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Sum rule of the correlation function

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Abstract

We derive a sum rule satisfied by the correlation function of two particles with small relative momenta, which results from the completeness condition of the quantum states.

The correlation functions of two identical or non-identical particles with 'small' relative momenta have been extensively studied in nuclear collisions for bombarding energies from tens of MeV [1] to hundreds of GeV [2]. These functions provide unique information about space-time characteristics of particle sources in the collisions. We show in this paper that the correlation function integrated over particle relative momentum satisfies a simple relation due to the completeness of the particle quantum states. A preliminary account of this work has been presented in [3].

The correlation function $\mathcal{R}$ is defined as

$$\frac{dn}{d\mathbf{p}_1 d\mathbf{p}_2} = \mathcal{R}(\mathbf{p}_1,\mathbf{p}_2)\frac{dn}{d\mathbf{p}_1}\frac{dn}{d\mathbf{p}_2},$$

where $dn/d\mathbf{p}_1 d\mathbf{p}_2$ and $dn/d\mathbf{p}_1$ is the two- and one-particle momentum distribution normalized to unity. It has been repeatedly argued [4] that the correlation function $\mathcal{R}$ can be expressed in the source rest frame in the following way

$$\mathcal{R}(\mathbf{p}_1,\mathbf{p}_2) = \int d^3r_1 dt_1 \int d^3r_2 dt_2 \times \mathcal{D}(\mathbf{r}_1,t_1)\mathcal{D}(\mathbf{r}_2,t_2)\; |\psi(\mathbf{r}'_1,\mathbf{r}'_2)|^2 , \qquad (2)$$

where the source function $\mathcal{D}(\mathbf{r},t)$, which is normalized as $\int d^3r\mathcal{D}(\mathbf{r},t) = 1$, gives the probability to emit a nucleon from a space-time point $(\mathbf{r},t)$ [2]; ψ is the final state wave function of the pair; $\mathbf{r}'_i \equiv \mathbf{r}_i - \mathbf{v}_i t_i$, $i = 1,2$ with $\mathbf{v}_i$ being the particle velocity relative to the source.

Eq. (2) determines the two particle correlation function as an overlap of the source function and the final state wave function squared of the two particles. The pair is assumed to be isolated from the rest of the system not only in the final state, but starting from the moment of *emission* or *freeze-out* when, due to the system decay or expansion, the (strong) interaction is switched off [3]. (If the long range Coulomb force is important after the 'strong' freeze-out, the pair motion in the external electromagnetic field should

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[2] It should be understood here that the coordinates $(\mathbf{r},t)$ determine the position of a particle wave-package center.

[3] The pair cannot be treated as isolated even for the pair, which does not interact with the rest of the system, in the case *many* identical particles. Then, the wave function of the system does not factorize into the pair wave function and the wave function of the rest, since the complete wave function must be (anti-)symmetrized with respect to all particles. However, the effect is significant only when the density of the identical particles is *large*. It does not happen at the currently available energies of nuclear collisions [5].

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be considered [6].) The overlap from Eq. (2) is computed at the freeze-out, which for the pair equals $\max(t_1, t_2)$, and then is averaged over t_1 and t_2. Thus, the correlation function carries the information on the system only at the moment of freeze-out and not at earlier times when the pair of the particles still interacts with the rest system.

Since we are interested in the correlations of particles with 'small' relative momenta, one can factorize the center-of-mass and relative motion of the two particles in the essentially nonrelativistic manner. Then, after eliminating the center-of-mass motion, Eqs. (2) can be rewritten as

$$\mathcal{R}(\mathbf{q}) = \int d^3r dt\, \mathcal{D}_r(\mathbf{r}, t)\; |\phi_{\mathbf{q}}(\mathbf{r}')|^2 , \tag{3}$$

with $\mathcal{D}_r(\mathbf{r}, t)$ being the distribution of the relative space-time position of the two particles,

$$\mathcal{D}_r(\mathbf{r}, t) = \int d^3R\, dT\, \mathcal{D}(\mathbf{R} + \mathbf{r}/2, T + t/2) \times \mathcal{D}(\mathbf{R} - \mathbf{r}/2, T - t/2) .$$

$\phi_{\mathbf{q}}(\mathbf{r}')$ is the nonrelativistic wave function of the relative motion with $\mathbf{q}$ denoting the particle momentum in the center-of-mass frame of the pair. While the particle relative motion is nonrelativistic, the center-of-mass motion with respect to the source is, in general, relativistic. Therefore, the particle relative distance measured in their center-of-mass frame $\mathbf{r}'$ is obtained by means of the Lorentz transformation i.e.

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r}\mathbf{n})\mathbf{n} - \gamma \mathbf{v} t , \tag{4}$$

where $\mathbf{v}$ is the pair velocity with respect to the source, $\mathbf{n} \equiv \mathbf{v}/|\mathbf{v}|$ and γ is the Lorentz factor of the center-of-mass motion relative to the source. The correlation function (3) also depends on the total momentum of the pair. This dependence, which is irrelevant for our considerations, did not show up.

Let us consider the correlation function integrated over the relative momentum. Since $\mathcal{R}(\mathbf{q}) \to 1$ when $\mathbf{q} \to \infty$, we rather discuss the integral of $\mathcal{R}(\mathbf{q}) - 1$. Using Eq. (3) one immediately finds

$$\int \frac{d^3q}{(2\pi)^3} \Big(\mathcal{R}(\mathbf{q}) - 1\Big) = \int d^3r dt\, \mathcal{D}_r(\mathbf{r}, t) \int \frac{d^3q}{(2\pi)^3} \Big(|\phi_{\mathbf{q}}(\mathbf{r}')|^2 - 1\Big) . \tag{5}$$

The wave functions satisfy the completeness condition

$$\int \frac{d^3q}{(2\pi)^3}\, \phi_{\mathbf{q}}(\mathbf{r})\phi^*_{\mathbf{q}}(\mathbf{r}') + \sum_\alpha \phi_\alpha(\mathbf{r})\phi^*_\alpha(\mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \pm \delta^{(3)}(\mathbf{r} + \mathbf{r}') , \tag{6}$$

where ϕ_α represents a bound state of the two particles. When the particles are not identical the second term in the r.h.s of Eq. (6) should be neglected. This term guarantees the right symmetry of both sides of the equation for the case of identical particles. The upper sign is for bosons while the lower one for fermions. The wave function of identical bosons (fermions) $\phi_{\mathbf{q}}(\mathbf{r})$ is (anti-)symmetric when $\mathbf{r} \to -\mathbf{r}$, and the r.h.s of Eq. (6) is indeed (anti-)symmetric when $\mathbf{r} \to -\mathbf{r}$ or $\mathbf{r}' \to -\mathbf{r}'$. If the particles of interests carry spin, the summation over the spin degrees of freedom in the l.h.s of Eq. (6) is implied.

When the integral representation of $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ is used, Eq. (6) can be rewritten as

$$\int \frac{d^3q}{(2\pi)^3} \Big(\phi_{\mathbf{q}}(\mathbf{r})\phi^*_{\mathbf{q}}(\mathbf{r}') - e^{i\mathbf{q}(\mathbf{r}-\mathbf{r}')}\Big) + \sum_\alpha \phi_\alpha(\mathbf{r})\phi^*_\alpha(\mathbf{r}') = \pm\delta^{(3)}(\mathbf{r} + \mathbf{r}') .$$

Now we take the limit $\mathbf{r} \to \mathbf{r}'$ and get the relation

$$\int \frac{d^3q}{(2\pi)^3} \Big(|\phi_{\mathbf{q}}(\mathbf{r}')|^2 - 1\Big) = \pm\, \delta^{(3)}(2\mathbf{r}') - \sum_\alpha |\phi_\alpha(\mathbf{r}')|^2 . \tag{7}$$

When Eq. (7) is substituted into Eq. (5), we get the desired sum rule

$$\int d^3q \Big(\mathcal{R}(\mathbf{q}) - 1\Big) = \pm \frac{\pi^3}{\gamma} \int dt\, \mathcal{D}_r(\mathbf{v}t/\gamma, t) - \sum_\alpha \mathcal{A}_\alpha , \tag{8}$$

where $\mathcal{A}_\alpha$ is the formation rate of a bound state α [7]

$$\mathcal{A}_\alpha = (2\pi)^3 \int d^3r dt\ \mathcal{D}_r(\mathbf{r},t)|\phi_\alpha(\mathbf{r}')|^2 ,$$

which connects the cross section to produce the bound state α carrying the momentum $\mathbf{P}$ with that one of the two particles with the momenta $\mathbf{P}/2$ as

$$\frac{d\sigma^\alpha}{d\mathbf{P}} = \gamma\mathcal{A}_\alpha \frac{d\tilde{\sigma}}{d(\mathbf{P}/2)d(\mathbf{P}/2)} .$$

The tilde means that the short range correlations are removed from the two-particle cross section, which is usually taken as a product of the single-particle cross sections.

If the particles are emitted simultaneously (more precisely, if $\langle\mathbf{r}^2\rangle \gg \langle\mathbf{v}^2t^2\rangle$) the source function is expressed as $\mathcal{D}_r(\mathbf{r},t) = \mathcal{D}_r(\mathbf{r})\ \delta(t)$, and the sum rule simplifies to

$$\int d^3q \left(\mathcal{R}(\mathbf{q}) - 1\right) = \pm\frac{\pi^3}{\gamma}\ \mathcal{D}_r(0) - \sum_\alpha \mathcal{A}_\alpha .$$

The completeness condition is, obviously, valid for any inter-particle interaction. It is also valid when the pair of particles interact with the time-independent external field, e.g. the Coulomb field, generated by the particle source. Thus, the sum rule (8) holds under very general conditions as long as the basic formula (2) is justified, in particular as long as the source function $\mathcal{D}_r(\mathbf{r},t)$ is $\mathbf{q}$-independent and spin independent. The validity of these assumptions can be only tested within a microscopic model of nucleus–nucleus collision. Below we consider three examples of the sum rule (8).

1) The correlation function of identical pions. In this case the sum rule reads

$$\int d^3q \left(\mathcal{R}_{\pi\pi}(\mathbf{q}) - 1\right) = \lambda\frac{\pi^3}{\gamma}\int dt\ \mathcal{D}_r(\mathbf{v}t/\gamma,t) , \tag{9}$$

where we have introduced ad hoc the chaoticity parameter λ. As well known, the interferometric formula (2) gives $\lambda = 1$ in conflict with the experimental data which provide $\lambda < 1$. The sum rule (9) was earlier found by Podgoretzky [9] who used the free wave function of pions and then explicitly integrated the correlation function.

The relation (9) is approximately satisfied by the experimental correlation function. The point is that the data are well described by the free wave functions of the two pions (with the Coulomb correction included) [2], which form the complete set of the quantum states.

2) The p-p and n-n correlation function. The sum rule (8) for the identical nucleons is

$$\int d^3q \left(\mathcal{R}_{NN}(\mathbf{q}) - 1\right) = -\frac{\pi^3}{\gamma}\int dt\ \mathcal{D}_r(\mathbf{v}t/\gamma,t) . \tag{10}$$

This relation, which, in particular, predicts exactly the same (negative) value of the integral of the n-n and p-p correlation function, is *not* satisfied by the experimental data [8]. The reason is probably the following.

The integration over $\mathbf{q}$ runs in the sum rule (10) to infinity. Thus even a small deviation of $\mathcal{R}_{NN}(\mathbf{q})$ from unity at 'large' $\mathbf{q}$ can provide a sizeable contribution to the integral (10). On the other hand, it is a serious problem to normalize the experimental correlation function, and one usually assumes that $\mathcal{R}_{NN}(\mathbf{q}) = 1$ at 'large' $\mathbf{q}$. Consequently, the sum rule (10) can be then easily violated.

3) The n-p correlation function. The sum rule (8) now reads

$$\int d^3q \left(\mathcal{R}_{np}(\mathbf{q}) - 1\right) = -\mathcal{A}_d ,$$

where the correlation function is averaged over spin. As expected, the number of correlated n-p pairs is directly related to the number of the produced deuterons. As in the previous case the sum rule is *not* satisfied by the data [8], and the reason is presumably the same.

We conclude our considerations as follows. Due to the completeness of the quantum states, the correlation functions satisfy the simple relation which, in particular, connects the number of correlated neutron-proton pairs with the number of deuterons produced in nuclear collisions. It appears difficult to apply the sum rule to the experimental data, however the relation is useful, at least, to test theoretical calculations.

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