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GREEN FUNCTION APPROACH TO TRANSPORT THEORY OF SCALAR FIELDS

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The contour Green function technique is used to derive the relativistic transport equations of neutral and charged scalar fields with the interaction lagrangian densities proportional to ϕ^3 and $(\phi\phi^*)^2$, respectively. The mean field and the collision terms in the equations are discussed in detail.

1. Introduction

Transport theory based on kinetic Boltzmann-type equations offers a natural framework to study nonequilibrium phenomena, whereas quantum field theory (QFT) provides the underlying dynamics for most physical systems. Therefore, the Lagrange equations of QFT should be the starting point for a derivation of transport theory, and we shall here perform such an analysis of the relativistic kinetic equations.

There are several fields of application of these equations in cosmology and astrophysics. However, of particular importance are nuclear collisions, where at low and intermediate initial energies one deals with hadron degrees of freedom (see e.g. ref. [1]), while at high energies the dynamics should be described in terms of quarks and gluons [2]. In fact, transport theory methods have been successfully used for a long time in the physics of nucleus–nucleus collisions over a wide range of energies [1, 2], although one usually makes drastic simplifications or approxima-

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tions to make the problem tractable, and quite often ad hoc assumptions are introduced to obtain an adequate description of experimental data. Thus, there is a double motivation for the derivation of the equations: firstly, the derivation shows which assumptions and approximations lead to the transport theory, and hence to what systems the theory can be applied. Secondly, the derivation indicates how to improve the theory by going beyond the approximations. This latter aspect is particularly important in the case of the systems with highly nontrivial dynamics, e.g. quark–gluon plasmas which are governed by quantum chromodynamics and where the simple intuitive arguments usually used to advocate one or another form of transport equation fail, and one has to employ a formal scheme to find the equation [2].

Although we are interested in physical theories like quantum electrodynamics, quantum chromodynamics, or the Walecka model of nuclear forces, we start with much simpler models of scalar fields to elucidate the problem and to form a basis for future considerations. Therefore, our present study is mainly of methodological character.

The derivation of relativistic transport equations by means of so-called reduction formulae has been described in detail by de Groot et al. [3]. In our study we use the Green function technique developed by Kadanoff and Baym [4], see also ref. [5]. There are only a few papers where this technique has been applied to relativistic systems: DuBois and Bezzerides have studied the electrodynamic plasma [6], the Walecka model has been analyzed by Li et al. [8] and by Botermans and Malfliet [8], and Li and McLerran considered the ϕ^3 model [7]. However, we find the studies [7, 8] far from complete. Recently, the $(\phi\phi^*)^2$ model has been carefully studied by Calzetta and Hu [9]. In a somewhat different context the transport of scalar fields has been also discussed by Carruthers and Zachariasen [10] and by Cooper and Feigenbaum [11].

The general scheme of our derivation is as follows: we start with the definition of the contour Green function, the time arguments of which lie on the contour in a space of complex time. After discussing its properties and relevance to the study of nonequilibrium systems, we write down the exact Green function equations of motion – the Dyson–Schwinger equations, which are the direct consequence of the Lagrange field equations. Assuming the macroscopic quasi-homogeneity of the system described, we perform the gradient expansion of the Dyson–Schwinger equations, i.e. expand in macroscopic variations. To make the resulting equations useful we must approximate the self-energies. Two types of approximations are discussed – the so-called pairing approximation and a perturbative expansion in the coupling constant. We discuss the latter case in detail and find significant differences between the ϕ^3 and $(\phi\phi^*)^2$ models. The problem of renormalization is treated rather superficially since the only divergent expressions, which explicitly appear in our considerations are those related to tadpole diagrams. These infinite terms can be removed by means of simple *physical* arguments without referring to

a more formal renormalization approach. We are not faced with the characteristic self-energy divergences known from vacuum field theory, because in our scheme only the finite imaginary parts of self-energies enter the quantity of interest. Further in our derivation we define the distribution functions and discuss their connection with the Green functions. Finally, we obtain the transport equations satisfied by the distribution functions. With the aid of the definitions introduced in sects. 2 and 3, more concrete formulation of the problem studied in this paper is given at the end of sect. 3.

In summary, we develop a scheme to derive the relativistic transport equations from QFT, and apply it to the simplest case of scalar fields. Although the scheme is a first approximation to the problem, it provides an insight into transport theory, which might be very useful in studies of more physical systems. Our work also suggests a method to improve the scheme to obtain more general transport equations.

In this article we use the units where $\hbar = c = 1$. The signature of the metric tensor is $(+,-,-,-)$.

2. Preliminaries

Since the lagrangian density is the starting point of QFT considerations let us begin by writing down the lagrangians of the models of interest. We choose the lagrangian density of the scalar neutral (real) field $\phi(x)$ in the standard form [12]

$$L(x) = \tfrac{1}{2}\partial^\mu\phi\partial_\mu\phi - \tfrac{1}{2}m^2\phi^2 + g\frac{1}{3!}\phi^3, \tag{2.1}$$

where m is the mass of the boson represented by the field ϕ and g is the coupling constant. The lagrangian density of the charged (complex) field is [12]

$$L(x) = \partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi + g\frac{1}{2!2!}(\phi\phi^*)^2. \tag{2.2}$$

The lagrangians (2.1) and (2.2) lead to the field equations

$$\left[\partial^\mu\partial_\mu + m^2\right]\phi(x) = g\tfrac{1}{2}\phi^2(x) \tag{2.3}$$

for a neutral field, and

$$\left[\partial^\mu\partial_\mu + m^2\right]\phi(x) = g\tfrac{1}{2}\phi^2(x)\phi^*(x), \tag{2.4a}$$

$$\left[\partial^\mu\partial_\mu + m^2\right]\phi^*(x) = g\tfrac{1}{2}\phi(x)\phi^{*2}(x), \tag{2.4b}$$

for charged fields.

Due to the invariance of the lagrangian (2.2) under U(1) global transformations, there is a conserved current which reads

$$j_\mu(x) = i\phi^*(x)\overleftrightarrow{\partial}_\mu\phi(x)\,. \tag{2.5}$$

Let us also write down the energy–momentum tensors of neutral and charged fields, respectively:

$$T^{\mu\nu}(x) = \partial^\mu\phi(x)\partial^\nu\phi(x) - g^{\mu\nu}L(x)\,,$$

$$T^{\mu\nu}(x) = \partial^\mu\phi^*(x)\partial^\nu\phi(x) + \partial^\nu\phi^*(x)\partial^\mu\phi(x) - g^{\mu\nu}L(x)\,.$$

Subtracting from the above energy–momentum tensors the total derivative terms

$$\tfrac{1}{4}\partial^\mu\partial^\nu\big(\phi^2(x)\big) - g^{\mu\nu}\tfrac{1}{4}\partial^\sigma\partial_\sigma\big(\phi^2(x)\big)$$

and

$$\tfrac{1}{2}\partial^\mu\partial^\nu\big(\phi^*(x)\phi(x)\big) - g^{\mu\nu}\tfrac{1}{4}\partial^\sigma\partial_\sigma\big(\phi^*(x)\phi(x)\big)\,,$$

respectively, we get the energy–momentum tensors which for noninteracting fields are of the form convenient for our purposes:

$$T_0^{\mu\nu}(x) = -\tfrac{1}{4}\phi(x)\overleftrightarrow{\partial}^\mu\overleftrightarrow{\partial}^\nu\phi(x)\,, \qquad T_0^{\mu\nu}(x) = -\tfrac{1}{2}\phi^*(x)\overleftrightarrow{\partial}^\mu\overleftrightarrow{\partial}^\nu\phi(x)\,, \tag{2.6a,b}$$

where the fields are assumed to satisfy eqs. (2.3) and (2.4), respectively.

The systems of scalar fields are quantized by postulating the following commutation relations [12]

$$\big[\dot\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})\big] = -i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,,$$

$$\big[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})\big] = 0$$

for real fields (the dot denotes the time derivative), and

$$\big[\dot\phi^*(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})\big] = -i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,,$$

$$\big[\phi(x),\phi(y)\big] = 0\,, \qquad \big[\phi^*(x),\phi^*(y)\big] = 0$$

for complex fields.

Let us also introduce the singular operator $D(x,y)$ defined for real and complex fields, respectively, as

$$\big[\phi(x),\phi(y)\big] = iD(x,y)\,, \qquad \big[\phi^*(x),\phi(y)\big] = iD(x,y)\,, \tag{2.7), (2.8}$$

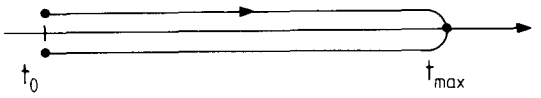

Fig. 1. The contour along the time axis for an evaluation of the operator expectation values.

which for free fields is a c-number function, and it equals [12]

$$iD(x,y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} 2\pi\delta(k^2 - m^2)(\Theta(k_0) - \Theta(-k_0))$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega} (e^{-ik(x-y)} - e^{ik(x-y)}), \tag{2.9}$$

where $k^\mu = (\omega, \boldsymbol{k})$ and $\omega = (m^2 + \boldsymbol{k}^2)^{1/2}$.

3. Green functions, their properties and interpretation

The central role in our considerations is played by the contour Green function

$$i\Delta(x,y) \stackrel{\text{def}}{=} \langle \tilde{T}\phi(x)\phi^*(y)\rangle - \langle\phi(x)\rangle\langle\phi^*(y)\rangle, \tag{3.1}$$

where $\langle \ldots \rangle$ denotes the average over an ensemble at time t_0, which is usually identified with minus infinity; $\tilde{T}$ is the time-ordering operation on the directed contour shown in fig. 1. The parameter t_{max} is shifted to $+\infty$ in calculations. An infinitesimal positive or negative imaginary parameter is attributed to the real-time arguments locating them on the upper or on the lower branch of the contour. The ordering operation is defined as

$$\tilde{T}\phi(x)\phi^*(y) \stackrel{\text{def}}{=} \Theta(x_0, y_0)\phi(x)\phi^*(y) + \Theta(y_0, x_0)\phi^*(y)\phi(x),$$

where $\Theta(x_0, y_0)$ equals 1 if x_0 succeeds y_0 on the contour, and it equals 0 when x_0 precedes y_0. The need to introduce contour Green functions for the description of nonequilibrium systems is discussed in e.g. ref. [5]. As seen from eq. (3.1) the product of the field expectation (classical) values is subtracted from $\langle \tilde{T}\phi(x)\phi^*(y)\rangle$ in the Green function's definition. Therefore, the Green function corresponds to field fluctuations around the classical value.

Apart from the function (3.1) we use four other functions with pure real-time arguments defined as follows:

$$i\Delta^{>}(x,y) \stackrel{\text{def}}{=} \langle\phi(x)\phi^{*}(y)\rangle - \langle\phi(x)\rangle\langle\phi^{*}(y)\rangle, \tag{3.2}$$

$$i\Delta^{<}(x,y) \stackrel{\text{def}}{=} \langle\phi^{*}(y)\phi(x)\rangle - \langle\phi(x)\rangle\langle\phi^{*}(y)\rangle, \tag{3.3}$$

$$i\Delta^{\mathrm{c}}(x,y) \stackrel{\text{def}}{=} \langle T^{\mathrm{c}}\phi(x)^{*}(y)\rangle - \langle\phi(x)\rangle\langle\phi^{*}(y)\rangle, \tag{3.4}$$

$$i\Delta^{\mathrm{a}}(x,y) \stackrel{\text{def}}{=} \langle T^{\mathrm{a}}\phi(x)\phi^{*}(y)\rangle - \langle\phi(x)\rangle\langle\phi^{*}(y)\rangle, \tag{3.5}$$

where T^{c} is chronological time ordering and T^{a} is antichronological time ordering defined as

$$T^{\mathrm{c}}\phi(x)\phi^{*}(y) \stackrel{\text{def}}{=} \Theta(x_0-y_0)\phi(x)\phi^{*}(y) + \Theta(y_0-x_0)\phi^{*}(y)\phi(x),$$

$$T^{\mathrm{a}}\phi(x)\phi^{*}(y) \stackrel{\text{def}}{=} \Theta(y_0-x_0)\phi(x)\phi^{*}(y) + \Theta(x_0-y_0)\phi^{*}(y)\phi(x).$$

The definitions of the Green functions for the real fields are analogous to eqs. (3.1)–(3.5) with $\phi^{*}(y)$ being replaced everywhere by $\phi(y)$. The functions (3.2)–(3.5) are related to $\Delta(x,y)$ in the following manner

$$\Delta^{\mathrm{c}}(x,y) \equiv \Delta(x,y) \quad \text{for } x_0, y_0 \text{ from the upper branch}, \tag{3.6}$$

$$\Delta^{\mathrm{a}}(x,y) \equiv \Delta(x,y) \quad \text{for } x_0, y_0 \text{ from the lower branch}, \tag{3.7}$$

$$\Delta^{>}(x,y) \equiv \Delta(x,y) \quad \text{for } x_0 \text{ from the upper branch and } y_0 \text{ from the lower one}, \tag{3.8}$$

$$\Delta^{<}(x,y) \equiv \Delta(x,y) \quad \text{for } x_0 \text{ from the lower branch and } y_0 \text{ from the upper one}. \tag{3.9}$$

One further finds the following relations

$$\Delta^{\mathrm{c}}(x,y) = \Theta(x_0-y_0)\Delta^{>}(x,y) + \Theta(y_0-x_0)\Delta^{<}(x,y), \tag{3.10}$$

$$\Delta^{\mathrm{a}}(x,y) = \Theta(y_0-x_0)\Delta^{>}(x,y) + \Theta(x_0-y_0)\Delta^{<}(x,y), \tag{3.11}$$

$$(i\Delta^{>}(x,y))^{\dagger} = i\Delta^{>}(x,y), \qquad (i\Delta^{<}(x,y))^{\dagger} = i\Delta^{<}(x,y), \tag{3.12, 3.13}$$

$$(i\Delta^{\mathrm{a}}(x,y))^{\dagger} = i\Delta^{\mathrm{c}}(x,y), \tag{3.14}$$

where † denotes hermitian conjugation, i.e. complex conjugation with an exchange of the Green function arguments. Because of the relations (2.7) and (2.8) the following identity holds:

$$\Delta^{>}(x,y) - \Delta^{<}(x,y) = \langle D(x,y)\rangle . \tag{3.15}$$

For the real fields the Green functions have a specific property

$$\Delta^{>}(x,y) = \Delta^{<}(y,x) . \tag{3.16}$$

The Green function $\Delta^{c}(x,y)$ describes the propagation of a disturbance in which a single particle is added to the many-particle system at space-time point y and then is removed from it at a space-time point x. An antiparticle disturbance is propagated backward in time. The meaning of $\Delta^{a}(x,y)$ is analogous but particles are propagated backward in time and antiparticles forward. In the zero-density limit $\Delta^{c}(x,y)$ coincides with the Feynman propagator [12].

The physical interpretation of functions $\Delta^{>}(x,y)$ and $\Delta^{<}(x,y)$ is more transparent when one considers Wigner transforms defined as

$$\Delta^{>(<)}(X,p) \overset{\text{def}}{=} \int \mathrm{d}^4u\, \mathrm{e}^{ipu}\, \Delta^{>(<)}\big(X+\tfrac{1}{2}u, X-\tfrac{1}{2}u\big) . \tag{3.17}$$

One then finds that the current (2.5) and free-field energy–momentum tensors (2.6) averaged over the ensemble can be expressed as

$$j^{\mu}(X) = -\int \frac{\mathrm{d}^4p}{(2\pi)^4} p^{\mu} i\Delta^{<}(X,p) + i\langle\phi^*(X)\rangle \overleftrightarrow{\partial_{\mu}} \langle\phi(X)\rangle , \tag{3.18}$$

$$T_0^{\mu\nu}(X) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} p^{\mu}p^{\nu} i\Delta^{<}(X,p) - \tfrac{1}{4}\langle\phi(X)\rangle \overleftrightarrow{\partial^{\mu}}\overleftrightarrow{\partial^{\nu}} \langle\phi(X)\rangle , \tag{3.19a}$$

$$T_0^{\mu\nu}(X) = 2\int \frac{\mathrm{d}^4p}{(2\pi)^4} p^{\mu}p^{\nu} i\Delta^{<}(X,p) - \tfrac{1}{2}\langle\phi^*(X)\rangle \overleftrightarrow{\partial^{\mu}}\overleftrightarrow{\partial^{\nu}} \langle\phi(X)\rangle . \tag{3.19b}$$

Eq. (3.19a) applies to the real fields while eq. (3.19b) applies to the complex ones. The meaning of the terms containing $\langle\phi\rangle$ is evident. They describe the current, or the energy–momentum tensor related to the nonvanishing expectation (classical) values of the fields, or in other words, the current and the energy–momentum carried by *classical* fields.

From eqs. (3.18) and (3.19) one sees that $i\Delta^<(X,p)$ corresponds to the density of particles with four-momentum p in a space-time point X, and consequently, it is a quantum analog of a classical distribution function. This interpretation is supported by the fact that $i\Delta^<(X,p)$ is hermitian [eq. (3.13)], however it is not positively definite and the probabilistic interpretation is only approximately valid. One should also observe that, in contrast to the classical distribution functions, $i\Delta^<(X,p)$ can be nonzero for the off-shell four-momenta.

Now we are in the position to explain the objective of our considerations. Starting from the Lagrange equations (2.3) and (2.4) we are going to derive the equations which describe the evolution of Green functions $\Delta^{>(<)}$, and then convert the obtained equations into the transport equations satisfied by distribution functions. Our aim is realized in several steps. First we write down the exact equation of the function $\Delta(x,y)$. This equation gives us the equations of $\Delta^{>(<)}$. We approximate the self-energies which enter these equations and we achieve our goal on expanding the equations in macroscopic variations. The discussion of the self-energy approximations constitutes the essential part of our considerations. In sect. 10 we discuss the assumptions and approximations leading to the transport equations.

4. Green function equations of motion

From eqs. (2.3) and (2.4) and definition (3.1) one finds two equations of motion of the contour Green function

$$\left[\partial_x^\mu\partial_\mu^x + m^2\right]\Delta(x,y) = -\delta^{(4)}(x,y) + \int_C \mathrm{d}^4x'\,\Pi(x,x')\Delta(x',y)\,, \tag{4.1}$$

$$\left[\partial_y^\mu\partial_\mu^y + m^2\right]\Delta(x,y) = -\delta^{(4)}(x,y) + \int_C \mathrm{d}^4x'\,\Delta(x,x')\Pi(x',y)\,, \tag{4.2}$$

where the integration over x_0' is performed on the contour. The function $\delta^{(4)}(x,y)$ is defined on the contour as

$$\delta^{(4)}(x,y) = \begin{cases} \delta^{(4)}(x-y) & \text{for } x_0, y_0 \text{ from the upper branch}, \\ 0 & \text{for } x_0, y_0 \text{ from the different branches}, \\ -\delta^{(4)}(x-y) & \text{for } x_0, y_0 \text{ from the lower branch}. \end{cases}$$

The self-energy $\Pi(x, x')$ can be defined as

$$\int_C \mathrm{d}^4x'\, \Pi(x,x')\Delta(x',y) \stackrel{\mathrm{def}}{=} -ig\tfrac{1}{2}\big(\langle \tilde{T}\phi^2(x)\phi(y)\rangle - \langle\phi^2(x)\rangle\langle\phi(y)\rangle\big) \quad (4.3a)$$

for the real fields, and

$$\int_C \mathrm{d}^4x'\, \Pi(x,x')\Delta(x',y) \stackrel{\mathrm{def}}{=} -ig\tfrac{1}{2}\big(\langle \tilde{T}\phi^2(x)\phi^*(x)\phi^*(y)\rangle - \langle\phi^2(x)\phi^*(x)\rangle\langle\phi^*(y)\rangle\big) \quad (4.3b)$$

for the complex fields. Because the Green function of the free fields $\Delta_0(x, y)$ satisfies the equation

$$\big[\partial_x^\mu\partial_\mu^x + m^2\big]\Delta_0(x,y) = -\delta^{(4)}(x,y)$$

we can now rewrite eqs. (4.1) and (4.2) in the symbolic operator notation

$$\Delta_0^{-1}\Delta = 1 - \Pi\Delta, \qquad \Delta_0^{-1}\Delta = 1 - \Delta\Pi$$

and obtain the familiar form of the Dyson–Schwinger equation

$$\Delta = \Delta_0 - \Delta_0\Pi\Delta\,. \quad (4.4)$$

Let us split the self-energy into three parts according to the formula

$$\Pi(x,y) = \Pi_{\mathrm{MF}}(x)\delta^{(4)}(x,y) + \Pi^{>}(x,y)\Theta(x_0,y_0) + \Pi^{<}(x,y)\Theta(y_0,x_0)\,. \quad (4.5)$$

As we shall see later, Π_{MF} corresponds to the mean-field effects while $\Pi^{>(<)}$ provides the collision terms in the transport equations. Therefore, we call Π_{MF} the mean-field self-energy and $\Pi^{>(<)}$ the collisional self-energy.

Using the relations (3.8), (3.9) and eq. (4.5), eqs. (4.1) and (4.2) can be manipulated to yield the equations of motion of the functions $\Delta^>$ and $\Delta^<$, known

as the Kadanoff–Baym equations [4]:

$$\left[\partial_x^\mu \partial_\mu^x + m^2 - \Pi_{\mathrm{MF}}(x)\right]\Delta^>(x,y) = \int_{-\infty}^{y_0} \mathrm{d}^4x'\, \Pi^>(x,x')[\Delta^<(x',y) - \Delta^>(x',y)]$$

$$+ \int_{-\infty}^{x_0} \mathrm{d}^4x'[\Pi^>(x,x') - \Pi^<(x,x')]\Delta^>(x',y), \tag{4.6}$$

$$\left[\partial_x^\mu \partial_\mu^x + m^2 - \Pi_{\mathrm{MF}}(x)\right]\Delta^<(x,y) = \int_{-\infty}^{y_0} \mathrm{d}^4x'\, \Pi^<(x,x')[\Delta^<(x,y') - \Delta^>(x',y)]$$

$$+ \int_{-\infty}^{x_0} \mathrm{d}^4x'[\Pi^>(x,x') - \Pi^<(x,x')]\Delta^<(x',y), \tag{4.7}$$

$$\left[\partial_y^\mu \partial_\mu^y + m^2 - \Pi_{\mathrm{MF}}(y)\right]\Delta^>(x,y) = \int_{-\infty}^{y_0} \mathrm{d}^4x'\, \Delta^>(x,x')[\Pi^<(x',y) - \Pi^>(x',y)]$$

$$+ \int_{-\infty}^{x_0} \mathrm{d}^4x'[\Delta^>(x,x') - \Delta^<(x,x')]\Pi^>(x',y), \tag{4.8}$$

$$\left[\partial_y^\mu \partial_\mu^y + m^2 - \Pi_{\mathrm{MF}}(y)\right]\Delta^<(x,y) = \int_{-\infty}^{y_0} \mathrm{d}^4x'\, \Delta^<(x,x')[\Pi^<(x',y) - \Pi^>(x',y)]$$

$$+ \int_{-\infty}^{x_0} \mathrm{d}^4x'[\Delta^>(x,x') - \Delta^<(x,x')]\Pi^<(x',y), \tag{4.9}$$

where the integration limits are for the time variable.

Upon defining the functions

$$\Pi^+(x,y) \stackrel{\mathrm{def}}{=} (\Pi^>(x,y) - \Pi^<(x,y))\Theta(x_0 - y_0), \tag{4.10}$$

$$\Pi^-(x,y) \stackrel{\mathrm{def}}{=} -(\Pi^>(x,y) - \Pi^<(x,y))\Theta(y_0 - x_0) \tag{4.11}$$

and functions Δ^+ and Δ^- in the analogous way, the Kadanoff–Baym equations

can be rewritten in the form which is more appropriate for our purposes

$$\left[\partial_x^\mu \partial_\mu^x + m^2 - \Pi_{\mathrm{MF}}(x)\right] \Delta^>(x,y)$$

$$= \int \mathrm{d}^4x' \left[\Pi^>(x,x')\Delta^-(x',y) + \Pi^+(x,x')\Delta^>(x',y)\right], \qquad (4.12)$$

$$\left[\partial_x^\mu \partial_\mu^x + m^2 - \Pi_{\mathrm{MF}}(x)\right] \Delta^<(x,y)$$

$$= \int \mathrm{d}^4x' \left[\Pi^+(x,x')\Delta^<(x',y) + \Pi^<(x,x')\Delta^-(x',y)\right], \qquad (4.13)$$

$$\left[\partial_y^\mu \partial_\mu^y + m^2 - \Pi_{\mathrm{MF}}(y)\right] \Delta^>(x,y)$$

$$= \int \mathrm{d}^4x' \left[\Delta^>(x,x')\Pi^-(x',y) + \Delta^+(x,x')\Pi^>(x',y)\right], \qquad (4.14)$$

$$\left[\partial_y^\mu \partial_\mu^y + m^2 - \Pi_{\mathrm{MF}}(y)\right] \Delta^<(x,y)$$

$$= \int \mathrm{d}^4x' \left[\Delta^+(x,x')\Pi^<(x',y) + \Delta^<(x,x')\Pi^-(x',y)\right], \qquad (4.15)$$

where the time integration runs from $-\infty$ to $+\infty$. Let us stress that the Kadanoff–Baym equations (4.6)–(4.9) or (4.12)–(4.15) are exact and they are equivalent to the field equations of motion.

For further convenience we also write down the equations satisfied by the functions $\Delta^\pm$

$$\left[\partial_x^\mu \partial_\mu^x + m^2\right] \Delta^\pm(x,y) = -\delta^{(4)}(x-y) + \int \mathrm{d}^4x' \, \Pi^\pm(x,x')\Delta^\pm(x',y), \qquad (4.16)$$

$$\left[\partial_y^\mu \partial_\mu^y + m^2\right] \Delta^\pm(x,y) = -\delta^{(4)}(x-y) + \int \mathrm{d}^4x' \, \Delta^\pm(x,x')\Pi^\pm(x',y). \qquad (4.17)$$

5. Towards transport equations

The transport equations are derived under the assumption that the Green functions $\Delta(x,y)$ and the self-energies $\Pi(x,y)$ weakly depend on the sums of arguments and that they are significantly different from zero only when the differences of arguments are close to zero. To express these properties it is convenient to define a new set of variables as

$$\Delta(X,u) \equiv \Delta\left(X - \tfrac{1}{2}u, X + \tfrac{1}{2}u\right).$$

Then, let us assume that $\Delta(X,u)$ and $\Pi(X,u)$ vary slowly with X and are strongly *peaked* for $u \cong 0$. Due to this assumption one can, in particular, approximate $\Delta(X+u,u)$ as

$$\Delta(X+u,u) \cong \Delta(X,u) + u^\mu \frac{\partial}{\partial X^\mu}\Delta(X,u)\,. \tag{5.1}$$

This assumption is discussed in sect. 11, where we analyze the whole procedure of the derivation of kinetic theory.

The Kadanoff–Baym equations are converted into transport equations implementing the above approximation and performing the Wigner transformation (3.17) of all Green functions and self-energies.

The approximation is introduced and the transformation is performed automatically by means of the following easily derivable formulas

$$\int \mathrm{d}^4x' f(x,x')g(x',y)$$

$$\to f(X,p)g(X,p) + i\frac{1}{2}\left[\frac{\partial f(X,p)}{\partial p_\mu}\frac{\partial g(X,p)}{\partial X^\mu} - \frac{\partial f(X,p)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu}\right], \tag{5.2}$$

$$h(x)g(x,y) \to h(X)g(X,p) - i\frac{1}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu}\,, \tag{5.3}$$

$$h(y)g(x,y) \to h(X)g(X,p) + i\frac{1}{2}\frac{\partial h(X)}{\partial X^\mu}\frac{\partial g(X,p)}{\partial p_\mu}\,, \tag{5.4}$$

$$\partial_x^\mu f(x,y) \to \left(-ip^\mu + \tfrac{1}{2}\partial^\mu\right)f(X,p)\,, \tag{5.5}$$

$$\partial_y^\mu f(x,y) \to \left(ip^\mu + \tfrac{1}{2}\partial^\mu\right)f(X,p)\,, \tag{5.6}$$

where $X=(x+y)/2$ and the functions $f(x,y)$ and $g(x,y)$ satisfy the assumption discussed above.

Using the formulas (5.2)–(5.6) one can change eqs. (4.12)–(4.15) into the form

$$\left[\tfrac{1}{4}\partial^\mu\partial_\mu - ip^\mu\partial_\mu - p^2 + m^2 - \Pi_{\mathrm{MF}}(X) + i\tfrac{1}{2}\partial_\mu\Pi_{\mathrm{MF}}(X)\partial_p^\mu\right]\Delta^>(X,p)$$
$$= \Pi^>(X,p)\Delta^-(X,p) + \Pi^+(X,p)\Delta^>(X,p)\,, \tag{5.7}$$

$$\left[\tfrac{1}{4}\partial^\mu\partial_\mu - ip^\mu\partial_\mu - p^2 + m^2 - \Pi_{\mathrm{MF}}(X) + i\tfrac{1}{2}\partial_\mu\Pi_{\mathrm{MF}}(X)\partial_p^\mu\right]\Delta^<(X,p)$$
$$= \Pi^+(X,p)\Delta^<(X,p) + \Pi^<(X,p)\Delta^-(X,p)\,, \tag{5.8}$$

$$\left[\tfrac{1}{4}\partial^\mu\partial_\mu + ip^\mu\partial_\mu - p^2 + m^2 - \Pi_{\mathrm{MF}}(X) - i\tfrac{1}{2}\partial_\mu\Pi_{\mathrm{MF}}(X)\partial_p^\mu\right]\Delta^>(X,p)$$
$$= \Delta^>(X,p)\Pi^-(X,p) + \Delta^+(X,p)\Pi^>(X,p)\,, \tag{5.9}$$

$$\left[\tfrac{1}{4}\partial^\mu\partial_\mu + ip^\mu\partial_\mu - p^2 + m^2 - \Pi_{\mathrm{MF}}(X) - i\tfrac{1}{2}\partial_\mu\Pi_{\mathrm{MF}}(X)\partial_p^\mu\right]\Delta^<(X,p)$$
$$= \Delta^+(X,p)\Pi^<(X,p) + \Delta^<(X,p)\Pi^-(X,p)\,. \tag{5.10}$$

On the right-hand-sides of eqs. (5.7)–(5.10) we have neglected the gradient terms like those from eq. (5.2). This approximation is discussed in sect. 11.

Subtracting eq. (5.9) from eq. (5.7) and eq. (5.10) from eq. (5.8), respectively, one finds

$$\left[p^\mu \partial_\mu - \tfrac{1}{2}\partial_\mu \Pi_{\mathrm{MF}}(X)\partial_p^\mu \right] \Delta^{>(<)}(X,p)$$

$$= i\tfrac{1}{2}\left[\Pi^{<}(X,p)\Delta^{>}(X,p) - \Pi^{>}(X,p)\Delta^{<}(X,p) \right], \qquad (5.11)$$

where we have used the equality

$$\Pi^{+}(X,p) - \Pi^{-}(X,p) = \Pi^{>}(X,p) - \Pi^{<}(X,p)$$

and an analogous equality for Δ^{+} and Δ^{-}, which follow from the definitions (4.10) and (4.11).

Summing eq. (5.7) with eq. (5.9), and eq. (5.8) with eq. (5.10), respectively, one finds

$$\left[\tfrac{1}{4}\partial^\mu \partial_\mu - p^2 + m^2 - \Pi_{\mathrm{MF}}(X) \right] \Delta^{>(<)}(X,p)$$

$$= \tfrac{1}{2}\left[\Pi^{>(<)}(X,p)\left[\Delta^{+}(X,p) + \Delta^{-}(X,p) \right] \right.$$

$$\left. + \left[\Pi^{+}(X,p) + \Pi^{-}(X,p) \right] \Delta^{>(<)}(X,p) \right]. \qquad (5.12)$$

The sum $\Delta^{+}(X,p) + \Delta^{-}(X,p)$ can be expressed as

$$\Delta^{+}(X,p) + \Delta^{-}(X,p) = \frac{1}{\pi i} \mathrm{P} \int \mathrm{d}\omega' \, \frac{\Delta^{>}(X,\omega',\boldsymbol{p}) - \Delta^{<}(X,\omega',\boldsymbol{p})}{\omega' - \omega} .$$

There is also an analogous formula for the self-energies.

Eqs. (5.11) and (5.12) are called the transport equation and the mass-shell equation, respectively. In order to make them definite one has to determine the self-energies. This is the subject of sects. 6, 7 and 8.

The conservation laws satisfied in the models represented by the lagrangian densities (2.1) and (2.2) impose certain restrictions on the approximations in eqs. (5.11) and (5.12).

Let us first discuss the conservation of current (2.5), that can be represented as in eq. (3.18). The two parts of current (3.18) – the *fluctuation* part and the *classical* part, are conserved separately when

$$\langle \phi^*(x) \rangle \langle \phi^2(x) \phi^*(x) \rangle - \langle \phi(x) \rangle \langle \phi(x) \phi^{*2}(x) \rangle = 0 .$$

This relation holds in the case of the pairing approximation discussed in sect. 6

and also follows from the perturbative expansion analyzed in sects. 7 and 8. From now on we assume that the above relation holds.

Since the function $\Delta^>(X,p)$ satisfies eq. (5.11), the conservation of the *fluctuation* current leads to the requirement

$$\int \frac{d^4p}{(2\pi)^4}\left[\Pi^<(X,p)\Delta^>(X,p)-\Pi^>(X,p)\Delta^<(X,p)\right]=0. \quad (5.13)$$

Because we have assumed that the functions $\Delta^{>(<)}(X,p)$ vanish at infinite four-momenta, the mean-field self-energy has been eliminated from eq. (5.13).

In kinetic theory one usually assumes [3] that the energy-momentum tensor can be approximated only by the kinetic part, which is given by eq. (2.6). Then, the energy–momentum conservation provides the relation

$$\int \frac{d^4p}{(2\pi)^4}p^\mu\left[\Pi^<(X,p)\Delta^>(X,p)-\Pi^>(X,p)\Delta^<(X,p)\right]=0. \quad (5.14)$$

Before analyzing further eqs. (5.11) and (5.12) let us briefly discuss the noninteracting fields. In this case the transport and mass-shell equations are

$$p^\mu\partial_\mu\Delta^{>(<)}(X,p)=0, \quad (5.15)$$

$$\left[\tfrac{1}{4}\partial^\mu\partial_\mu-p^2+m^2\right]\Delta^{>(<)}(X,p)=0. \quad (5.16)$$

However, it should be stressed that for free fields eqs. (5.15) and (5.16) can be obtained from the Lagrange equations without using the approximations discussed at the beginning of this section.

Eq. (5.15) might be identified with the classical relativistic kinetic equation (see e.g. [3]), however, following eq. (5.16), the Green function $\Delta^{>(<)}(X,p)$ can be nonzero for the off-shell four-momenta. Nonetheless, the kinetic theory deals with the system characteristics averaged over scales larger than the particle Compton wavelength of the order of m^{-1}. Formally, we impose the condition

$$|\Delta^{>(<)}(X,p)| \gg \left|\frac{1}{m^2}\partial^\mu\partial_\mu\Delta^{>(<)}(X,p)\right|, \quad (5.17)$$

which is further discussed in sect. 11. We then get from (5.16)

$$(p^2-m^2)\Delta^{>(<)}(X,p)=0. \quad (5.18)$$

Condition (5.17) makes the Green functions $\Delta^{>(<)}(X,p)$ zero for the four-momenta which do not satisfy the mass-shell equation $p^2=m^2$.

Let us also discuss the equations of Green functions $\Delta^{c(a)}(X,p)$. For the noninteracting fields the equations analogous to (5.15) and (5.16) are

$$p^\mu \partial_\mu \Delta^c(X,p) = 0, \qquad \left[\tfrac{1}{4}\partial^\mu \partial_\mu - p^2 + m^2\right]\Delta^c(X,p) = -1. \quad (5.19), (5.20)$$

For the function $\Delta^a(X,p)$ the right-hand-side of an equation analogous to (5.20) equals 1. Let us also mention that the functions $\Delta^{c(a)}(X,p)$ are X-independent in the vacuum QFT [12] because of the translational invariance assumed. Imposing the condition (5.17) on the functions $\Delta^{c(a)}(X,p)$, the solutions of eqs. (5.19) and (5.20) and the analogous solutions for function $\Delta^a(X,p)$, can be written as

$$\Delta^c(X,p) = \frac{1}{p^2 - m^2 + i0^+} + \Theta(-p_0)\Delta^>(X,p) + \Theta(p_0)\Delta^<(X,p), \quad (5.21a)$$

$$\Delta^a(X,p) = \frac{-1}{p^2 - m^2 - i0^+} + \Theta(-p_0)\Delta^>(X,p) + \Theta(p_0)\Delta^<(X,p), \quad (5.21b)$$

where the functions $\Delta^{>(<)}(X,p)$ satisfy the equations (5.15) and (5.18). The solution (5.21a) admits the standard Feynman propagator initial conditions [12]. The form of the solutions (5.21) has been chosen to satisfy the relations (3.10)–(3.14).

From the solutions (5.21) one can also find that

$$\Delta^+(X,p) = \frac{1}{p^2 - m^2 + ip_0 0^+}, \qquad \Delta^-(X,p) = \frac{1}{p^2 - m^2 - ip_0 0^+}, \quad (5.22a, b)$$

where we have used the identities

$$\Delta^+(X,p) + \Delta^-(X,p) = \Delta^c(X,p) - \Delta^a(X,p)$$

$$\Delta^+(X,p) - \Delta^-(X,p) = \Delta^>(X,p) - \Delta^<(X,p), \quad (5.23)$$

which directly follow from the definitions of functions Δ^+ and Δ^-. In the case of noninteracting fields, the relation (3.15) can be written as

$$i\Delta^>(X,p) - i\Delta^<(X,p) = 2\pi\delta(p^2 - m^2)(\Theta(p_0) - \Theta(-p_0)), \quad (5.24)$$

cf. eq. (2.9).

6. The pairing approximation

Let us now go back to eq. (4.3) defining the self-energies. Following refs. [10, 11] we approximate the expectation values of the field operator products with the sum

of products of expectation values of products of no more than two operators, i.e. with the sum of all possible pairings. Physically it means that interparticle correlations are neglected. Then, for the real fields one finds

$$\langle \tilde{T}\phi^2(x)\phi(y)\rangle \cong \langle\phi(x)\rangle^2\langle\phi(y)\rangle + \big(\langle\phi^2(x)\rangle - \langle\phi(x)\rangle^2\big)\langle\phi(y)\rangle + 2\langle\phi(x)\rangle\big(\langle\tilde{T}\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle\big) \tag{6.1}$$

and

$$\int_C \mathrm{d}^4x'\,\Pi(x,x')\Delta(x',y) = gA(x)\Delta(x,y)\,,$$

where

$$A(x) \equiv \langle\phi(x)\rangle\,.$$

Therefore

$$\Pi(x,y) = g\delta^{(4)}(x,y)A(x)$$

and one finds by means of eq. (4.5), that the approximation (6.1) gives

$$\Pi_{\mathrm{MF}}(x) = gA(x) \quad \text{and} \quad \Pi^{>}(x,y) = \Pi^{<}(x,y) = 0\,. \tag{6.2}$$

The analogous approximation to (6.1) for the complex field reads

$$\begin{aligned}
&\langle \tilde{T}\phi^2(x)\phi^*(x)\phi^*(y)\rangle - \langle\phi^2(x)\phi^*(x)\rangle\langle\phi^*(y)\rangle \\
&\quad\cong \langle\phi(x)\rangle^2\langle\phi^*(x)\rangle\langle\phi^*(y)\rangle + \big(\langle\phi^2(x)\rangle - \langle\phi(x)\rangle^2\big)\langle\phi^*(x)\rangle\langle\phi^*(y)\rangle \\
&\qquad + 2\big(\langle\phi(x)\phi^*(x)\rangle - \langle\phi(x)\rangle\langle\phi^*(x)\rangle\big)\langle\phi(x)\rangle\langle\phi^*(y)\rangle \\
&\qquad + \langle\phi(x)\rangle^2\big(\langle\tilde{T}\phi^*(x)\phi^*(y)\rangle - \langle\phi^*(x)\rangle\langle\phi^*(y)\rangle\big) \\
&\qquad + 2\big(\langle\tilde{T}\phi(x)\phi^*(y)\rangle - \langle\phi(x)\rangle\langle\phi^*(y)\rangle\big)\langle\phi(x)\rangle\langle\phi^*(x)\rangle \\
&\qquad + 2\big(\langle\phi(x)\phi^*(x)\rangle - \langle\phi(x)\rangle\langle\phi^*(x)\rangle\big) \\
&\qquad\quad \times\big(\langle\tilde{T}\phi(x)\phi^*(y)\rangle - \langle\phi(x)\rangle\langle\phi^*(y)\rangle\big) \\
&\qquad + \big(\langle\phi^2(x)\rangle - \langle\phi(x)\rangle^2\big)\big(\langle\tilde{T}\phi^*(x)\phi^*(y)\rangle - \langle\phi^*(x)\rangle\langle\phi^*(y)\rangle\big) \\
&\qquad - \langle\phi(x)\rangle^2\langle\phi^*(x)\rangle\langle\phi^*(y)\rangle - \big(\langle\phi^2(x)\rangle - \langle\phi(x)\rangle^2\big)\langle\phi^*(x)\rangle\langle\phi^*(y)\rangle \\
&\qquad - 2\big(\langle\phi(x)\phi^*(x)\rangle - \langle\phi(x)\rangle\langle\phi^*(x)\rangle\big)\langle\phi(x)\rangle\langle\phi^*(y)\rangle \\
&\quad = 2\big(\langle\tilde{T}\phi(x)\phi^*(y)\rangle - \langle\phi(x)\rangle\langle\phi^*(y)\rangle\big)\langle\phi(x)\phi^*(x)\rangle \\
&\qquad + \langle\phi^2(x)\rangle\big(\langle\tilde{T}\phi^*(x)\phi^*(y)\rangle - \langle\phi^*(x)\rangle\langle\phi^*(y)\rangle\big)\,.
\end{aligned}$$

In addition, we assume here that the commuting fields like $\phi^*(x)$, $\phi^*(y)$ are independent from each other, i.e. the expectation value of the product of such fields equals the product of the expectation values of the fields. Then,

$$\langle \tilde{T}\phi^2(x)\phi^*(x)\phi^*(y)\rangle - \langle \phi^2(x)\phi^*(x)\rangle\langle\phi(y)\rangle$$
$$\cong 2\big(\langle \tilde{T}\phi(x)\phi^*(y)\rangle - \langle\phi(x)\rangle\langle\phi^*(y)\rangle\big)\langle\phi(x)\phi^*(x)\rangle \tag{6.3}$$

and

$$\int_C \mathrm{d}^4x'\, \Pi(x,x')\Delta(x',y) = g\big(i\Delta^>(x,x) + |A(x)|^2\big)\Delta(x,y)\,,$$

where, as in the real field case,

$$A(x) \equiv \langle\phi(x)\rangle\,.$$

The Green function $\Delta(x,y)$ is not well defined for $x=y$ and must be renormalized. To fix the way in which the renormalization will be carried out, we substitute the function $\Delta^<(x,x)$ for the function $\Delta(x,x)$. As we shall see, using the function $\Delta^>(x,x)$ instead of $\Delta^<(x,x)$ leads to the same result after renormalization.

The mean-field self-energy is

$$\Pi_{\mathrm{MF}}(x) = g\big(i\Delta^<(x,x) + |A(x)|^2\big)\,. \tag{6.4}$$

The transport and the mass-shell equations in the pairing approximation are

$$\big[p^\mu\partial_\mu - \tfrac{1}{2}g\partial_\mu A(X)\partial_p^\mu\big]\Delta^{>(<)}(X,p) = 0\,, \tag{6.5a}$$

$$\big[\tfrac{1}{4}\partial^\mu\partial_\mu - p^2 + m^2 - gA(X)\big]\Delta^{>(<)}(X,p) = 0\,, \tag{6.5b}$$

for the real fields and

$$\big[p^\mu\partial_\mu - \tfrac{1}{2}g\partial_\mu\big(i\Delta^<(X,X) + |A(X)|^2\big)\partial_p^\mu\big]\Delta^{>(<)}(X,p) = 0\,, \tag{6.6a}$$

$$\big[\tfrac{1}{4}\partial^\mu\partial_\mu - p^2 + m^2 - g\big(i\Delta^<(X,X) + |A(X)|^2\big)\big]\Delta^{>(<)}(X,p) = 0\,, \tag{6.6b}$$

for the complex fields.

To make both sets of eqs. (6.5) and (6.6) complete, one has to add the equation for the generation of the field $A(x)$. In the case of real fields this equation, which directly follows from eq. (2.3), is

$$\big[\partial^\mu\partial_\mu + m^2\big]A(X) = g\tfrac{1}{2}\big[i\Delta^<(X,X) + A^2(X)\big]\,. \tag{6.7}$$

For the complex fields one has to approximate the right-hand-side of a similar equation following from eq. (2.4) as in eq. (6.3). Then, one finds

$$\left[\partial^\mu\partial_\mu + m^2\right]A(X) = g\tfrac{1}{2}\left[2i\Delta^<(X,X)A(X) + A^*(X)A^2(X)\right], \quad (6.8a)$$

$$\left[\partial^\mu\partial_\mu + m^2\right]A^*(X) = g\tfrac{1}{2}\left[2i\Delta^<(X,X)A^*(X) + A(X)A^{*2}(X)\right]. \quad (6.8b)$$

Eq. (6.8b) is conjugated with respect to eq. (6.8a).

It is interesting to consider eqs. (6.7) and (6.8) for homogenous (equilibrium) systems. Then, the Green function $\Delta^<$ and the field A are X-independent. In this case eq. (6.7) provides

$$A_\pm = \frac{m^2}{g} \pm \left(\frac{m^4}{g^2} - i\Delta^<\right)^{1/2}. \quad (6.9)$$

Solution A_+ should be rejected on the basis of the following *physical* argumentation. This solution gives a finite value of the field A in the limit of zero density, i.e. when $\Delta^< = 0$. To cure this the vacuum contribution equal to $2m^2/g$ can be subtracted from the solution A_+. However, another problem remains. The effective mass, defined according to eq. (5.12) as

$$m^{*2} = m^2 - \Pi_{\mathrm{MF}}, \quad (6.10)$$

has strange behaviour when a system density and/or coupling constant increases. Namely, the real part of m^{*2} equals $2m^2$ at the point where the self-energy acquires an imaginary part, cf. eqs. (6.2) and (6.9). On the contrary, the solution A_- behaves in agreement with *physical* intuition. At zero density $A_- = 0$. When an imaginary contribution to Π_{MF} appears the effective mass equals zero. Such behaviour is well known in the theory of strong fields, see e.g. ref. [13]. The appearance of a self-energy imaginary part signals instability due to particle–anti-particle pair generation from a vacuum.

Finally, the mean-field self-energy of a homogenous neutral field system is

$$\Pi_{\mathrm{MF}} = m^2 - \left(m^4 - ig^2\Delta^<\right)^{1/2},$$

which in the small coupling limit gives

$$\Pi_{\mathrm{MF}} = i\frac{g^2}{2m^2}\Delta^<. \quad (6.11)$$

Let us now consider a homogenous complex field system. Then, one finds the solution of eqs. (6.8) to be

$$|A|^2 = 2\frac{m^2}{g} - 2i\Delta^<, \quad (6.12)$$

which should be renormalized to give a zero value of the field A at zero density. It is an interesting feature of eq. (6.12) that after the renormalization, $|A|^2$ is independent of the coupling constant, i.e. the field A does not vanish in the zero coupling limit. However, the mean-field self-energy is coupling-constant dependent and it equals

$$\Pi_{\mathrm{MF}} = -ig\Delta^{<} . \tag{6.13}$$

7. Perturbative approach to the mean field

As discussed e.g. in refs. [5, 9], the contour Green functions admit a perturbative expansion very similar to that known from the vacuum QFT [12]. One difference is that the time integrations do not run from $-\infty$ to $+\infty$, but along the contour. The top of the contour ($t_{\max}$) must be above the largest time argument of the evaluated Green function. In practice, t_0 is shifted to $-\infty$ and $t_{\max}$ to $+\infty$. The second difference is the appearance of the tadpoles, i.e. the loops formed by single lines which do not appear in the vacuum QFT, because of the operator normal ordering present in the Green function definition. A tadpole corresponds to the Green function of space-time arguments equal to each other. However, the Green function $\Delta(x, x)$ is not well defined and we ascribe the function $i\Delta^{<}(x, x)$ to each tadpole. One should note that $\Delta^{<}(x, x) = \Delta^{>}(x, x)$, because the function $D(x, y)$ from eq. (2.9) equals zero for $x = y$. The renormalization of tadpole graphs is briefly discussed in sect. 9.

The definition of a Green function (3.1) suggests that the perturbative expansion should be constructed not around the zero value of field $\phi(x)$, but rather around the zero value of field $\tilde{\phi}(x) = \phi(x) - A(x)$, where $A(x) \equiv \langle \phi(x) \rangle$. Substituting the field $\tilde{\phi}(x)$ into the lagrangian densities (2.1) or (2.2) we find that the field $\tilde{\phi}(x)$ interacts with the field $A(x)$. For example, in the case of real fields we have the couplings proportional to $\tilde{\phi}^2(x)A(x)$ and to $\tilde{\phi}(x)A^2(x)$. A finite expectation value of a field can appear for several reasons. It may happen due to symmetry breakdown as in the Higgs model, see e.g. ref. [14], or due to finite particle density as in the Walecka model [15]. However, as long as the field $A(x)$, or more precisely, the self-energy due to interaction with this field, is of *perturbative* character, i.e. it disappears when the coupling constant goes to zero, the perturbative expansions around $\phi(x) - A(x) = 0$ and around $\phi(x) = 0$ are expected equivalent. Below we consider the latter case.

The Green functions which enter the Feynman diagrams correspond, as usual, to the noninteracting fields and the respective free functions satisfy eqs. (5.15), (5.16) and (5.19), (5.20). To simplify the notation we omit the index 0 everywhere.

It follows from eq. (5.18) that the functions $\Delta^{>(<)}(X, p)$ of noninteracting fields are different from zero only for on-shell four-momenta. In the case of interacting fields this is not strictly true, but one expects that this is approximately true when

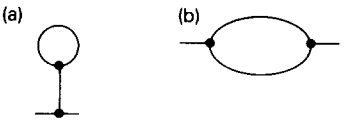

Fig. 2. The lowest-order diagrams for the self-energy in the ϕ^3 model.

the perturbative expansion is justified. Then, on calculating the self energies $\Pi^{>(<)}(X, p)$ we are interested in their values only for on-shell momenta. As we will see below this situation essentially simplifies calculations. However it occurs only in the case of massive fields studied here, see sect. 11.

In this section we consider the lowest-order (in coupling constant) contributions to the self-energies. As we shall see these contributions correspond to the mean field effects.

In the case of real fields the lowest-order self-energy, represented by the graph shown in fig. 2a, is

$$\Pi_{\mathrm{a}}(x, y) = -i\tfrac{1}{2}(-ig)^2\delta^{(4)}(x-y)\int_{\mathrm{C}} \mathrm{d}^4x'\, i\Delta(x, x')i\Delta^{<}(x', x')\,.$$

It should be noted that the expression corresponding to a respective diagram should be multiplied by a coefficient $-i$ in order it gives the self-energy.

On locating the argument x on the upper branch of the contour, one finds

$$\Pi_{\mathrm{a}}(x, y) = -i\tfrac{1}{2}g^2\delta^{(4)}(x-y)\int \mathrm{d}^4x'\left[\Delta^{\mathrm{c}}(x, x')\Delta^{>}(x', x') - \Delta^{<}(x, x')\Delta^{<}(x', x')\right], \tag{7.1}$$

where the time integration runs from $-\infty$ to $+\infty$. On using eq. (5.21) one can prove that an equivalent result is obtained if the x argument is located on the lower branch.

Let us discuss the second term from eq. (7.1). On performing the Wigner transformation, one finds that this term is proportional to

$$\int \mathrm{d}^4x'\,\Delta^{<}(x, x')\Delta^{<}(x', x')$$

$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{d}^4x'\,\mathrm{e}^{-ip(x'-x)}\Delta^{<}\left(\frac{x+x'}{2}, p\right)\Delta^{<}(x', k)\,. \tag{7.2}$$

One observes that the main contribution to the integral from (7.2) comes from x' which is close to x. Therefore, in accordance with the approximations made in

sect. 5, we expand the functions from eq. (7.2) around $x' = x$, neglecting the second and higher gradients. Then, one finds

$$\int \mathrm{d}^4x'\, \Delta^<(x,x')\Delta^<(x',x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\big(\Delta^<(x,p=0)\Delta^<(x,k)$$

$$+ i\big(\partial_p^\mu \partial_\mu \Delta^<(x,p=0)\big)\Delta^<(x,k)$$

$$+ i\partial_p^\mu \Delta^<(x,p=0)\partial_\mu \Delta^<(x,k)\big).$$

The function $\Delta^<(x,p=0)$ and its momentum derivative equal zero if we impose the mass-shell constraints on the Green functions. Consequently, the second term from eq. (7.1) gives zero contribution. One finds that the integral (7.2) differs from zero when the characteristic length of space-time variation of the functions $\Delta^{>(<)}(x,p)$ is comparable with the characteristic inverse particle four-momentum. However, as discussed in sect. 5, see also sect. 11, such quickly varying functions have been excluded from our considerations. Thus, the self-energy (7.1) gives the mean field term as

$$\Pi_{\mathrm{MF}}(x) = -i\tfrac{1}{2}g^2 \int \mathrm{d}^4x'\, \Delta^{\mathrm{c}}(x,x')\Delta^<(x',x'). \tag{7.3}$$

Let us note that eq. (7.1) can be rewritten by means of the function Δ^+, and then the mean-field self-energy equals

$$\Pi_{\mathrm{MF}}(x) = -i\tfrac{1}{2}g^2 \int \mathrm{d}^4x'\, \Delta^+(x,x')\Delta^<(x',x').$$

The graph from fig. 2b corresponds to

$$\Pi_{\mathrm{b}}(x,y) = -i\tfrac{1}{2}g^2 \Delta(x,y)\Delta(y,x)$$

with no contribution to the mean-field self-energy, cf. eq. (4.5), and

$$\Pi_{\mathrm{b}}^{>(<)}(x,y) = -i\tfrac{1}{2}g^2 \Delta^{>(<)}(x,y)\Delta^{<(>)}(y,x). \tag{7.4}$$

One can notice that the self-energies (7.4) equal zero when the functions $\Delta^{>(<)}$ are nonzero only for the on-shell momenta.

The lowest-order contribution to the self-energy for complex fields which corresponds to the graph from fig. 3 is

$$\Pi(x,y) = -i(-ig)\delta^{(4)}(x,y) i\Delta^<(x,x)$$

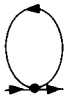

Fig. 3. The lowest-order diagram for the self-energy in the $(\phi\phi^*)^2$ model.

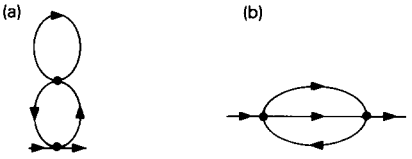

Fig. 4. The second-order diagrams for the self-energy in the $(\phi\phi^*)^2$ model.

giving

$$\Pi_{\mathrm{MF}}(x) = -ig\Delta^<(x,x)\,. \tag{7.5}$$

Let us compare the results (7.3) and (7.5) with those obtained in sect. 6. The pairing approximation should be equivalent to the perturbative one when a coupling constant goes to zero. One immediately finds that the self-energy (7.3) calculated for a homogeneous system reproduces the result (6.11). Further, eq. (7.5) coincides with eq. (6.13).

8. Higher-order self-energies

In this section we calculate next-to-lowest-order perturbative contributions to self-energies (for the complex fields this is the contribution of order g^2 while for the real ones it is of order g^4). In that way we will get the lowest-order nonvanishing contributions to the collision self-energies.

8.1. COMPLEX FIELDS

The second order contributions to the self-energy are represented by the two diagrams shown in fig. 4. One should remember that the self-energy defined as in eq. (4.4) relates only to one-particle-irreducible diagrams. The graph from fig. 4a corresponds to the mean field, and is of higher order than (7.5); it is neglected here. The diagram (b) gives

$$\Pi_{\mathrm{b}}(x,y) = -i(-ig)^2\tfrac{1}{2}i\Delta(x,y)i\Delta(x,y)i\Delta(y,x)\,,$$

which provides

$$\Pi_{\mathrm{b}}^{>(<)}(x,y) = -(-ig)^2\tfrac{1}{2}\Delta^{>(<)}(x,y)\Delta^{>(<)}(x,y)\Delta^{<(>)}(y,x)\,.$$

Upon introducing the Wigner transformation one obtains

$$\Pi_{\mathrm{b}}^{>(<)}(X,p) = -(-ig)^2\tfrac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{d}^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+k-q-r)$$

$$\times\Delta^{>(<)}(X,q)\Delta^{>(<)}(X,r)\Delta^{<(>)}(X,k)\,. \tag{8.1}$$

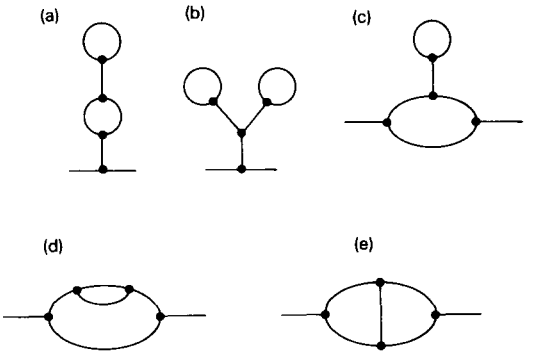

Fig. 5. The fourth-order diagrams for the self-energy in the ϕ^3 model.

It can be easily checked that the self-energies (8.1) satisfy the current and energy–momentum conservation conditions (5.13) and (5.14).

8.2. REAL FIELDS

The fourth-order contributions to the self-energy are represented by five one-particle-irreducible graphs shown in fig. 5. The contributions from diagrams (a) and (b) give the higher-order mean-field terms and are neglected. The other diagrams in fig. 5 require more careful analysis.

Up to now we have calculated the contour self-energy and then used definition (4.5) to extract the functions Π_{MF}, $\Pi^{>}$ and $\Pi^{<}$ expressed in terms of Δ^{c}, Δ^{a}, $\Delta^{>}$ and $\Delta^{<}$. In the case of more difficult diagrams, such as those from fig. 5, the latter step can be quite difficult. Therefore it is better to calculate $\Pi^{>}$ and $\Pi^{<}$ from the beginning by means of the following graphical method.

We draw a line dividing the plane into parts, left and right, that correspond to the two time-branches, the left part to the chronological (upper) branch and the right part to the antichronological (lower) branch. Then we draw all topologically distinct diagrams locating the interaction vertices on both half-planes in all possible ways. For example, on calculating the self-energy $\Pi^{>}(x, y)$ related to the diagram from fig. 5c we place the x point in the left half-plane and the y point in the right one. The remaining two vertices can be placed in four possible ways as it is shown in fig. 6. By virtue of the relations (3.6)–(3.9), the lines in the diagrams are identified with the functions $i\Delta^{\mathrm{c}}$, $i\Delta^{\mathrm{a}}$, $i\Delta^{>}$ and $i\Delta^{<}$ according to the following rules:

(i) When both end points are at the left (right) side of the plane, the line represents $i\Delta^{\mathrm{c}}$ ($i\Delta^{\mathrm{a}}$).

(ii) When the start point is at the left (right) side of the plane and the end point is at the right (left) side, the line represents $i\Delta^{>}$ ($i\Delta^{<}$).

We perform integration over the vertex positions with the time integration running

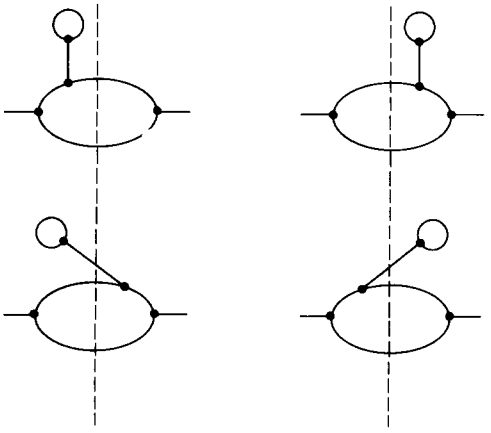

Fig. 6. The diagrams for the self energy $\Pi^{>(<)}$ corresponding to the graph from fig. 5c.

from $-\infty$ to $+\infty$. With each integration of a vertex placed on the antichronological (right) half-lane, there is associated a factor -1.

Because the functions $\Pi^{>(<)}(x, p)$ are only considered for momentum arguments that satisfy the mass-shell constraints, and the functions $\Delta^{>(<)}(X, p)$ are finite only for the arguments that satisfy such a constraint, it is possible to isolate the diagrams which give a zero contribution as a result of energy–momentum conservation.

Now we can return to the analysis of the diagrams from fig. 5. The graphs for the self-energy $\Pi^{>}$ corresponding to the diagram from fig. 5c are shown in fig. 6. One finds that each of them provides zero contribution if the arguments of the functions represented with the lines crossing the plane division line, satisfy the mass-shell constraints.

The graphs for the self-energy $\Pi^{>}$ corresponding to the graphs from fig. 5d and fig. 5e are shown in fig. 7 and fig. 8, respectively. We immediately find that among the eight graphs only three give a finite contribution. Namely, the graphs from fig. 7c, fig. 8c and fig. 8d. The sum of these graphs yields

$$\Pi^{>(<)}(X,p) = -(-ig)^4 \tfrac{1}{2} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{\mathrm{d}^4r}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+k-q-r)$$

$$\times \Delta^{>(<)}(X,q)\Delta^{>(<)}(X,r)\Delta^{<(>)}(X,k)$$

$$\times \left[2\Delta^{c(a)}(X,p-r)\Delta^{a(c)}(X,p-q)\right.$$

$$\left. +\Delta^{c(a)}(X,p-r)\Delta^{a(c)}(X,p-r)\right]. \tag{8.2}$$

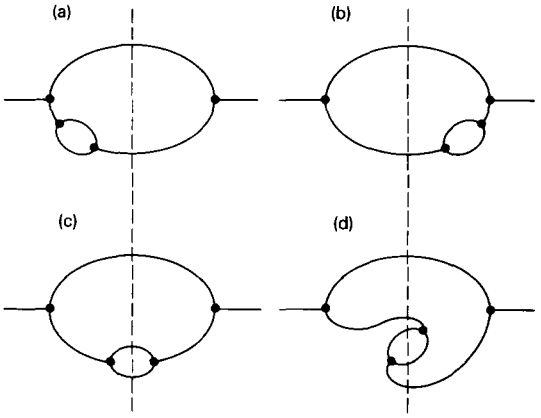

Fig. 7. The diagrams for the self-energy $\Pi^{>(<)}$ corresponding to the graph from fig. 5d.

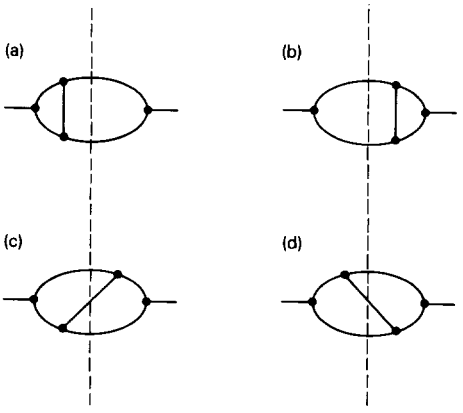

Fig. 8. The diagrams for the self-energy $\Pi^{>(<)}$ corresponding to the graph from fig. 5e.

In the more involved but explicitly causal analysis [16], the functions $\Delta^{c(a)}$ are replaced by the functions $\Delta^{+(-)}$. All these functions differ only in their imaginary parts [see eqs. (5.21) and (5.22)] that vanish for the off-shell function arguments, making the distinction between the functions irrelevant for our purposes.

The perturbative expansion of contour Green functions, as other field-theory perturbative expansions, suffers from the appearance of infinite expressions. The specific divergences are due to the tadpole diagrams. We discuss them in sect. 9, where the distribution functions are introduced. The renormalization is based on the *physical* argument that the tadpole contributions should vanish in the vacuum limit, i.e. these contribution should be compensated by respective counterterms.

Divergences other than tadpole divergences have not explicitly appeared in our considerations due to our *practical* approach to the problem. The mean-field divergent diagrams from fig. 4a and fig. 5a have been neglected since they are not of leading order. The contributions from the rest of graphs in figs. 4 and 5 were finite because we imposed the mass-shell constraints.

The discussion of the renormalization procedure of the contour Green function expansion for the ϕ^4-model is given in ref. [9].

9. Distribution functions

The derivative term in eq. (5.12) and the term proportional to $\Pi^{>(<)}$ may be considered as of higher order [5] than the other terms in the mass-shell equation within the gradient expansion (5.1). Upon dropping these terms, the mass-shell equation can be written as

$$\left[p^2 - m^2 + \mathrm{Re}\,\Pi(X,p)\right]\Delta^{>(<)}(X,p) = 0\,, \tag{9.1}$$

where $\mathrm{Re}\,\Pi(X,p) = \Pi_{\mathrm{MF}}(X,p) + \frac{1}{2}(\Pi^{+},(X,p) + \Pi^{-}(X,p))$.

The particle energy $E_p(X)$ is defined as the positive solution, $p = (E_p, \boldsymbol{p})$, of the equation

$$p^2 - m^2 + \mathrm{Re}\,\Pi(X,p) = 0\,, \tag{9.2a}$$

and the antiparticle energy $\bar{E}_p(X)$ as the positive solution of the equation

$$p^2 - m^2 + \mathrm{Re}\,\Pi(X,-p) = 0\,, \tag{9.2b}$$

with $p = (\bar{E}_p, \boldsymbol{p})$. If only the mean field is retained, then $E_p = \bar{E}_p = (\boldsymbol{p}^2 + m^{*2})^{1/2}$, where $m^{*2}(X) = m^2 - \Pi_{\mathrm{MF}}(X)$.

The distribution functions of particles $f(X,p)$ and of antiparticles $\bar{f}(X,p)$ defined only for on-shell four-momenta, are introduced with

$$\begin{aligned}\Theta(p_0)i\Delta^{<}(X,p) &= \Theta(p_0)2\pi\delta\left(p^2 - m^2 + \mathrm{Re}\,\Pi(X,p)\right)f(X,p) \\ &= \frac{Z_p\pi}{E_p}\delta(E_p - p_0)f(X,p)\,,\end{aligned} \tag{9.3a}$$

$$\begin{aligned}\Theta(p_0)i\Delta^{>}(X,-p) &= \Theta(p_0)2\pi\delta\left(p^2 - m^2 + \mathrm{Re}\,\Pi(X,-p)\right)\bar{f}(X,p) \\ &= \frac{\bar{Z}_p\pi}{\bar{E}_p}\delta(E_p - p_0)\bar{f}(X,p)\,.\end{aligned} \tag{9.3b}$$

Here the factors Z_p and $\bar{Z}_p$ are

$$Z_p^{-1} = 1 + \frac{1}{2E_p}\left.\frac{\partial\,\mathrm{Re}\,\Pi(X,p)}{\partial p_0}\right|_{p_0 = E_p}, \tag{9.4a}$$

and

$$\bar{Z}_p^{-1} = 1 + \frac{1}{2\bar{E}_p} \frac{\partial \operatorname{Re} \Pi(X, -p)}{\partial p_0}\bigg|_{p_0=\bar{E}_p}. \tag{9.4b}$$

The function $\Delta^<(X,p)$ for $p_0<0$ and $\Delta^>(X,p)$ for $p_0>0$, may be further expressed with $f(X,p)$ and $\bar{f}(X,p)$ by making use of the relation (5.23). Specifically, upon Wigner transformation and dropping the gradient terms eqs. (4.16) and (4.17) provide

$$\big(p^2 - m^2 + \Pi_{\mathrm{MF}}(X) + \Pi^{\pm}(X,p)\big)\Delta^{\pm}(X,p) = 1.$$

One further observes [5] that

$$\operatorname{Re}\Pi^{\pm}(X,p) = \tfrac{1}{2}\big(\Pi^+(X,p) + \Pi^-(X,p)\big),$$

$$\operatorname{Im}\Pi^{\pm}(X,p) = \pm\frac{1}{2i}\big(\Pi^+(X,p) - \Pi^-(X,p)\big) = \pm\frac{1}{2i}\big(\Pi^>(X,p) - \Pi^<(X,p)\big).$$

However, the self-energies $\Pi^{>(<)}$ vanish at the zeroth order in the gradient expansion, and consequently

$$\Delta^{\pm}(X,p) = \big(p^2 - m^2 + \operatorname{Re}\Pi(X,p) \pm ip_0 0^+\big)^{-1},$$

where the infinitesimal imaginary terms are the remanents of $\operatorname{Im}\Pi^{\pm}$. Finally, one finds from the relation (5.23)

$$i\Delta^>(X,p) - i\Delta^<(X,p) = 2\pi\delta\big(p^2 - m^2 + \operatorname{Re}\Pi(X,p)\big)\big(\Theta(p_0) - \Theta(-p_0)\big), \tag{9.5}$$

cf. eq. (5.24).

From relation (9.5) and definition (9.3), we obtain

$$\begin{aligned}\Theta(p_0) i\Delta^>(X,p) &= \Theta(p_0) 2\pi\delta\big(p^2 - m^2 + \operatorname{Re}\Pi(X,p)\big)\big[f(X,p) + 1\big] \\ &= \frac{Z_p\pi}{E_p}\delta(E_p - p_0)\big[f(X,p) + 1\big],\end{aligned} \tag{9.6a}$$

$$\begin{aligned}\Theta(p_0) i\Delta^<(X,-p) &= \Theta(p_0) 2\pi\delta\big(p^2 - m^2 + \operatorname{Re}\Pi(X,-p)\big)\big[\bar{f}(X,p) + 1\big] \\ &= \frac{\bar{Z}_p\pi}{\bar{E}_p}\delta(E_p - p_0)\big[\bar{f}(X,p) + 1\big].\end{aligned} \tag{9.6b}$$

In the case of real fields, particles and antiparticles are indistinguishable. The relation (3.16) for the Wigner-transformed functions takes the form

$$\Delta^{>}(X,p) = \Delta^{<}(X,-p)\,. \tag{9.7}$$

Eqs. (4.1) and (4.2) may be used to show that $\Pi^{>(<)}$ satisfy a relation analogous to eq. (3.16). This further implies that

$$\mathrm{Re}\,\Pi^{>}(X,p) = \mathrm{Re}\,\Pi^{<}(X,-p)\,. \tag{9.8}$$

From eqs. (9.2), (9.7) and (9.8) we then have $E_p = \bar{E}_p$ and $Z_p = \bar{Z}_p$, and finally

$$f(X,p) = \bar{f}(X,p)\,.$$

When only the mean field in retained in the self-energy Re Π, then $Z_p = \bar{Z}_p = 1$. This will be assumed to be the case further on.

Using the formulas (9.3) and (9.6) one finds

$$i\Delta^{<}(X,p) = \frac{\pi}{E_p}\delta(E_p - p_0)f(X,p) + \frac{\pi}{E_p}\delta(E_p + p_0)\big[\bar{f}(X,-p) + 1\big]\,, \tag{9.9a}$$

$$i\Delta^{>}(X,p) = \frac{\pi}{E_p}\delta(E_p - p_0)\big[f(X,p) + 1\big] + \frac{\pi}{E_p}\delta(E_p + p_0)\bar{f}(X,-p)\,. \tag{9.9b}$$

Expressing the current (3.18) with the Green functions of the form (9.3) one finds

$$j^{\mu}(X) = -2\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} p^{\mu}\big[f(X,p) - \bar{f}(X,p) + 1\big] + i\langle\phi^*(X)\rangle\overleftrightarrow{\partial}_{\mu}\langle\phi(X)\rangle\,. \tag{9.10}$$

The integral from eq. (9.10) is divergent and in the vacuum limit ($f(X,p)$, $\bar{f}(X,p) \to 0$), where the current for the *physical* reasons should be zero, it gives

$$-\int \frac{\mathrm{d}^3p}{(2\pi)^3 E_p} p^{\mu}\,.$$

This type of divergences, which also appear in the tadpole contributions, is well known in the field theory. In the case of the vacuum QFT they do not appear because of the operator normal-ordering present in the Green function definition [12]. Upon subtracting of the vacuum value from the right-hand side of eq. (9.10),

the current becomes

$$j^{\mu}(X) = -2\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} p^{\mu}\left[f(X,p) - \bar{f}(X,p)\right] + i\langle\phi^*(X)\rangle\overleftrightarrow{\partial^{\mu}}\langle\phi(X)\rangle\,. \tag{9.11}$$

In a similar way, one has to subtract the vacuum part to obtain the finite expression of the energy–momentum tensor (3.19). Then, one gets for the real and complex fields, respectively:

$$T_0^{\mu\nu}(X) = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} p^{\mu}p^{\nu} f(X,p) - \tfrac{1}{4}\langle\phi(X)\rangle\overleftrightarrow{\partial^{\mu}}\overleftrightarrow{\partial^{\nu}}\langle\phi(X)\rangle\,, \tag{9.12a}$$

$$T_0^{\mu\nu}(X) = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} p^{\mu}p^{\nu}\left[f(X,p) + \bar{f}(X,p)\right] - \tfrac{1}{2}\langle\phi^*(X)\rangle\overleftrightarrow{\partial^{\mu}}\overleftrightarrow{\partial^{\nu}}\langle\phi(X)\rangle\,. \tag{9.12b}$$

The mean-field self-energies (7.3) and (7.5) corresponding to the tadpole diagrams should also be renormalized. After subtracting the vacuum values we get the self-energies of real and complex fields, respectively:

$$\Pi_{\mathrm{MF}}(x) = -\tfrac{1}{2}g^2\int \mathrm{d}^4x'\,\Delta^{\mathrm{c}}(x,x')d(x') = -\tfrac{1}{2}g^2\int \mathrm{d}^4x'\,\Delta^{+}(x,x')d(x')\,, \tag{9.13a}$$

$$\Pi_{\mathrm{MF}}(x) = -gd(x)\,, \tag{9.13b}$$

where $d(x)$ equals

$$d(x) = 2\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p} f(x,p) \tag{9.14a}$$

for the real fields and

$$d(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\left[f(x,p) + \bar{f}(x,p)\right] \tag{9.14b}$$

for the complex ones. In the nonrelativistic limit, $d(x)$ multiplied by $2m$ is equal to the particle density.

The real-field self-energy (9.13a) requires a further discussion. Let us express the Green function $\Delta^{\mathrm{c}}(x,y)$ in terms of the distribution functions. From eq. (5.21a) one finds

$$\Delta^{\mathrm{c}}(x,y) = \Delta^{\mathrm{c}}_{\mathrm{vacuum}}(x,y) + \Delta^{\mathrm{c}}_{\mathrm{medium}}(x,y)\,, \tag{9.15}$$

where

$$\Delta^{c}_{\text{vacuum}}(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - m^2 + i0^+},$$

$$\Delta^{c}_{\text{medium}}(x,y) = -i\int \frac{d^3p}{(2\pi)^3 2E_p}(e^{-ip(x-y)} + e^{ip(x-y)}) f\left(\frac{x+y}{2}, p\right).$$

Upon substituting the function $\Delta^c(x,y)$ of the form (9.15) into (9.13a), one finds two terms of the mean-field self-energy corresponding to the *vacuum* and *medium* parts of $\Delta^c(x,y)$. The arguments analogous to that presented after eq. (7.1) lead us to the conclusion that $\Pi^{\text{medium}}_{\text{MF}}$ can be ignored.

10. Transport equations

In sects. 7, 8 we have expressed the self-energies through the free Green functions by means of the perturbative expansion. Treating these Green functions as exact ones and substituting the self-energies into eqs. (5.11) and (5.12) we get the closed set of equations. However, one should remember that these equations are valid up to the order of g^2 for the complex fields and to g^4 for the real ones.

The final form of transport equations satisfied by the distribution functions is obtained by substituting the Green functions of the form (9.9) into eq. (5.11) with an explicit form of the self energies.

10.1. COMPLEX FIELDS

Let us first obtain an expression for the right-hand-side of eq. (5.11). From eq. (8.1) one finds that

$$\begin{aligned}
&\Theta(p_0)[\Pi^<(X,p)\Delta^>(X,p) - \Pi^>(X,p)\Delta^<(X,p)] \\
&\quad = -(-ig)^{2\frac{1}{2}}\int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}(2\pi)^4\delta^{(4)}(p+k-q-r) \\
&\qquad \times[\Delta^<(X,q)\Delta^<(X,r)\Delta^>(X,k)\Delta^>(X,p) \\
&\qquad\quad -\Delta^>(X,q)\Delta^>(X,r)\Delta^<(X,k)\Delta^<(X,p)] \\
&\quad = -\frac{g^2\pi}{2E_p}\delta(E_p - p_0)\int \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}\frac{d^3r}{(2\pi)^3 2E_r}(2\pi)^4 \\
&\qquad \times\big[\delta^{(4)}(p+k-q-r)[f(X,q)f(X,r)[f(X,p)+1][f(X,k)+1] \\
&\qquad -[f(X,q)+1][f(X,r)+1]f(X,p)f(X,k)] \\
&\qquad +\delta^{(4)}(p-k+q-r)\big[[\bar f(X,q)+1]f(X,r)[f(X,p)+1]\bar f(X,k) \\
&\qquad -\bar f(X,q)[f(X,r)+1]f(X,p)[\bar f(X,k)+1]\big] \\
&\qquad +\big[\delta^{(4)}(p-k-r+r)\big[f(X,q)[\bar f(X,r)+1][f(X,p)+1]\bar f(X,k) \\
&\qquad -[f(X,q)+1]\bar f(X,r)f(X,p)[\bar f(X,k)+1]\big]\big]. \qquad (10.1)
\end{aligned}$$

On deriving eq. (10.1) we have changed the sign of antiparticle four-momenta. Subsequently, we have ignored terms as proportional to $\delta^{(4)}(p+k+q+r)$, $\delta^{(4)}(p-k-q-r)$, or $\delta^{(4)}(p-k-q-r)$. Since all four-momenta are on the mass shell and the zero components of them are positive, these terms give zero contribution to the integral from eq. (10.1).

Finally, changing the variables $k \leftrightarrow q$ in the second term and $r \leftrightarrow q$ in the third one, eq. (10.1) can be rewritten as

$$\begin{aligned}
&\Theta(p_0)\left[\Pi^<(X,p)\Delta^>(X,p)-\Pi^>(X,p)\Delta^<(X,p)\right] \\
&\quad=\frac{\pi}{2E_p}\delta(E_p-p_0)\int\frac{\mathrm{d}^3k}{(2\pi)^3 2E_k}\frac{\mathrm{d}^3q}{(2\pi)^3 2E_q}\frac{\mathrm{d}^3r}{(2\pi)^3 2E_r}(2\pi)^4 \\
&\qquad\times\delta^{(4)}(p+k-q-r) \\
&\qquad\times\Big[|M_1|^2\big[f(X,p)f(X,k)\big[f(X,q)+1\big]\big[f(X,r)+1\big] \\
&\qquad\qquad -f(X,q)f(X,r)\big[f(X,p)+1\big]\big[f(X,k)+1\big]\big] \\
&\qquad\quad +2|M_2|^2\big[f(X,p)\bar f(X,k)\big[f(X,q)+1\big]\big[\bar f(X,r)+1\big] \\
&\qquad\qquad -f(X,q)\bar f(X,r)\big[f(X,p)+1\big]\big[\bar f(X,k)+1\big]\big]\Big], \qquad (10.2)
\end{aligned}$$

where M_1 is the particle–particle amplitude scattering and M_2 is the antiparticle–particle amplitude scattering;

$$M_1 = M_2 = -ig\,.$$

The left-hand-side of eq. (5.11) can also be split into two parts corresponding to particles and antiparticles, respectively, and finally the transport equation for the particle distribution function is obtained

$$\begin{aligned}
&\left[p^\mu\partial_\mu+\tfrac{1}{2}g\partial_\mu d(X)\partial_p^\mu\right]f(X,p) \\
&\quad=\tfrac{1}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2E_k}\frac{\mathrm{d}^3q}{(2\pi)^3 2E_q}\frac{\mathrm{d}^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(4)}(p+k-q-r) \\
&\qquad\times\Big[\tfrac{1}{2}|M_1|^2\big[f(X,p)f(X,k)\big[f(X,q)+1\big]\big[f(X,r)+1\big] \\
&\qquad\qquad -f(X,q)f(X,r)\big[f(X,p)+1\big]\big[f(X,k)+1\big]\big] \\
&\qquad\quad +|M_2|^2\big[f(X,p)\bar f(X,k)\big[f(X,q)+1\big]\big[\bar f(X,r)+1\big] \\
&\qquad\qquad -f(X,q)\bar f(X,r)\big[f(X,p)+1\big]\big[\bar f(X,k)+1\big]\big]\Big], \qquad (10.3a)
\end{aligned}$$

where $d(X)$ is given by eq. (9.14b).

Analogously we get the transport equation for the antiparticle distribution function

$$\begin{aligned}
&\left[p^\mu\partial_\mu + \tfrac{1}{2}g\partial_\mu d(X)\partial_p^\mu\right]\bar{f}(X,p) \\
&\quad = \tfrac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_k}\frac{\mathrm{d}^3q}{(2\pi)^3 2E_q}\frac{\mathrm{d}^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(4)}(p+k-q-r) \\
&\qquad \times\Big[\tfrac{1}{2}|M_1|^2\big[\bar{f}(X,p)\bar{f}(X,k)\big[\bar{f}(X,q)+1\big]\big[\bar{f}(X,r)+1\big] \\
&\qquad\qquad -\bar{f}(X,q)\bar{f}(X,r)\big[\bar{f}(X,p)+1\big]\big[\bar{f}(X,k)+1\big]\big] \\
&\qquad + |M_2|^2\big[\bar{f}(X,p)f(X,k)\big[\bar{f}(X,q)+1\big]\big[f(X,r)+1\big] \\
&\qquad\qquad -\bar{f}(X,q)f(X,r)\big[\bar{f}(X,p)+1\big]\big[f(X,k)+1\big]\big]\Big]. \qquad (10.3b)
\end{aligned}$$

10.2. REAL FIELDS

The derivation of the transport equation of the real-field distribution function proceeds as in the complex-field case. However, the form of the self-energies (8.3) is now more complicated. The difficulty is to recognize the sum of nine terms like $\Delta^{\mathrm{c}}(X,q)\Delta^{\mathrm{a}}(X,k)$ as the squared amplitude for particle–particle scattering. One should remember that according to the relation (3.14) $i\Delta^{\mathrm{a}}(X,k)$ is a hermitian conjugate of $i\Delta^{\mathrm{c}}(X,k)$. After quite long manipulations involving changes of the momentum variables one finds

$$\begin{aligned}
&\left[p^\mu\partial_\mu - \tfrac{1}{2}\partial_\mu\Pi_{\mathrm{MF}}(X)\partial_p^\mu\right]f(X,p) \\
&\quad = \tfrac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_k}\frac{\mathrm{d}^3q}{(2\pi)^3 2E_q}\frac{\mathrm{d}^3r}{(2\pi)^3 2E_r}(2\pi)^4\delta^{(4)}(p+k-q-r) \\
&\qquad \times |M(k,p,q,r)|^2\big[f(X,p)f(X,k)\big[f(X,q)+1\big]\big[f(X,r)+1\big] \\
&\qquad\qquad -f(X,q)f(X,r)\big[f(X,p)+1\big]\big[f(X,k)+1\big]\big], \qquad (10.4)
\end{aligned}$$

where the mean-field self-energy is given by eq. (9.13a) and the amplitude $M(k,p,q,r)$ is

$$M(k,p,q,r) = i(-ig)^2\big[\Delta^{\mathrm{c}}(x,q-p) + \Delta^{\mathrm{c}}(X,p-r) + \Delta^{\mathrm{c}}(X,k+p)\big]. \qquad (10.5)$$

This amplitude corresponds to the sum of diagrams shown in fig. 9. Because the on-shell part of the free Green function $\Delta^{\mathrm{c}}(X,p)$, eq. (5.21), does not contribute

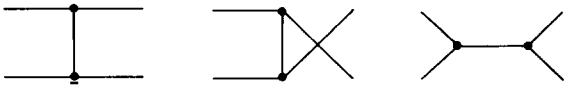

Fig. 9. The diagrams for the lowest-order scattering amplitude in the ϕ^3-model.

to the amplitude (10.5), the function $\Delta^c(X, p)$ effectively coincides with the vacuum Feynman propagator.

The transport equations following from the pairing approximation, which are satisfied by the distribution functions, can be immediately obtained from eqs. (6.5a) and (6.6a). These equations, however, are not very interesting because of the lack of collision terms.

On writing down eqs. (10.3) and (10.4) we have achieved our goal to derive the kinetic equations.

11. Discussion and summary

Our derivation of the transport equations is based on several restrictive assumptions and approximations. Let us now discuss the most important of them keeping in mind that these assumptions and approximations, on one hand, impose some restrictions on the physical systems which can be described in the framework of transport theory, but on the other hand they limit the amount of information about the system which can be obtained from this theory.

The essential simplifications have been made in sect. 5 where we have assumed that $\Delta(X, u)$ is a slowly varying function of X and it is strongly *peaked* for $u \cong 0$. This assumption can be written as a condition

$$|\Delta(X,p)| \gg \left|\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial p_\mu}\Delta(X,p)\right| \gg \left|\left(\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial p_\mu}\right)^2\Delta(X,p)\right| \gg \ldots, \quad (11.1)$$

which is equivalent to the requirement

$$\Delta X^\mu \Delta p_\mu \gg 1, \quad (11.2)$$

where ΔX^μ and Δp^μ are the characteristic lengths at which the function $\Delta(X, p)$ varies in position and momentum space. In more standard units the right-hand-side of inequality (11.2) equals $\hbar$.

If $\Delta(X, p)$ provides the *exact* description of a single-particle system, the relation (11.2) cannot be satisfied since $\Delta X^\mu \sim 1/\Delta p^\mu$ in this case. For a single-particle system condition (11.2) is equivalent to the one which justifies the classical description of the system, i.e. the description with poor position and/or momentum resolutions. Therefore, to satisfy condition (11.2), the single-particle function $\Delta(X, p)$ should carry the information averaged over the space-time cells which are much larger than the single-particle de Broglie wavelength. In the case of a many-particle system the function $\Delta(X, p)$ carries only the averaged information due to the very meaning of $\Delta(X, p)$. Consequently, the size of the averaging cell can be, in principle, smaller than a single-particle de Broglie wavelength.

One should remember that condition (11.1) should be also fulfilled at the *macroscopic* level, i.e. the distribution function, which caries only averaged information, should satisfy the relation

$$|f(X,p)| \gg \left|\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial p_\mu} f(X,p)\right| \gg \left|\left(\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial p_\mu}\right)^2 f(X,p)\right| \gg \ldots . \quad (11.3)$$

Because the distribution of a many-particle system is never momentum independent (Δp_μ never approaches infinity), the requirement (11.3) limits the kinetic description to systems where the rate of the temporal changes is much smaller than the particle energies.

In sect. 5 we have assumed that the self-energy satisfies the condition analogous to (11.1), i.e.

$$|\Pi(X,p)| \gg \left|\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial p_\mu}\Pi(X,p)\right| \gg \left|\left(\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial p_\mu}\right)^2 \Pi(X,p)\right| \gg \ldots . \quad (11.4)$$

The characteristic length Δp^μ, at which the self-energy $\Pi(X,p)$ varies in four-momentum space, corresponds to the inverse space-time interaction range. Therefore, the requirement $\Delta X^\mu \Delta p_\mu \gg 1$, applied to the self-energy case, demands shortness of the space-time interaction range when compared with the system's space-time unhomogeneity scale.

The conditions (11.1) and (11.4) justify the expansion *in gradients* and, in particular, the formulas (5.2)–(5.6). Deriving the transport equations we have kept only the quantities which are of no more than the second order in gradients. However, the mean-field and collision parts of self-energy have been treated in a different way. In the case of the mean-field self-energy we have taken into account the gradient terms and we have neglected analogous terms of the collision self-energy. Such a procedure is justified when the interaction in the system is *weak* and the perturbative expansion is allowed, since the mean-field contribution appears at a lower order in coupling constant than the collision one.

On calculating perturbatively the self-energy, we have extensively made use of the mass-shell constraints for the free Green functions $\Delta^{>(<)}(X,p)$. As has been shown following eq. (5.16) the functions, $\Delta^{>(<)}(X,p)$ are finite only for the on-shell momenta, if the functions $\Delta^{>(<)}(x,p)$ weakly depend on X on the scale of the particle Compton wave. It appears when the ensemble averaging present in the definition of $\Delta^{>(<)}(X,p)$ is performed over the space-time cells the size of which is much greater than the particle Compton length.

Also when the functions $\Pi^{>(<)}(X,p)$ have been calculated in the second order in coupling constant, it has been assumed, that incoming four-momentum p satisfies the free-field relation $p^2 = m^2$. This has allowed us to neglect several diagrams because of energy–momentum conservation which, in particular, forbids

the decay $A \to B + C$ when the masses of the particles A, B and C are equal to each other. When $p^2 = m^{*2}$ and m^* differs from m, our analysis remains valid as long as the difference between m^* and m is much smaller than m. This is the case for a *perturbative* interaction of massive fields. When one considers massless fields, our arguments do not hold since a small (*perturbative*) modification of the particle dispersion relation can make the process, which is forbidden in vacuum, allowed in a medium. An example is Cherenkov radiation which plays an important role in the transport theory of electrodynamic plasma, see e.g. ref. [4].

The derivation of transport equation usually leads to the so-called BBGKY hierarchy of kinetic equations, see e.g. ref. [3], which is further truncated under the assumptions of smallness of interparticle correlations in the system. In our derivation the BBGKY hierarchy has not appeared because in the perturbative expansion the correlations are assumed to vanish in the remote past [5]. In that case any correlations in the system can be expressed with integrals extending over the history of the system, and containing only single-particle functions.

At the end let us recapitulate our considerations. We have started with the definition of the contour Green function and its exact equation of motion – the Dyson–Schwinger equation. Then, making some assumptions on the Green function properties, which we discussed in this section, we have obtained the approximate equations of motion of the functions $\Delta^{>(<)}$. These equations, which have been recognized as the transport equation and the mass-shell equation, have no meaning unless the self-energies are determined. For the self-energies we have considered the pairing approximation and the perturbative expansion. The pairing approximation, as the perturbative expansion in the lowest order, has provided the equation where the interaction effects enter only through the mean field. The nonvanishing contributions to the collision self-energies have been found in the next to the lowest order calculations. However, the analysis of the $(\phi\phi^*)^2$ model happened to be much simpler than that of the ϕ^3 one. Subsequently, we have introduced the distribution functions defined only for on-shell four-momenta. Finally, expressing the functions $\Delta^{>(<)}$ through the distribution functions we have arrived at the transport equations.

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