Spectral Quark Model

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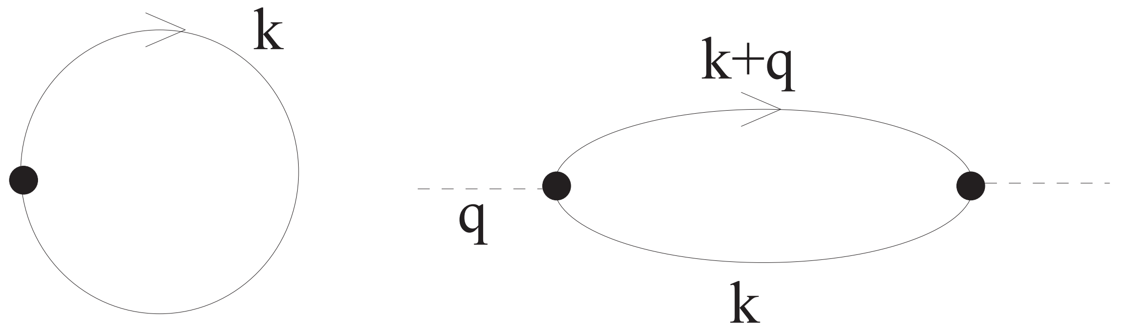
- ERA+WB, Spectral quark model and low-energy hadron phenomenology, hep-ph/0301202, Phys. Rev. **D67** (2003) 074021
- ERA+WB, Pion light-cone wave function and pion distribution amplitude in the Nambu–Jona-Lasinio model, hep-ph/0207266, Phys. Rev. **D66** (2002) 094016
- ERA, in Proc. of the Workshop on Lepton Scattering, Hadrons, and QCD, Adelaide, 2001

What is a Chiral Quark Model?

Prototype: Nambu-Jona–Lasinio

One-loop (leading-N_c)



The momentum running around the loop is cut, $k < \Lambda$

This is not what we are going to do!

Requirements

1. Give finite values for hadronic observables

2. Satisfy the Ward-Takahashi identities, thus reproducing all necessary symmetry requirements

3. Satisfy the anomaly conditions

4. Comply to the QCD factorization property, in the sense that the expansion of a correlator at a large Q is a pure twist-expansion, involving only the inverse powers of Q^2, without the $\log Q^2$ corrections

5. Have the usual dispersion relations

All simultaneously – far from trivial!

The spectral representation

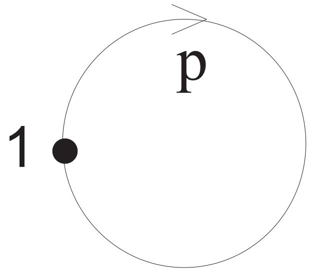
A novel approach, the spectral regularization of the chiral quark model, is based on the Lehmann representation

$$S(p) = \int_C d\omega \frac{\rho(\omega)}{\not{p} - \omega}$$

$\rho(\omega)$ – the spectral function, C – a suitable contour in the complex ω plane

Example: free theory has $\rho(\omega) = \delta(\omega - m)$,

Quark condensate



$$\langle \bar{q} q \rangle \equiv -iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} S(p) = -4iN_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega}{p^2 - \omega^2}$$

The integral over p is quadratically divergent, which requires the use of an auxiliary regularization, *removed* at the end

$$\langle \bar{q} q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \omega \rho(\omega) \left[2\Lambda^2 + \omega^2 \log\left(\frac{\omega^2}{4\Lambda^2}\right) + \omega^2 + \mathcal{O}(1/\Lambda) \right]$$

The finiteness of the result at $\Lambda \to \infty$ requires the conditions The Arriola conditions

$$\int d\omega \omega \rho(\omega) = 0, \quad \int d\omega \omega^3 \rho(\omega) = 0$$

and thus

$$\langle \bar{q} q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \log(\omega^2) \omega^3 \rho(\omega)$$

The spectral condition allowed us to rewrite $\log(\omega^2/\Lambda^2)$ as $\log(\omega^2)$, hence no scale dependence is present

Spectral moments

Postulate

$$\rho_0 \equiv \int d\omega \rho(w) = 1,$$

$$\rho_n \equiv \int d\omega \omega^n \rho(\omega) = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Observables are given by the inverse moments

Such a $\rho(\omega)$ exists!

$$\rho_{-k} \equiv \int d\omega \omega^{-k} \rho(\omega), \quad \text{for } k = 1, 2, 3, \ldots$$

as well as by the "log moments",

$$\rho'_n \equiv \int d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n = 2, 3, 4, \ldots$$

Gauge technique and the vertex functions

Delburgo & West '77

CVC and PCAC imply the Ward-Takahashi identities (WTI)

The gauge technique consists of writing a solution for the unamputated vector and axial vertices

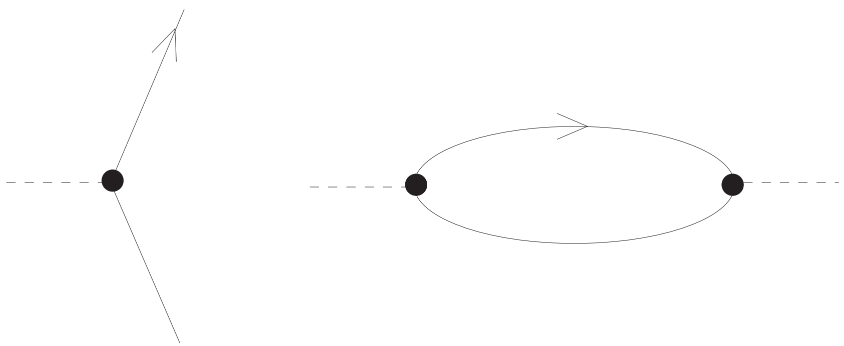
not unique!

$$\Lambda_V^{\mu,a}(p',p) = \int d\omega \rho(\omega) \frac{i}{\not p' - \omega} \gamma^\mu \frac{\lambda_a}{2} \frac{i}{\not p - \omega}$$

$$\Lambda_A^{\mu,a}(p',p) = \int d\omega \rho(\omega) \frac{i}{\not p' - \omega} \left(\gamma^\mu - \frac{2\omega q^\mu}{q^2} \right) \gamma_5 \frac{\lambda_a}{2} \frac{i}{\not p - \omega}$$

pion pole

Similar for more vertices, one $\rho(\omega)$ for each quark line



$e^+e^- \to$ hadrons

At large s we find

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha_{\text{QED}}^2}{3s}\left(\sum_{i=u,d,\ldots} e_i^2\right)\int d\omega\rho(\omega)$$

This is the proper asymptotic QCD result if

$$\int d\omega\rho(\omega) = 1$$

Pion properties

Finiteness of f_π requires the condition $\rho_2 = 0$. Then

$$f_\pi^2 = -\frac{N_c}{4\pi^2}\int d\omega \log(\omega^2)\omega^2\rho(\omega) \equiv -\frac{N_c}{4\pi^2}\rho_2'$$

The electromagnetic form factor

$$F_\pi^{em}(q^2) = \frac{4N_c}{f_\pi^2}\int dw \rho(\omega)\omega^2 I(q^2,\omega)$$

At low-momenta $F_\pi^{em}(0) = 1$

$$F_\pi^{em}(q^2) = 1 + \frac{1}{4\pi^2 f_\pi^2}\left(\frac{q^2\rho_0}{6} + \frac{q^4\rho_{-2}}{60} + \frac{q^6\rho_{-4}}{240} + \dots\right)$$

The mean squared radius reads $\langle r_\pi^2\rangle = \frac{N_c}{4\pi^2 f_\pi^2}$

At large momenta

$$F_\pi^{em}(q^2) \sim \frac{N_c}{4\pi^2 f_\pi^2} \int d\omega \rho(\omega) \{ \frac{2\omega^4}{q^2} \left[\log(-q^2/\omega^2) + 1 \right] + \ldots \}$$

With help of the spectral conditions for $n = 2, 4, 6, \ldots$ we get

$$F_\pi^{em}(q^2) \sim -\frac{N_c}{4\pi^2 f_\pi^2} \left[\frac{2\rho'_4}{q^2} + \frac{2\rho'_6}{q^4} + \frac{4\rho'_8}{q^6} + \ldots \right]$$

All spectral conditions needed!

Pure twist expansion, no logs !

Odd-parity processes

$\pi^0 \to \gamma\gamma$

$$F_{\pi\gamma\gamma}(0,0,0) = \frac{1}{4\pi^2 f_\pi} \int d\omega \rho(\omega) = \frac{1}{4\pi^2 f_\pi}$$

which coincides with the QCD result. Not true when the loop momentum is cut!

$\gamma \to \pi^+\pi^0\pi^-$

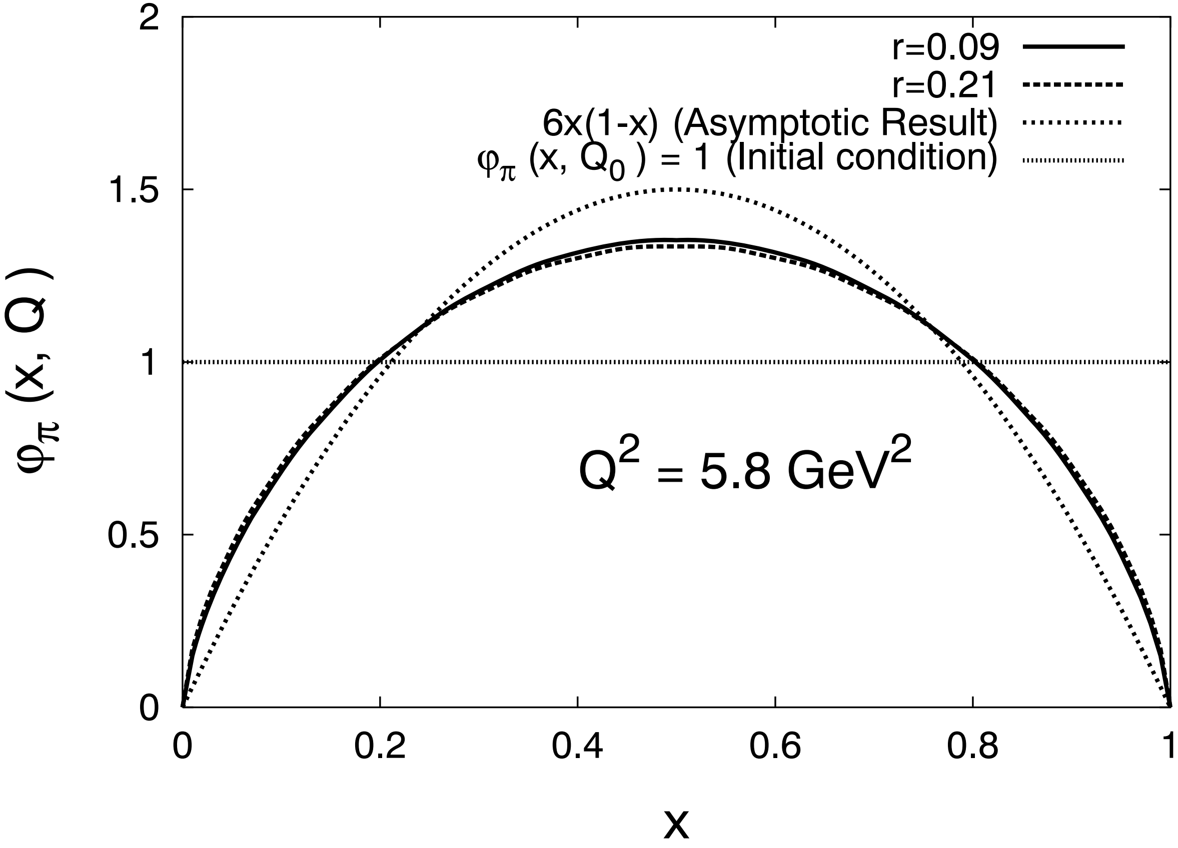
$$F(0,0,0) = \frac{1}{4\pi^2 f_\pi^3} \int d\omega \rho(\omega) = \frac{1}{4\pi^2 f_\pi^3}$$

which is the correct result

High-energy processes

The leading-twist pion distribution amplitude at $Q_0 \sim 320$ MeV

$$\varphi(x; Q_0) = 1$$



Pion light-cone wave function (at the model's scale)

$$\Psi(x,k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}\int d\omega\rho(\omega)\frac{\omega^2}{k_\perp^2+\omega^2}\theta(x)\theta(1-x)$$

Pion structure function (for π^+, at the model's scale)

$$u_\pi(x) = \bar{d}_\pi(1-x) = \theta(x)\theta(1-x),$$

independently of $\rho(\omega)$. One recovers the Bjorken scaling, the Callan-Gross relation, the proper support, the correct normalization, and the momentum sum rule. After DGLAP evolution very good reproduction of the Durham parameterization!

Further results/predictions

Gasser-Leutwyler coefficients: Leading-N_c quark model values

Magnetic permeability of the vacuum, χ

$$\langle 0|\bar{q}(0)\sigma_{\alpha\beta}q(0)|\gamma^{(\lambda)}(q)\rangle = ie_q\chi\,\langle\bar{q}q\rangle\left(q_\beta\varepsilon_\alpha^{(\lambda)} - q_\alpha\varepsilon_\beta^{(\lambda)}\right)$$

$$\chi = \frac{N_c}{4\pi^2}\rho_1'/\langle\bar{q}q\rangle$$

First log-moment

Tensor susceptibility of the vacuum

$$\Pi = i\langle 0|\int d^4z\,T\{\overline{q}(z)\sigma^{\mu\nu}q(z),\overline{q}(0)\sigma_{\mu\nu}q(0)\}|0\rangle = -12f_\pi^2$$

Broniowski, Polyakov & Goeke '98

Résumé

Spectral condition	Physical significance
zeroth moment	normalization
$\rho_0 = 1$	proper normalization of the quark propagator
	preservation of anomalies
	proper normalization of the pion distribution amplitude
	proper normalization of the pion structure function
	reproduction of the large-N_c quark-model values of the Gasser-Leutwyler coefficients
positive moments	finiteness/pure twist
$\rho_1 = 0$	finiteness of the quark condensate, $\langle \bar{q}q \rangle$
	vanishing quark mass at asymptotic Euclidean momenta,
$\rho_2 = 0$	finiteness of the vacuum energy density, B
	finiteness of the pion decay constant, f_π
$\rho_3 = 0$	finiteness of the quark condensate, $\langle \bar{q}q \rangle$
$\rho_4 = 0$	finiteness of the vacuum energy density, B
$\rho_n = 0,\ n = 2, 4 \ldots$	absence of logs in the twist expansion of vector amplitudes
$\rho_n = 0,\ n = 5, 7 \ldots$	finiteness of nonlocal quark condensates, $\langle \bar{q}(\partial^2)^{(n-3)/2} q \rangle$
	absence of logs the twist expansion of the scalar pion form factor

Spectral condition	Physical significance
negative moments	values of observables
$\rho_{-2} > 0$	positive quark wave-function normalization at vanishing momentum
$\rho_{-1}/\rho_{-2} > 0$	positive value of the quark mass at vanishing momentum, $M(0) > 0$
ρ_{-n}	low-momentum expansion of correlators
log-moments	values of observables
ρ'_1	magnetic permeability of the vacuum
$\rho'_2 < 0$	$f_\pi^2 = -N_c/(4\pi^2)\rho'_2$
$\rho'_3 > 0$	negative value of the quark condensate, $\langle \bar{q}q \rangle = -N_c/(4\pi^2)\rho'_3$
$\rho'_4 > 0$	negative value of the vacuum energy density, $B = -N_c/(4\pi^2)\rho'_4$
$\rho'_5 < 0$	positive value of the squared vacuum virtuality of the quark, $\lambda_q^2 = -\rho'_5/\rho'_3$
ρ'_n	high-momentum (twist) expansion of correlators

Meson dominance model

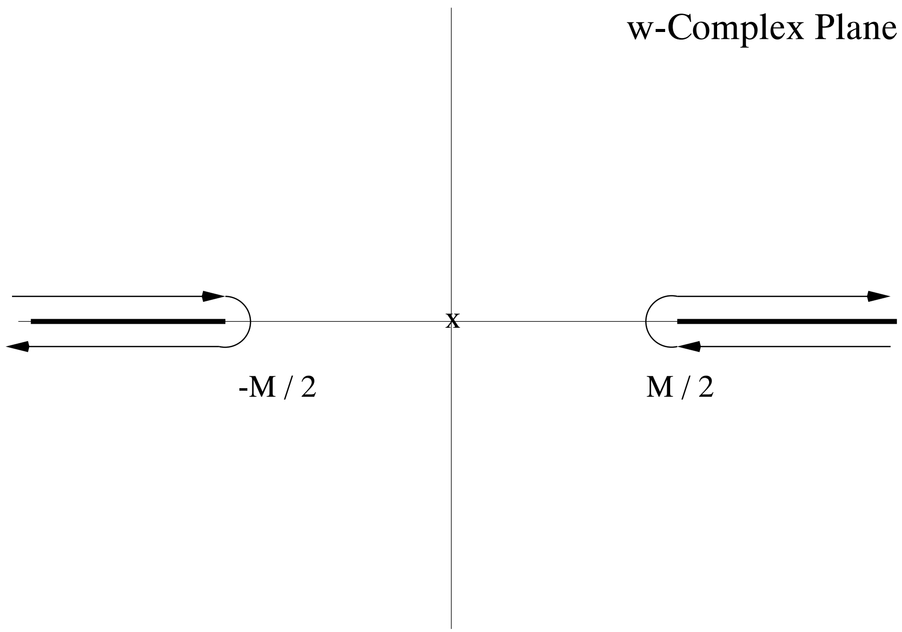
Explicit example of $\rho(\omega)$! Vector-meson dominance (VMD) of the pion form factor is assumed (works up to $t \sim 2$ GeV^2)

$$F_V(t) \;=\; \frac{M_V^2}{M_V^2 + t}$$

with $M_V = m_\rho$. Given this, we get the part of $\rho(\omega)$ responsible for the even moments

$$\rho_V(\omega) = \frac{1}{2\pi i}\frac{3\pi^2 M_V^3 f_\pi^2}{4N_c}\,\frac{1}{\omega}\,\frac{1}{(M_V^2/4-\omega^2)^{5/2}}.$$

The function $\rho_V(\omega)$ has a single pole at the origin and branch cuts starting at $\pm M_V/2$.



The condition $\rho_0 = 1$ gives $M_V^2 = 24\pi^2 f_\pi^2 / N_c$ (matching quark models to VMD) The positive even moments fulfill the spectral conditions Miracle!

$$\rho_{2n} = 0, \qquad n = 1, 2, 3 \ldots$$

The log-moments and negative even moments are finite

For the case of the scalar spectral function (controlling odd moments) we proceed heuristically, by proposing its form in analogy to ρ_V

$\rho = \rho_V + \rho_S$

$$\rho_S(\omega) = \frac{1}{2\pi i} \frac{-48\pi^2 \langle \bar{q} q \rangle}{N_c M_S^4 (1 - 4\omega^2 / M_S^2)^{5/2}}$$

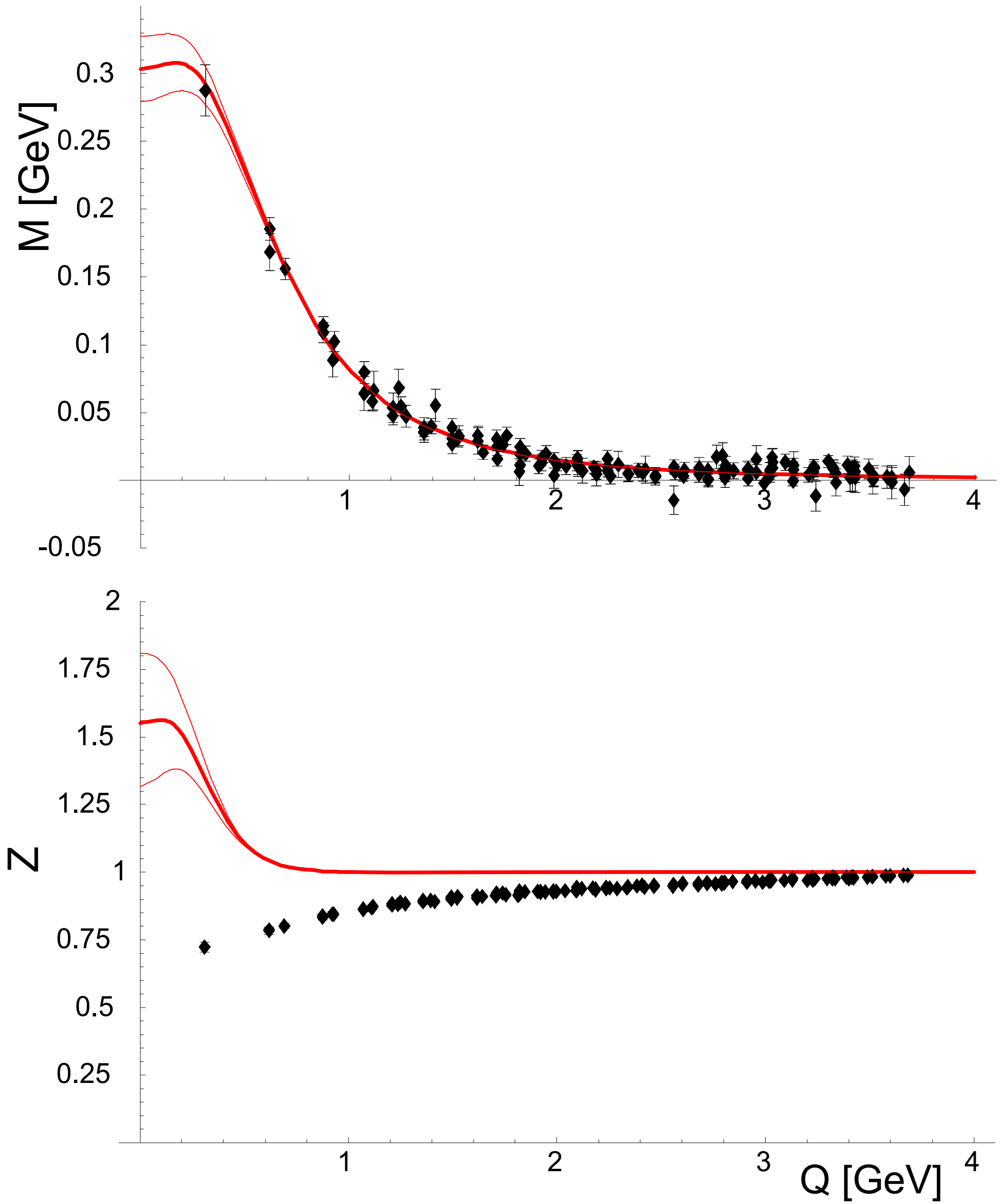
M_S is a scale parameter. The analytic structure similar to $\rho_V(\omega)$, except for the absence of the pole at $\omega = 0$. Odd positive moments vanish!

The quark propagator

$$
\begin{aligned}
S(p) &= A(p)\not{p} + B(p) = Z(p)\frac{\not{p} + M(p)}{p^2 - M^2(p)} \\
A(p^2) &\equiv \int_C d\omega \frac{\rho_V(\omega)}{p^2 - \omega^2} = \frac{1}{p^2}\left[1 - \frac{1}{(1 - 4p^2/M_V^2)^{5/2}}\right] \\
B(p^2) &\equiv \int_C d\omega \frac{\omega \rho_S(\omega)}{p^2 - \omega^2} = \frac{48\pi^2\langle\bar{q}q\rangle}{M_S^4 N_c (1 - 4p^2/M_S^2)^{5/2}}
\end{aligned}
$$

No poles in the whole complex plane! Only branch cuts starting at $p^2 = M_{V,S}^2/4$. The absence of poles is sometimes called "the analytic confinement"

Poles would lead to cuts in form f.



data:

Bowman, Heller,

& Williams '02

$M(Q^2)$ decreases as $1/Q^3$ at large Euclidean momenta, which is favored by the recent lattice calculations. The fit results in

$$
\begin{aligned}
M_S &= 970 \pm 21 \text{ MeV}, \\
M(0) &= 303 \pm 24 \text{ MeV}
\end{aligned}
$$

with $\chi^2/\text{DOF} = 0.72$. The corresponding value of $\langle \bar{q} q \rangle$ is

$$
\langle \bar{q} q \rangle = -(243.0^{+0.1}_{-0.8} \text{ MeV})^3
$$

Summary

1. Assumptions: generalized spectral representation, one quark loop (large N_c), gauge technique (WTI), **spectral conditions** (finiteness)

2. Symmetries, anomalies, normalization, pure twist expansion, preserved

3. **Dynamics encoded in moments**, the approach itself is non-dynamical

4. Specific relations follow, since all observables are expressed in moments of the spectral function

5. The method is technically very simple (computations are short) and predictive (lots of applications)

6. What does not work (at the moment): second Weinberg sum rule (modify vertices?, freedom)

7. Applicable to **high-energy** processes. Very reasonable results follow after evolution

8. Interesting particular realization of the spectral method: the **meson-dominance model**

9. **Analytic confinement** in the sense of the absence of poles in the quark propagator

10. Surprisingly good $M(Q^2)$ vs. lattice results, $Z(Q^2)$ could be better

11. Specific predictions of the VMD model for unintegrated PDF, non-local quark condensate, ...

Other predictions

Pion light-cone wave function and PDF:

$$\Psi(x,k_\perp) = q(x,k_\perp) = \frac{3M_V^3}{16\pi(k_\perp^2 + M_V^2/4)^{5/2}}\theta(x)\theta(1-x)$$

Nonlocal quark condensate:

$$Q(z) = \exp(-M_S\sqrt{-z^2}/2)$$

Magnetic permeability of the vacuum:

$$\chi = \frac{2}{M_S^2}$$

After evolution $\chi(1\ \mathrm{GeV}) = 3.3\ \mathrm{GeV}^2$ in agreement with other estimates

c.f. Ball, Braun & Kivel '03

BACKUP SLIDES

Perturbative QCD yields at LO

Non-perturbative?

$$\rho(\omega) = \delta(\omega - m) + \mathrm{sign}(\omega)\frac{\alpha_S C_F}{4\pi}\frac{1-\xi}{\omega}\theta(\omega^2 - m^2)$$

In the perturbative phase with no spontaneous symmetry breaking, where $\rho(\omega) = \rho(-\omega) = \delta(\omega)$, we have $\langle \bar{q}q \rangle = 0$.

With the accepted value of

$$\langle \bar{q}q \rangle = \simeq -(243 \text{ MeV})^3$$

we infer the value of the third log-moment. The negative sign of the quark condensate shows that

$$\int d\omega \log(\omega^2)\omega^3\rho(\omega) > 0.$$

The vector and axial-vector currents of QCD are:

$$J_V^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \frac{\lambda_a}{2} q(x), \quad J_A^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x)$$

CVC and PCAC:

$$\partial_\mu J_V^{\mu,a}(x) = 0, \quad \partial_\mu J_A^{\mu,a}(x) = \bar{q}(x)\hat{M}_0 i\gamma_5 \frac{\lambda_a}{2} q(x)$$

A number of results are then obtained essentially for free:

- Pions arise as Goldstone bosons, with standard current-algebra properties

for free!

- At high energies parton-model features, such as the spin-1/2 nature of hadronic constituents, are recovered

The vector and axial unamputated vertex functions:

$$\Lambda_{V,A}^{\mu,a}(p',p) = \int d^4x d^4x' \langle 0|T\left\{J_{V,A}^{\mu,a}(0)q(x')\bar{q}(x)\right\}|0\rangle e^{ip'\cdot x' - ip\cdot x}$$

WTI

$$
\begin{aligned}
(p'-p)_\mu \Lambda_V^{\mu,a}(p',p) &= S(p')\frac{\lambda_a}{2} - \frac{\lambda_a}{2}S(p) \\
(p'-p)_\mu \Lambda_A^{\mu,a}(p',p) &= S(p')\frac{\lambda_a}{2}\gamma_5 + \gamma_5\frac{\lambda_a}{2}S(p)
\end{aligned}
$$

“Transverse ambiguity”

The above ansätze fulfil the WTI’s. They are determined up to *transverse pieces*.

This ambiguity appears in all effective models. Current conservation fixes only the longitudinal pieces. Example:

$$j_\mu = \bar{\psi}\left(f_1 \gamma_\mu + i f_2 \sigma_{\mu\nu} q^\nu\right)\psi$$

The condition $q^\mu j_\mu = 0$ does not constrain the f_2-term, since $\sigma_{\mu\nu} q^\nu q^\mu = 0$ from antisymmetry.

Vertices with two currents

Vertices with two currents, axial or vector, are constructed similarly. The vacuum polarization is

$$
\begin{aligned}
i\Pi^{\mu a,\nu b}_{VV}(q) &= \delta^{ab}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\bar{\Pi}_{VV}(q) = \int d^4x e^{-iq\cdot x}\langle 0|T\left\{J^{\mu a}_V(x)J^{\nu b}_V(0)\right\}|0\rangle \\
&= -N_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not p - \not q - \omega}\gamma_\mu \frac{\lambda_a}{2}\frac{i}{\not p - \omega}\gamma_\nu \frac{\lambda_b}{2}\right]
\end{aligned}
$$

transverse

$$
\begin{aligned}
\bar{\Pi}_{VV}(q) &= \ldots \\
I(q^2,\omega) &= -\frac{1}{(4\pi)^2}\int_0^1 dx \log\left[\omega^2 + x(1-x)q^2\right]
\end{aligned}
$$

Dispersion relation

The twice-subtracted dispersion relation holds:

$$\bar{\Pi}_V(q^2) = \frac{q^4}{\pi}\int_0^\infty \frac{dt}{t^2}\frac{\mathrm{Im}\bar{\Pi}_V(t)}{t-q^2-i0^+}$$

This is in contrast to quark models formulated in the Euclidean space, where the usual dispersion relations do not hold

The pion decay constant, defined as

$$\langle 0\,|J_A^{\mu a}(x)|\,\pi_b(q)\rangle = i f_\pi q_\mu \delta_{a,b} e^{iq\cdot x},$$

can be computed from the axial-axial correlation function. The result is

Vacuum energy density

$$\langle \theta^{\mu\nu} \rangle = -iN_cN_f \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \times$$
$$\mathrm{Tr} \frac{1}{\not p - \omega} \left[\frac{1}{2} (\gamma^\mu p^\nu + \gamma^\nu p^\mu) - g^{\mu\nu} (\not p - \omega) \right] = B g^{\mu\nu} + \langle \theta^{\mu\nu} \rangle_0,$$

where $\langle \theta^{\mu\nu} \rangle_0$ is the energy-momentum tensor for the free theory, evaluated with $\rho(\omega) = \delta(\omega)$, and B (bag constant) is the vacuum energy density:

$$B = -iN_cN_f \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega^2}{p^2 - \omega^2},$$

The conditions that have to be fulfilled for B to be finite are

$$\rho_2 = 0, \quad \rho_4 = 0$$

Then

$$B = -\frac{N_c N_f}{16\pi^2}\rho_4' \equiv -\frac{3N_c}{16\pi^2}\int d\omega \log(\omega^2)\omega^4\rho(\omega)$$

for $N_f = 3$.

According to QCD sum rules

$$B = -\frac{9}{32}\langle\frac{\alpha}{\pi}G^2\rangle = -(224^{+35}_{-70}\mathrm{MeV})^4$$

The negative sign of B enforces

$$\rho_4' > 0$$

Pion-quark coupling

Near the pion pole ($q^2 = 0$) we get

$$\Lambda_A^{\mu,a}(p+q,p) \to -\frac{q^\mu}{q^2}\Lambda_\pi^a(p+q,p),$$

where

$$\Lambda_\pi^a(p+q,p) = \int d\omega \rho(\omega) \frac{i}{\not{p}+\not{q}-\omega} \frac{\omega}{f_\pi} \gamma_5 \lambda_a \frac{i}{\not{p}-\omega}$$

We recognize in our formulation the Goldberger-Treiman relation for quarks:

$$g_\pi(\omega) = \frac{\omega}{f_\pi}$$

QCD evolution of PDA

All results of the effective, low-energy model, refer to a soft energy scale, Q_0. In order to compare to experimental results, obtained at large scales, Q, the QCD evolution must be performed. Initial condition:

$$\varphi(x; Q_0) = \theta(x)\theta(1-x).$$

The evolved distribution amplitude reads

$$\begin{aligned} \varphi(x; Q) &= 6x(1-x) \sum_{n=0}^{\infty} C_n^{3/2}(2x-1) a_n(Q) \\ a_n(Q) &= \frac{2}{3} \frac{2n+3}{(n+1)(n+2)} \left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right)^{\gamma_n^{(0)}/(2\beta_0)} \end{aligned}$$

where $C_n^{3/2}$ are the Gegenbauer polynomials, $\gamma_n^{(0)}$ are appropriate anomalous dimensions, and $\beta_0 = 9$.

Results extracted from the experimental data of CLEO provide

$a_2(2.4\mathrm{GeV}) = 0.12 \pm 0.03$, which we use to fix

$$\alpha(Q = 2.4\mathrm{GeV})/\alpha(Q_0) = 0.15 \pm 0.06$$

At LO this corresponds to $Q_0 = 322 \pm 45$ MeV

Now we can predict

$$\begin{aligned} a_4(2.4\mathrm{GeV}) &= 0.06 \pm 0.02 \quad (\mathrm{exp}: -0.14 \pm 0.03 \mp 0.09) \\ a_6(2.4\mathrm{GeV}) &= 0.02 \pm 0.01 \end{aligned}$$

Encouraging, with leading-twist and LO QCD evolution!

QCD evolution of PDF

The QCD evolution of the constant PDF has been treated in detail by Davidson & ERA at LO and NLO. In particular, the non-singlet contribution to the energy-momentum tensor evolves as

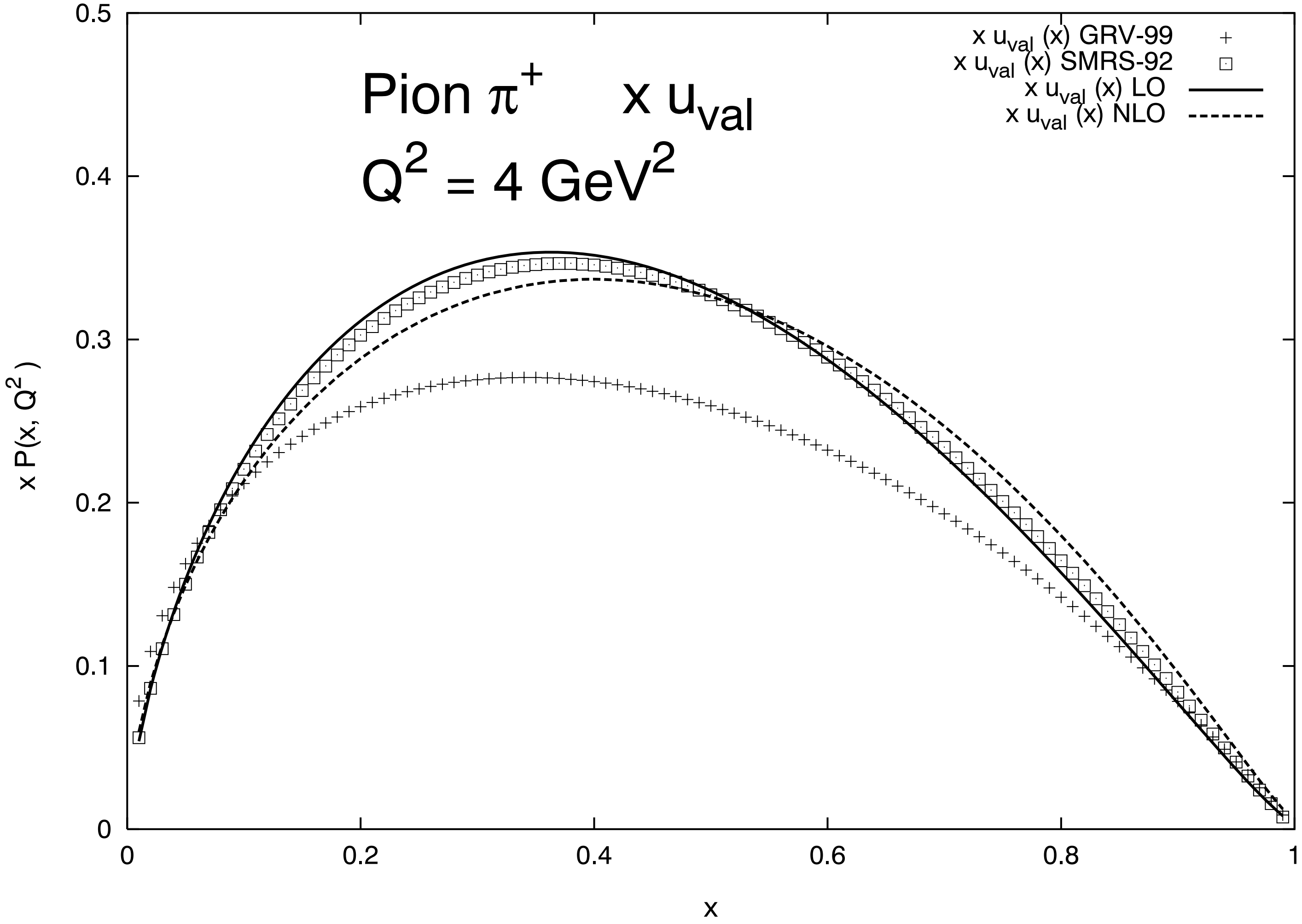
$$\frac{\int dx\, xq(x,Q)}{\int dx\, xq(x,Q_0)} = \left(\frac{\alpha(Q)}{\alpha(Q_0)}\right)^{\gamma_1^{(0)}/(2\beta_0)} ,$$

In has been found that at $Q^2 = 4\mathrm{GeV}^2$ the valence quarks carry $47 \pm 0.02\%$ of the total momentum fraction in the pion. Downward LO evolution yields that at the scale

$$Q_0 = 313^{+20}_{-10}\mathrm{MeV}$$

the quarks carry 100% of the momentum. The agreement of the evolved PDF with the SMRS data analysis is impressive

DGLAP evolution



Gasser-Leutwyler coefficients

The one-quark-loop effective action that incorporates the quark-pion coupling obeying the Goldberger-Treiman relation can be written as

$$S = -iN_c \int d^4x \int d\omega \rho(\omega) \mathrm{Tr} \log \left[i\partial\!\!\!/ - \omega \exp\left(i\gamma_5 \tau_a \phi_a(x)/f_\pi \right) \right]$$

This form is manifestly chirally symmetric, with ϕ denoting the non-linearly realized pion field. One may evaluate the Gasser-Leutwyler coefficients through the use of standard derivative expansion techniques. With the spectral normalization condition imposed, the calculation is equivalent to standard quark-model calculations with the cut-off removed. One gets

$$\begin{aligned} \bar{l}_1 &= -N_c, \\ \bar{l}_2 &= N_c. \end{aligned}$$

Other low energy constants, such as $\bar{l}_3$ and $\bar{l}_4$ require a specification of explicit chiral symmetry breaking within the quark model.