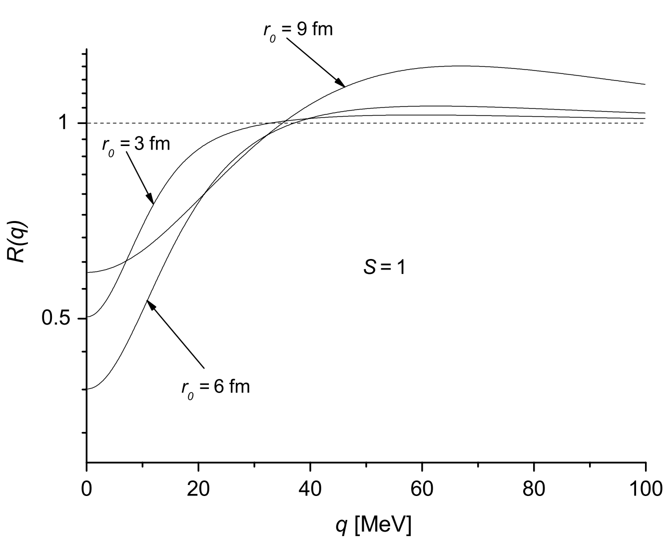

FIG. 2. Same as Fig. 1, but for triplet correlation function.

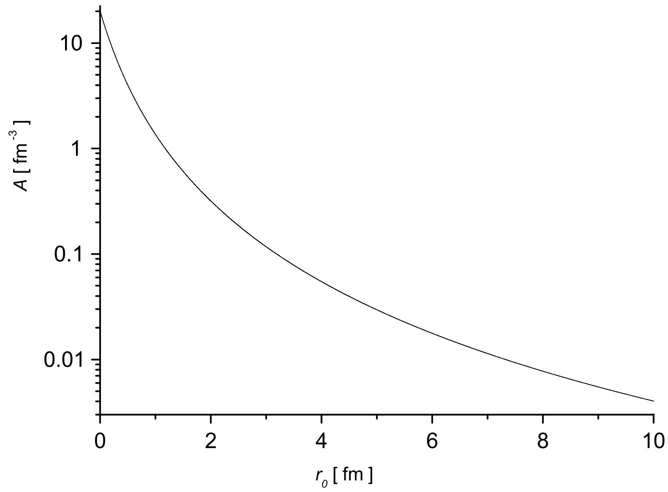

FIG. 3. Deuteron formation rate as a function of source size parameter r_0.

with $\alpha = 0.23$ fm^{-1} and $\beta = 1.61$ fm^{-1} [13]. Substituting the wave function (18) and the source function (4) into Eq. (9) results in

$$A_d = \frac{2\pi^2}{r_0^2}\,\frac{\alpha\beta(\alpha+\beta)}{(\alpha-\beta)^2}[K(2\alpha r_0) - 2K((\alpha+\beta)r_0) + K(2\beta r_0)], \tag{18}$$

where

$$K(x) \equiv e^{x^2}\mathrm{erfc}(x), \qquad \mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}}\int_x^\infty dt\, e^{-t^2}.$$

In Fig. 3, we present the deuteron formation rate (18) as a function of the source size parameter r_0. As seen, A_d monotonously decreases when the source grows.

In Figs. 4 and 5 we display the function $S(q_{\max})$, defined by Eq. (11), found for the singlet and triplet correlation functions presented in Figs. 1 and 2, respectively. Although $S(q_{\max})$

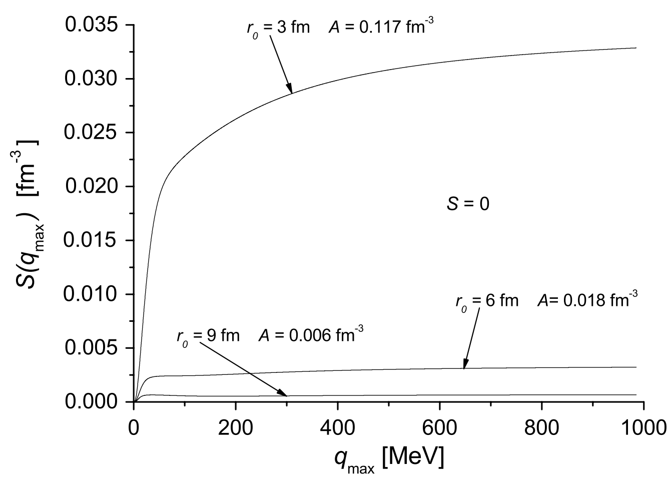

FIG. 4. Function $S(q_{\max})$ for singlet correlation function of neutron and proton for three values of source size parameter r_0. To set the scale, the corresponding values of the deuteron formation rate are given.

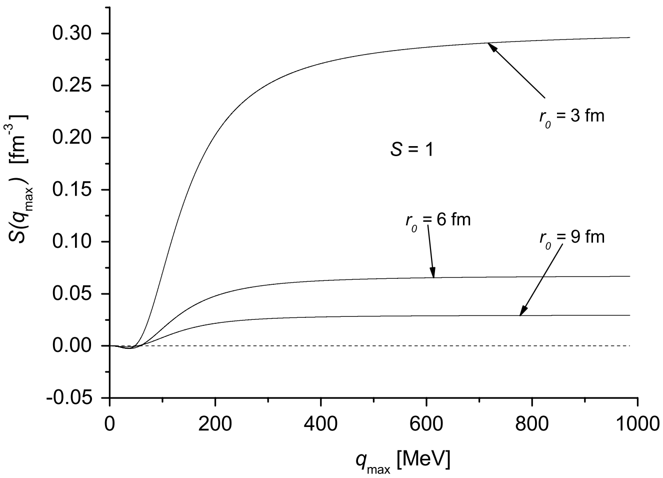

FIG. 5. Same as Fig. 4, but for triplet correlation function.

saturates at large $q_{\max}$ for both the singlet and the triplet correlation function, neither the sum rule (12) nor (13) is satisfied. The singlet $S(q_{\max})$ does not vanish at large $q_{\max}$, as it should according to Eq. (12), while the triplet $S(q_{\max})$ is not negative, as required by Eq. (13). However, comparing the numerical values of $S(q_{\max})$ to the corresponding deuteron formation rate, which sets the characteristic scale, one sees that with growing r_0, the sum rule is violated less and less dramatically. It is not surprising, because the asymptotic form of the wave function (14) and the s-wave approximation then become more accurate. Formula (11) shows that because of the factor q^2, even small deviations of the correlation function from unity at large q generate sizable contributions to S. Therefore, we can expect that higher partial waves have to be taken into account to satisfy the sum rule.

VI. HARD SPHERES

We consider here a pair of the so-called hard spheres interacting via the potential that is zero at distances larger than a and is infinite otherwise; a is the sphere's diameter. As is well known (see, e.g., [14]), scattering of the spheres is analytically tractable, and the lth phase shift δ_l is given by the equation

$$\tan\delta_l = \frac{j_l(qa)}{n_l(qa)}, \tag{19}$$

where $j_l(z)$ and $n_l(z)$ are the spherical Bessel and Neumann functions, respectively. The scattering amplitude equals

$$f(\Theta) = \frac{1}{q}\sum_{l=0}^{\infty}(2l+1)\,P_l(\cos\Theta)\frac{\cot\delta_l + i}{\cot^2\delta_l + 1}, \tag{20}$$

where Θ is the scattering angle and $P_l(z)$ is the lth Legendre polynomial. Substituting amplitude (20) into the equation analogous to Eq. (14) produces the (asymptotic) wave function. Using this function and the Gaussian source (4), we obtain the correlation function of nonidentical hard spheres as

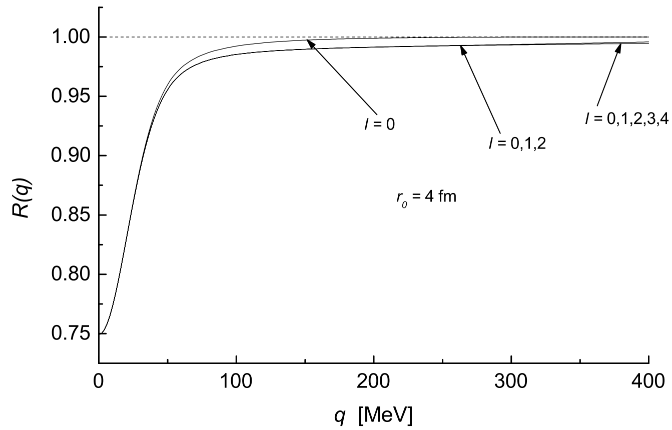

FIG. 6. Correlation function of nonidentical hard spheres for source size parameter $r_0 = 4$ fm.

$$\begin{aligned}R(q) = 1 &+ \frac{1}{\sqrt{\pi}\,r_0^3 q}\sum_l (2l+1)\,\frac{j_l(qa)}{n_l^2(qa)+j_l^2(qa)}\\ &\times\left\{\left[n_l(qa)\cos\left(\frac{\pi l}{2}\right) + j_l(qa)\sin\left(\frac{\pi l}{2}\right)\right]\right.\\ &\times\int_0^\infty drr\,e^{-\frac{r^2}{4r_0^2}}\cos(qr)\,j_l(qr)\\ &+\left[n_l(qa)\sin\left(\frac{\pi l}{2}\right) - j_l(qa)\cos\left(\frac{\pi l}{2}\right)\right]\\ &\times\left.\int_0^\infty drr\,e^{-\frac{r^2}{4r_0^2}}\sin(qr)\,j_l(qr)\right\}\\ &+\frac{1}{2r_0^2q^2}\sum_l(2l+1)\frac{j_l^2(qa)}{n_l^2(qa)+j_l^2(qa)}.\end{aligned} \tag{21}$$

For identical hard spheres, the wave function should be (anti-)symmetrized as $\frac{1}{\sqrt{2}}[\phi_{\mathbf{q}}(\mathbf{r}) \pm \phi_{\mathbf{q}}(-\mathbf{r})]$, and the correlation function equals

$$\begin{aligned}R(q) = 1 \pm e^{-4r_0^2q^2} &+ \frac{1}{\sqrt{\pi}\,r_0^3 q}\sum_l(2l+1)\frac{j_l(qa)}{n_l^2(qa)+j_l^2(qa)}\\ &\times[1\pm(-1)^l]\left\{\left[n_l(qa)\cos\left(\frac{\pi l}{2}\right) - j_l(qa)\sin\left(\frac{\pi l}{2}\right)\right]\right.\\ &\times\int_0^\infty drr\,e^{-\frac{r^2}{4r_0^2}}\cos(qr)\,j_l(qr) - \left[n_l(qa)\sin\left(\frac{\pi l}{2}\right)\right.\\ &+\left.j_l(qa)\cos\left(\frac{\pi l}{2}\right)\right]\left.\int_0^\infty drr\,e^{-\frac{r^2}{4r_0^2}}\sin(qr)\,j_l(qr)\right\}\\ &+\frac{1}{2r_0^2q^2}\sum_l(2l+1)\frac{j_l^2(qa)}{n_l^2(qa)+j_l^2(qa)}\,[1\pm(-1)^l].\end{aligned} \tag{22}$$

We note that there is a specific asymmetry of the signs of the corresponding terms in Eqs. (21) and (22). This happens because $[(-1)^l \pm 1]/2 = \mp(-1)^l$.

In Figs. 6–9, we show the correlation functions for nonidentical spheres (21) and identical bosonic ones (22). The

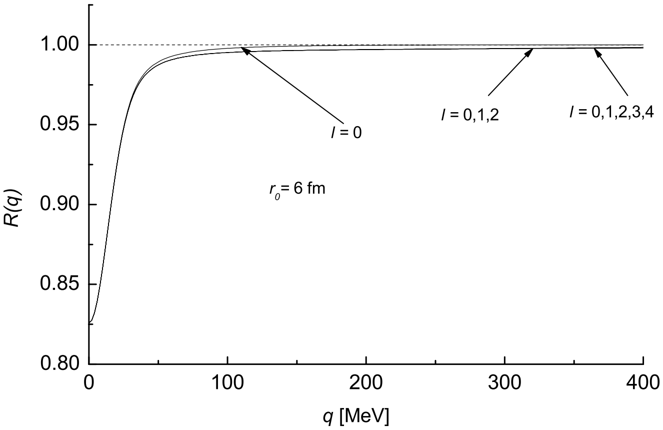

FIG. 7. Same as Fig. 6, but for $r_0 = 6$ fm.

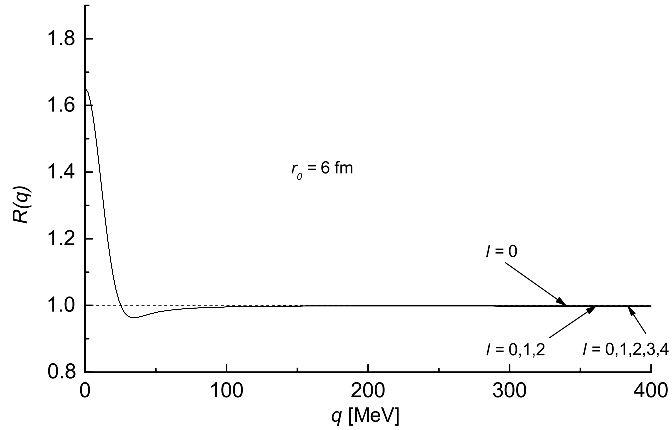

FIG. 9. Same as Fig. 8, but for $r_0 = 6$ fm.

functions are computed numerically for the sphere diameter $a = 1$ fm and two values of the source size parameter $r_0 = 4$ fm and $r_0 = 6$ fm. We have considered rather large sources because the wave function, which is used to compute the correlation function, is of the asymptotic form (14). This requires $r_0 \gg a$. The functions $S(q_{\max})$ defined by Eq. (11), which correspond to the correlation functions shown in Figs. 6–9, are presented in Figs. 10–13. In Figs. 6–13, three families of curves represent the s-wave approximation, the sum of partial waves with $l = 0,\ 1,\ 2$, and the sum of $l = 0,\ 1,\ 2,\ 3,\ 4$. As seen, going beyond the s-wave approximation minimally modifies the correlation functions shown in Figs. 6–9. This is rather expected, as the correlation functions are significantly different from unity only at small momenta where contributions of higher partial waves are strongly suppressed. The higher partial waves are more important for the functions $S(q_{\max})$, but the integrals are still far from being saturated. According to the sum rule (8), the function $S(q_{\max})$ should tend to zero for nonidentical spheres and to $\pi^3\mathcal{D}_r(0) = (\sqrt{\pi}/2r_0)^3$ for identical ones when $q_{\max} \to \infty$. Such a behavior is not observed in our calculations, and taking into account the partial waves of higher l does not improve the situation. We see two possible explanations for the problem. First, the sum rule is in principle fulfilled, but one should go to much larger momenta $q_{\max}$ to saturate the integral (11). However, higher relative momenta require taking into account more and more partial waves. If this is indeed the case, the sum rule is formally correct but useless because Eq. (1), which is a starting point of the sum-rule derivation, assumes that the relative momentum of particles q is much smaller than the typical particle momentum. Second, the integral (11) is divergent as $q_{\max}$ goes to infinity. Then, the sum rule is simply meaningless. We discuss the second possibility in more detail in the next section, where we study the Coulomb interaction.

VII. COULOMB INTERACTION

As is well known, the Coulomb problem is exactly solvable within the nonrelativistic quantum mechanics [14]. The exact wave function of two nonidentical particles interacting as a result of the repulsive Coulomb force is given as

$$\phi_{\mathbf{q}}(\mathbf{r}) = e^{-\frac{\pi\lambda}{2q}}\, \Gamma\left(1 + i\frac{\lambda}{q}\right) e^{iqz/2}\, F\left(-i\frac{\lambda}{q}, 1, iq\eta\right), \quad (23)$$

where $\lambda \equiv \mu e^2/8\pi$, with μ being the reduced mass of the two particles and $\pm e$ the charge of each of them; F denotes

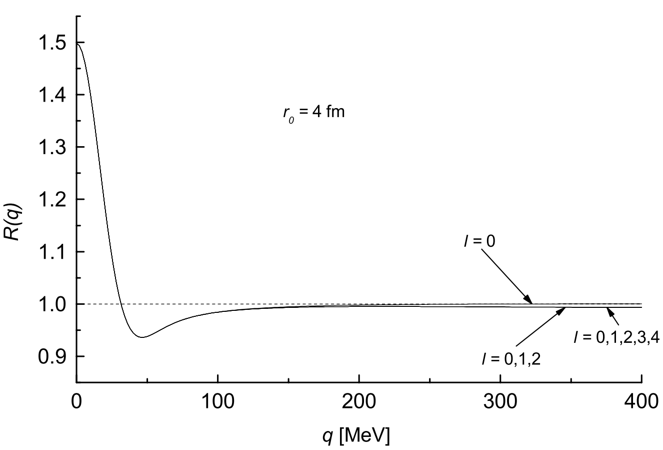

FIG. 8. Correlation function of identical hard spheres for source size parameter $r_0 = 4$ fm.

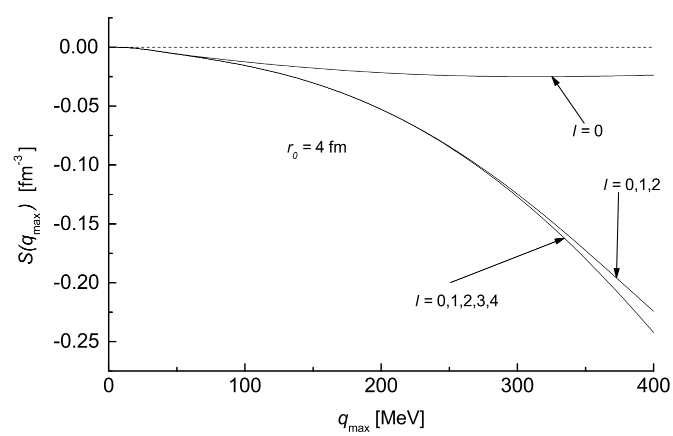

FIG. 10. Function $S(q_{\max})$ corresponding to correlation function of nonidentical hard spheres for source size parameter $r_0 = 4$ fm.

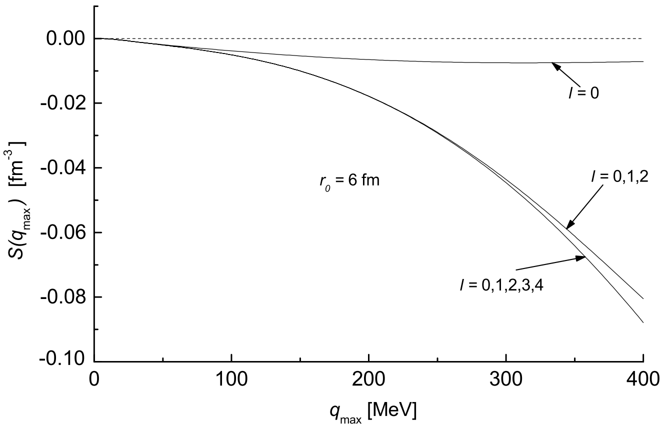

FIG. 11. Same as Fig. 10, but for $r_0 = 6$ fm.

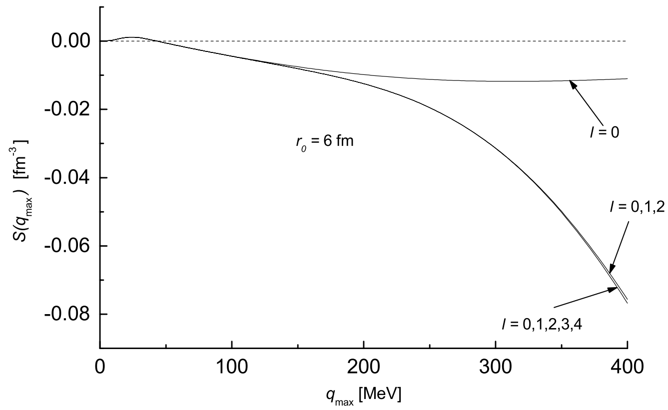

FIG. 13. Same as Fig. 12, but for $r_0 = 6$ fm.

the hypergeometric confluent function, and η is the parabolic coordinate (see below). The wave function for the attractive interaction is obtained from (23) by means of the substitution $\lambda \to -\lambda$. When one deals with identical particles, the wave function $\phi_{\mathbf{q}}(\mathbf{r})$ should be replaced by its (anti-)symmetrized form. The modulus of the wave function (23) equals

$$|\phi_q(\mathbf{r})|^2 = G(q) \left| F\left(-i\frac{\lambda}{q}, 1, iq\eta\right)\right|^2,$$

where $G(q)$ is the so-called Gamov factor defined as

$$G(q) = \frac{2\pi\lambda}{q} \frac{1}{\exp\left(\frac{2\pi\lambda}{q}\right) - 1}. \tag{24}$$

As seen, the modulus of the wave function of nonidentical particles solely depends on the parabolic coordinate η. Therefore, it is natural to calculate the Coulomb correlation function in the parabolic coordinates: $\eta \equiv r - z$, $\xi \equiv r + z$, and ϕ which is the azimuthal angle. Then, the correlation function computed for the Gaussian source function (4) equals

$$R(q) = \frac{G(q)}{2\sqrt{\pi}\, r_0} \int_0^\infty d\eta \exp\left(-\frac{\eta^2}{16r_0^2}\right) \left| F\left(-i\frac{\lambda}{q}, 1, iq\eta\right)\right|^2, \tag{25}$$

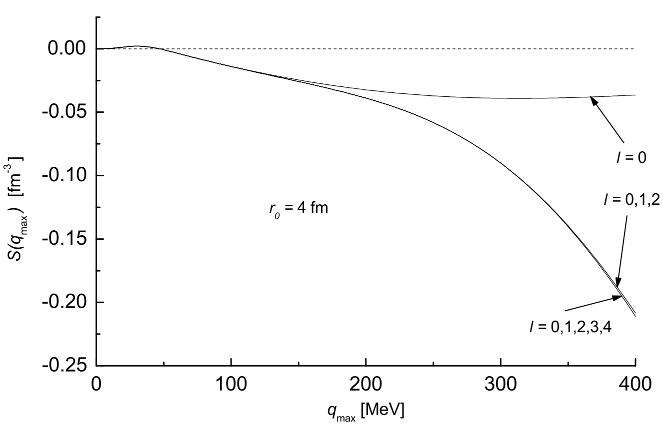

FIG. 12. Function $S(q_{max})$ corresponding to correlation function of identical hard spheres for source size parameter $r_0 = 4$ fm.

where the integration over ξ has been performed. Note here that in contrast to the neutron-proton and hard sphere cases, the Coulomb correlation function is calculated with the *exact* wave function not with the asymptotic form of it.

The modulus of the (anti-)symmetrized Coulomb wave function equals

$$\begin{aligned}
|\phi_{\mathbf{q}}(\mathbf{r})|^2
&= \frac{1}{2} G(q)\Bigg\{ \left| F\left(-i\frac{\lambda}{q}, 1, iq\eta\right)\right|^2 + \left| F\left(-i\frac{\lambda}{q}, 1, iq\xi\right)\right|^2 \\
&\quad \pm 2\,\mathrm{Re}\left[e^{iqz}\, F\left(-i\frac{\lambda}{q}, 1, iq\eta\right) F^*\left(-i\frac{\lambda}{q}, 1, iq\xi\right)\right]\Bigg\},
\end{aligned}$$

and the correlation function analogous to (25) is

$$\begin{aligned}
R(q) &= \frac{G(q)}{2\sqrt{\pi}\, r_0}\Bigg\{\int_0^\infty d\eta \exp\left(-\frac{\eta^2}{16r_0^2}\right)\left| F\left(-i\frac{\lambda}{q}, 1, iq\eta\right)\right|^2 \\
&\quad \pm \frac{1}{4r_0^2}\int_0^\infty d\eta \int_0^\infty d\xi(\eta+\xi)\exp\left(-\frac{(\eta+\xi)^2}{16r_0^2}\right) \\
&\quad \times \mathrm{Re}\left[e^{iq(\xi-\eta)/2} F\left(-i\frac{\lambda}{q}, 1, iq\eta\right) F^*\left(-i\frac{\lambda}{q}, 1, iq\xi\right)\right]\Bigg\}.
\end{aligned} \tag{26}$$

In Figs. 14 and 15, we demonstrate the correlation function of nonidentical particles and identical pions, respectively, which are given by Eqs. (25) and (26). In the case of nonidentical particles, we have taken the reduced mass, which enters the parameter λ, to be equal to that of identical pions. Therefore, the two systems differ only because of the effects of quantum statistics. In both cases, the Coulomb interaction is repulsive. While the correlation of nonidentical particles is everywhere negative, the correlation of identical pions is negative at small q because of the Coulomb repulsion, but it is positive for larger q as a result of the effect of quantum statistics.

Figures 16–18 show the functions $S(q_{max})$ defined by Eq. (11), which are computed for the correlation functions presented in Figs. 14 and 15. According to the sum rule (8), $S(q_{max})$ should vanish for sufficiently large q_{max} in the case of nonidentical particles that repel each other, and $S(q_{max})$ should

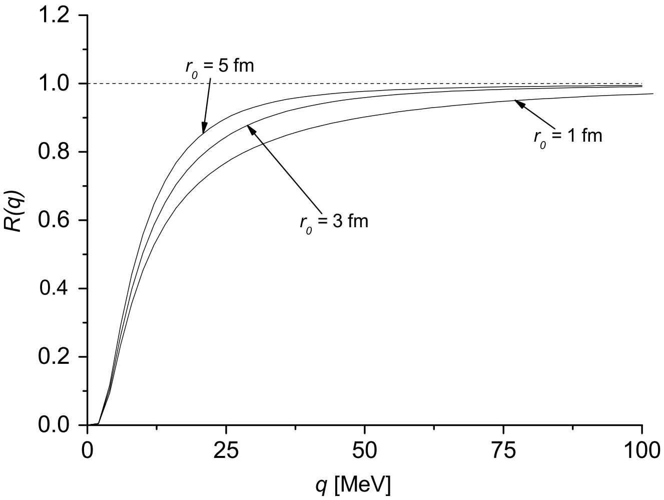

FIG. 14. Coulomb correlation function of nonidentical repelling particles for three values of source size parameter r_0.

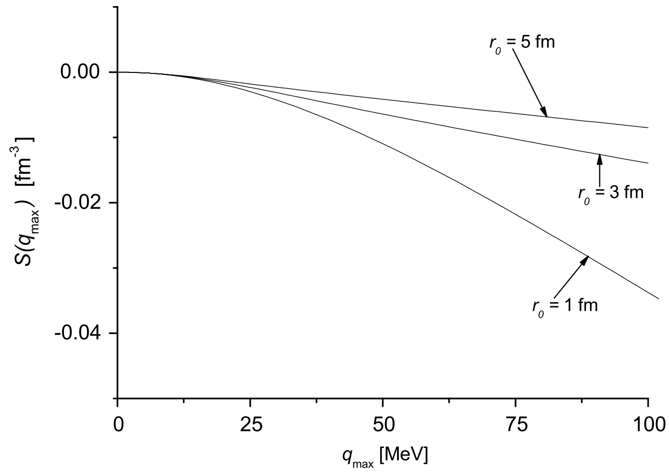

FIG. 16. Function $S(q_{\rm max})$ corresponding to Coulomb correlation function of nonidentical repelling particles for three values of source size parameter r_0.

tend to $\pi^3\,\mathcal{D}_r(0)$ for identical pions. As seen in Figs. 16–18, the functions $S(q_{\rm max})$ do not seem to saturate at large $q_{\rm max}$, and the sum rule is badly violated.

What is wrong here? The derivation of the sum rule (8) implicitly assumes that the integral in the lhs exists, i.e., it is convergent. Otherwise, interchanging the integrations over **q** and **r**, which leads to Eq. (5), is mathematically illegal. Unfortunately, the Coulomb correlation functions appear to decay too slowly with q; consequently, the integral in the lhs of Eq. (8) diverges.

To clarify how the integral diverges, one has to find the asymptotics of the correlation function at large q. This is a difficult problem, as one has to determine a behavior of the wave function (23) at large q for any **r** and then perform the integration over **r**. To our knowledge, the problem of large q asymptotics of the Coulomb correlation function has not been satisfactorily solved, although it has been discussed in several papers [15–19]. We have not found a complete solution of the problem, but our rather tedious analysis, which uses the analytic approximate expressions of the hypergeometric confluent function at small and large distances, suggests, in agreement with [15–19], that $[R(q)-1] \sim 1/q^2$ when $q \to \infty$. Then, the integral in the lhs of Eq. (8) linearly diverges, and the sum rule does not hold. We note that the Gamov factor (24), which represents a zero size source and decays as $1/q$ at large q, leads to the quadratic divergence of the integral (8). We also note here that the asymptotics $1/q^2$ of the correlation function does not have much to do with the well-known classical limit of the correlation function [15–19]. Since the large q limit of the correlation function corresponds to the small separation of the charged particles, which at sufficiently large q is smaller than the de Broglie wavelength, the classical approximation breaks down.

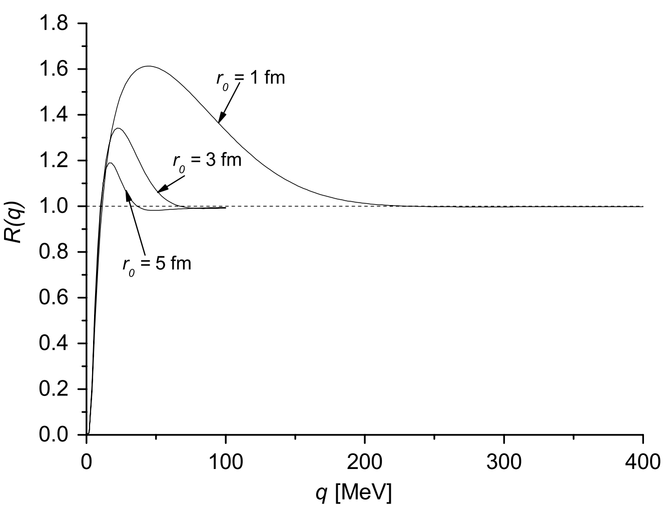

FIG. 15. Same as Fig. 14, but for identical pions.

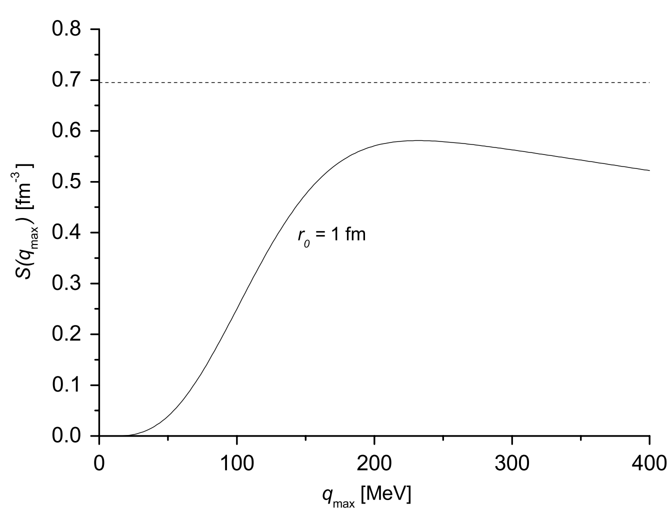

FIG. 17. Function $S(q_{\rm max})$ corresponding to Coulomb correlation function of identical pions for source size parameter $r_0 = 1$ fm. The dotted line represents the value of $\pi^3\,\mathcal{D}_r(0)$.

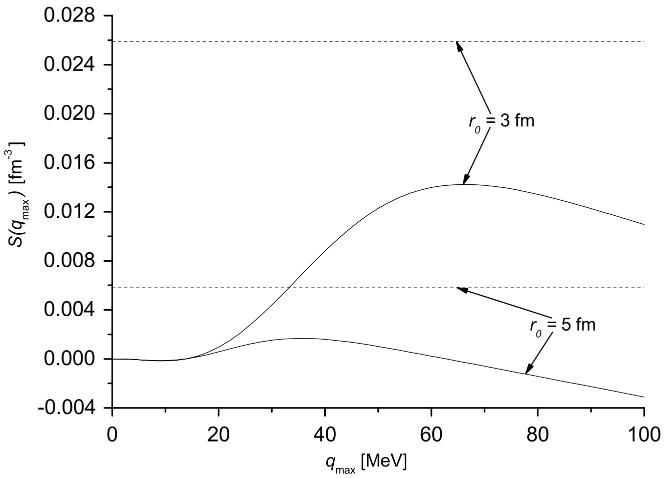

FIG. 18. Same as Fig. 17, but for $r_0 = 3$ fm and 5 fm.

VIII. FINAL REMARKS

The sum rule (8) provides a rigorous constraint on the correlation function if the momentum integral in Eq. (8) exists. The rule is trivially satisfied by the correlation function of noninteraction particles. The model calculations of the n-p correlation function in the s-wave approximation fail to fulfill the sum rule. One suspects that the approximation, which is sufficient to properly describe a general shape of the correlation function, distorts its tail. Since even small deviations of the correlation function from unity at large q generate a sizable contribution to the sum-rule integral, it seems reasonable to expect that the higher partial waves have to be included to comply with the sum rule. Paradoxically, when the higher partial waves are taken into account for the interacting hard spheres, the situation does not improve but gets worse. This suggests that either one should go to really large momenta to saturate the sum-rule integral or the integral in Eq. (8) is divergent. In the case of Coulomb repelling interaction, we certainly deal with the second option. Because of the strong electrostatic repulsion at small distances, the correlation function decays too slowly at large momenta; consequently, the sum-rule integral does not exist.

Being rather useless, the sum rule nevertheless explains some qualitative features of the correlation function. In particular, it shows that in spite of the attractive interaction, the correlation can be negative, as observed in the triplet state of the neutron-proton pair and in the $\bar{p}\Lambda$ system.

ACKNOWLEDGMENTS

We are very grateful to Wojciech Florkowski for his generous help in performing numerical calculations and to David Brown, Cristina Manuel, Richard Lednický, and Scott Pratt for fruitful discussions and correspondence.

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