Phenomenology of ultrarelativistic nuclear collisions I

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Six e-lectures for PhD students

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The pdf file with the covered material is accessible via Dropbox

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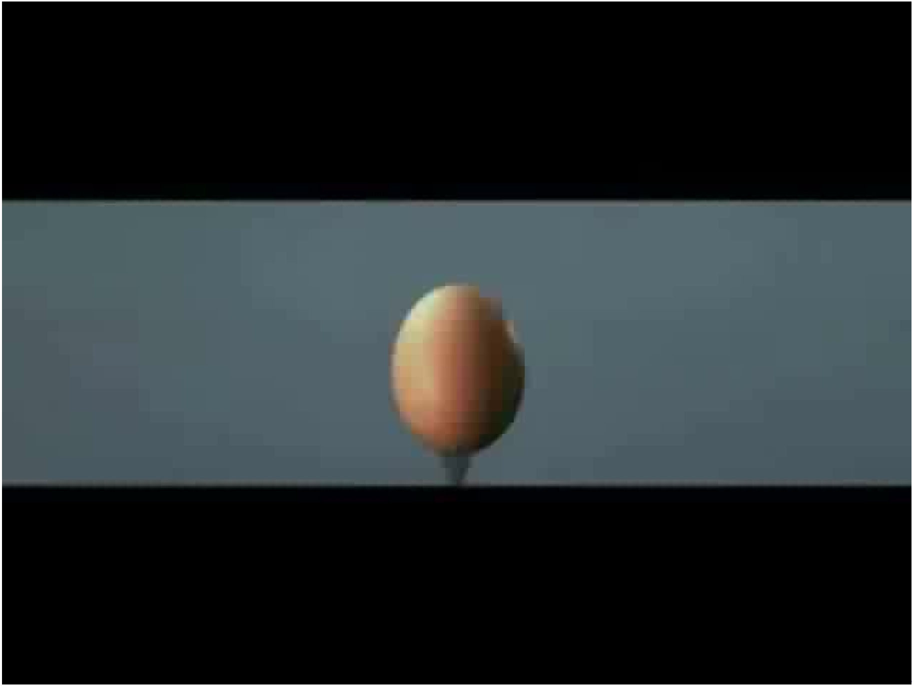
Lecture 1

Foreword

Feynman: Scattering of protons on protons is like colliding Swiss watches to find out how they are built.



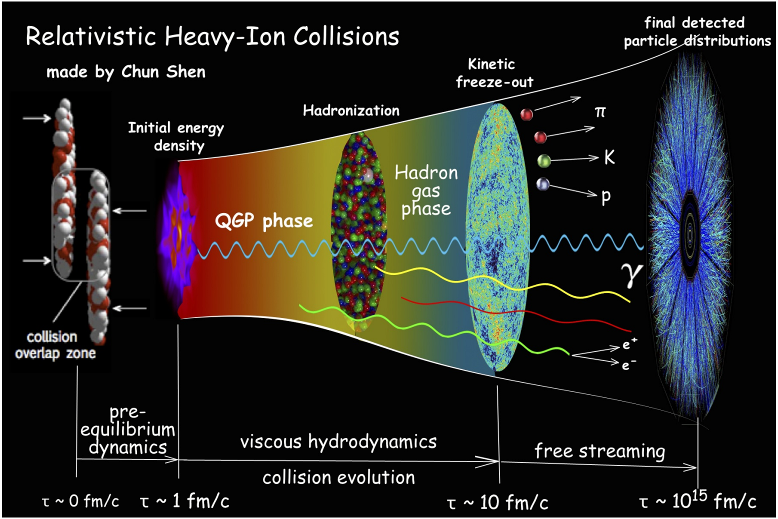
Studying the hydrodynamics of water by shooting at a watermelon!

- What is the equation of state, sound velocity, viscosity . . . ?
- What was the shape before destruction?

Not a completely impossible task . . .

Note that in the explosion some matter is ejected backwards!

Little bangs



Three stages of the "Standard Model" of Little Bangs

partons hydrodynamization quark-gluon plasma freeze-out hadrons

time: ∼1 fm/c ∼10 fm/c

These rather basic lectures are given for an audience that is familiar with many aspects of the field, but I hope that nevertheless they will be useful. **I am open to suggestions on the way what to cover in a greater detail.**

Introduction

Some basic kinematics

Coordinates of 4-vectors

Consider any 4-vector (take momentum for definiteness)

$$p^\mu = (p_0, \vec{p}) = (E, p_T \cos\phi, p_T \sin\phi, p_{||})$$

(energy, transverse momentum, azimuthal angle, longitudinal momentum) with

$$p_{||} = p\cos\theta,\ p_T = p\sin\theta,\ p^\mu p_\mu = m^2 \geq 0,\ E^2 = m^2 + p^2$$

($p = |\vec{p}|$, polar angle, mass, time-like 4-vector).
A set of 4 variables is *complete*, e.g., $\{m, p, \phi, \theta\}$, or $\{E, p_T, \phi, p_{||}\}$.
Quite generally, one may pass to another set of coordinates, i.e.,

$$(E, p_{||}) \to (E(c_1, c_2), p_{||}(c_1, c_2))$$

with the functions chosen in such a way that the whole (relevant) space is covered.

Rapidity

Most important case of such a transformation:

$$E = m_T \cosh y, \quad p_\parallel = m_T \sinh y$$

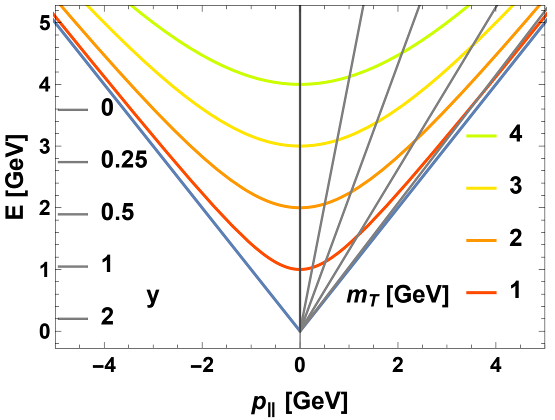
Immediately $\tanh y = p_\parallel / E$
(recall $E = \gamma m$, $\vec{p} = \vec{v}\gamma m$, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$, hence $\tanh y = v/c$)
The transverse mass m_T can be obtained from

$$m^2 = p^\mu p_\mu = E^2 - p_\parallel^2 - p_T^2 = m_T^2(\cosh^2 y - \sinh^2 y) - p_T^2 = m_T^2 - p_T^2,$$

hence $m_T^2 = m^2 + p_T^2$.
Inverse transformation:

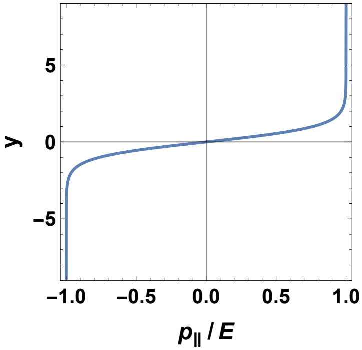
$$y = \text{arctanh}\frac{p_\parallel}{E} = \frac{1}{2}\log\frac{E + p_\parallel}{E - p_\parallel}, \quad m_T = \sqrt{E^2 - p_\parallel^2}.$$

Rapidity cont.



Hyperboloids $E^2 - p_\parallel^2 = m_T^2$ and straight lines with slope $\tanh y$, light cone $E = p_\parallel$, high resolution in y near the light cone. Lowest value for m_T is, of course, m. Completeness: all accessible space is covered with the new coordinates.

Forward/backward resolution



The curve $y = \text{arctanh}\, p_{\|}/E$. As much is happening in the forward $(E/p_{\|} \to 1)$ or backward $(E/p_{\|} \to -1)$ direction, passing to y provides the necessary resolution.

Phase space

Integration over the phase space (of a single particle on the mass shell) involves the Lorentz-invariant measure, $I = \int d^4p\delta(p^2 - m^2)\theta(p^0)f(p)$. Passing to (y, m_T) variables we have $dp^0 dp^3 = |J| dy\, dm_T$, where the Jacobian is

$$J = \begin{vmatrix} dp_0/dm_T & dp_0/dy \\ dp_3/dm_T & dp_3/dy \end{vmatrix} = \begin{vmatrix} \cosh y & m_T \sinh y \\ \sinh y & m_T \cosh y \end{vmatrix} = m_T$$

Substituting and integrating over m_T, and using $\delta(g(x)) = \sum_i \delta(x - z_i)/|g'(z_i)|$

$$\begin{aligned} I &= \int d^2p_T dy\, dm_T m_T \delta(m_T^2 - p_T^2 - m^2)\theta(m_T)f(p) \\ &= \int d^2p_T dy\, dm_T m_T \frac{\delta(m_T - \sqrt{m^2 + p_T^2})}{2m_T} f(p) \\ &= \int \tfrac{1}{2} d^2p_T dy\, f(p_T, y) \end{aligned}$$

Phase space 2

Simpler derivation:

$$
\begin{aligned}
I &= \int dp_0 d^3p \delta(p_0^2 - p_T^2 - m^2)\theta(p_0) f(p) = \int \frac{d^3p}{2E} f(p) \\
&= \int d^2p_T \frac{dp_3}{2E} f(p), \quad E = \sqrt{m^2 + p^2}
\end{aligned}
$$

We have (for fixed p_T) $dp_3 = m_T d(\mathrm{sinh} y) = m_T \mathrm{cosh} y \, dy$, hence again

$$I = \int \tfrac{1}{2} d^2p_T dy \, f(p_T, y)$$

Boost invariance

Lorentz transformation (system boosted along the z-axis with velocity $v/c = \tanh\zeta$):

$$\begin{aligned} E &\rightarrow E\cosh\zeta + p_{||}\sinh\zeta \\ p_{||} &\rightarrow p_{||}\cosh\zeta + E\sinh\zeta \end{aligned}$$

Then

$$\begin{aligned} E &\rightarrow m_T\cosh y\cosh\zeta + m_T\sinh y\sinh\zeta = m_T\cosh(y+\zeta) \\ p_{||} &\rightarrow m_T\sinh y\cosh\zeta + m_T\cosh y\sinh\zeta = m_T\sinh(y+\zeta) \\ y &\rightarrow y+\zeta \end{aligned}$$

(rapidities are additive). Thus the measure $d^2p_T dy$ is invariant with respect to boosts along z. In particular, the spectrum

$$\frac{d^3N}{d^2p_T dy}(p_T, y)$$

is simply shifted in the y variable by ζ, with no change of shape.

The above additivity is the main reason to use the rapidity variable y!

Going between Lab and CM frames

Mandelstam's invariant $s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$.
In collider experiments one typically quotes $\sqrt{s_{NN}}$, which relates to nucleons from two different projectiles. In CM frame of this NN system,

$$s_{NN} = (E_1 + E_2)^2 = [m_N(\cosh y_b^{\rm CM} + \cosh(-y_b^{\rm CM}))]^2 = [2m_N \cosh y_b^{\rm CM}]^2,$$

hence the rapidity of the beam in CM is

$$y_b^{\rm CM} = \mathrm{arccosh}\left(\frac{\sqrt{s_{NN}}}{2m_N}\right)$$

(the other beam has $-y_b^{\rm CM}$). Going to the Lab frame means carrying a boost such that one nucleon is at rest, i.e. a boost with rapidity y_b. Then

$$E_1 = E_{N,\rm Lab} = E_{\rm Lab}/A = m\cosh(y_b^{\rm Lab}), \quad y_b^{\rm Lab} = 2y_b^{\rm CM}, \quad E_2 = m,$$

where A is the mass number. Since $\cosh(2\alpha) = 2\cosh^2\alpha - 1$, or $\cosh(2\,\mathrm{arccosh}\,z) = 2z^2 - 1$, we get

$$E_{\rm Lab}/A = \frac{s_{NN}}{2m_N} - m_N, \quad \sqrt{s_{NN}} = \sqrt{2m_N}\sqrt{E_{\rm Lab}/A + m_N}$$

Examples

Pb+Pb at SPS, $E_{\rm Lab} = 158$ GeV A

$m_N = (938.27 + 939.57)/2$ GeV $\simeq 939$ MeV

$$\sqrt{s_{NN}} = \sqrt{2m_N}\sqrt{158\ \mathrm{GeV} + m_N} \simeq 17.3\ \mathrm{GeV}, \quad y_b^{\rm Lab} \simeq 5.8$$

p+p at LHC, $\sqrt{s_{NN}} = 14$ TeV

$$E_{\rm Lab} = \frac{(14\ \mathrm{TeV})^2}{2m_p} - m_p \simeq 1 \times 10^5\ \mathrm{TeV}, \quad y_b^{\rm CM} = 9.6$$

Pseudorapidity

Similar game:

$$p = |\vec{p}| = p_T \cosh \eta, \quad p_{\|} = p_T \sinh \eta$$

Then $\tanh \eta = p_{\|}/p$. In the polar representation

$$p_{\|} = p \cos \theta, \quad p_T = p \sin \theta$$

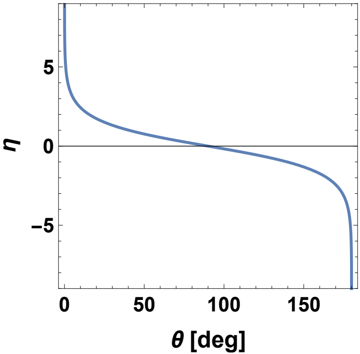
hence

$$\eta = \text{arctanh} \cos \theta, \quad \theta = \arccos \tanh \eta$$

$$\eta = \frac{1}{2} \log \frac{p + p_{\|}}{p - p_{\|}}, \quad p_T = \sqrt{p^2 - p_{\|}^2}.$$

Remark: for $m = 0$ we immediately have $\eta = y$

Pseudorapidity 2



Large resolution in η for very forward or backward motion

$$y = \log\left(\frac{2}{\theta}\right) + O\left(\theta^2\right),\ \frac{\pi}{2} - \theta + O\left(\left(\theta - \frac{\pi}{2}\right)^2\right),\ -\log\left(\frac{2}{\pi-\theta}\right) + O\left(\pi-\theta\right)^2\right),$$

Homework

- Plot the analog of the figure from slide 14 for the present case of pseudorapidity
- Derive the above expansion

Pseudorapidity vs rapidity

From previous expressions we readily get

$$\eta = \tfrac{1}{2}\log\frac{\sqrt{m_T^2\cosh^2 y - m^2} + m_T \mathrm{sinh} y}{\sqrt{m_T^2\cosh^2 y - m^2} - m_T \mathrm{sinh} y}$$

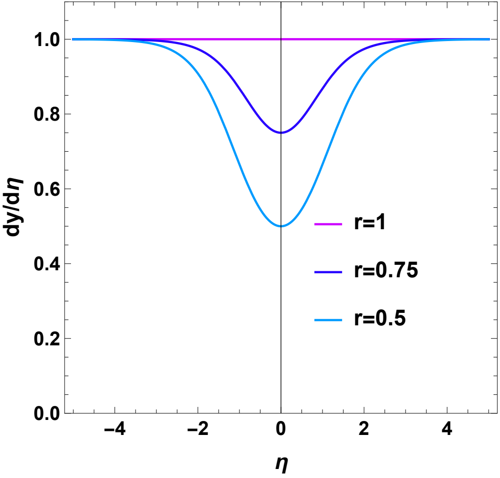
$$y = \tfrac{1}{2}\log\frac{\sqrt{p_T^2\cosh^2 \eta + m^2} + p_T \mathrm{sinh} \eta}{\sqrt{p_T^2\cosh^2 \eta + m^2} - p_T \mathrm{sinh} \eta}$$

Because $p_{\parallel} = m_T \sinh y = p_T \sinh \eta$, then $m_T \cosh y \, dy = p_T \cosh \eta \, d\eta$, hence

$$\frac{dy}{d\eta} = \frac{p}{E} = \frac{\sqrt{p_T^2 + m_T^2 \sinh^2 y}}{m_T \cosh y} = \frac{p_T \cosh \eta}{\sqrt{m_T^2 + p_T^2 \sinh^2 \eta}} \quad (\leq 1)$$

$dy/d\eta$

$r = p_T/m_T$



Pseudorapidity vs rapidity 2

With $r = p_T/m_T$ we can rewrite

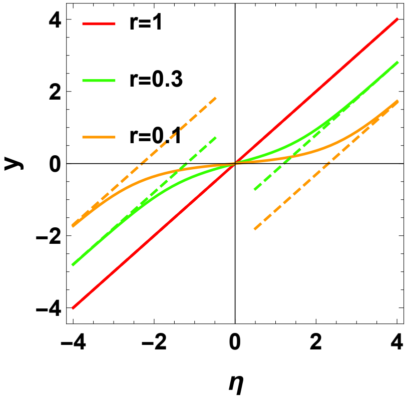
$$\eta = \mathrm{arcsinh}(1/r \sinh y) = \tfrac{1}{2} \log \frac{\sqrt{\sinh^2 y + r^2} + \mathrm{sinh} y}{\sqrt{\sinh^2 y + r^2} - \mathrm{sinh} y}$$

$$y = \mathrm{arcsinh}(r \sinh \eta) = \tfrac{1}{2} \log \frac{\sqrt{\sinh^2 \eta + 1/r^2} + \mathrm{sinh} \eta}{\sqrt{\sinh^2 \eta + 1/r^2} - \mathrm{sinh} \eta}$$

$$\frac{dy}{d\eta} = \frac{\sqrt{r^2 + \sinh^2 y}}{\cosh y} = \frac{\cosh \eta}{\sqrt{1/r^2 + \sinh^2 \eta}}$$

Note that the symmetry $y \leftrightarrow \eta$, $m_T \leftrightarrow p_T$ ($r \leftrightarrow 1/r$) is explicit.

Pseudorapidity vs rapidity 3



$|y| \leq |\eta| \to$ spectra are broader in η than in y

Asymptotically, $y = \eta + \text{sgn}(\eta) \log r$ (dashed lines)

Note that since $r = p_T/m_T$, the relation depends on p_T and m.

Experimentally, to pass from η to y one needs to identify particles.

Mid-pseudorapidity depletion

Since $dy/d\eta \leq 1$ (equality for $m = 0$), possibility of minimum at $\eta = 0$ (see 3 slides up). Generally, what is extracted experimentally are the double spectra in p_T and y. Obviously,

$$\frac{d^3N}{d^2p_T d\eta} = \frac{d^3N}{d^2p_T dy}\frac{dy}{d\eta}$$

At $\eta = 0$:

$$\begin{aligned}
\left.\frac{dN}{d\eta}\right|_{\eta=0} &\equiv \int d^2p_T \left.\frac{d^3N}{d^2p_T d\eta}\right|_{\eta=0} = \int d^2p_T \left.\frac{d^3N}{d^2p_T dy}\frac{dy}{d\eta}\right|_{y=0} \\
&\leq \int d^2p_T \left.\frac{d^3N}{d^2p_T dy}\right|_{y=0} \equiv \left.\frac{dN}{dy}\right|_{y=0}
\end{aligned}$$

$$\left.\frac{dN}{d\eta}\right|_{\eta=0} \leq \left.\frac{dN}{dy}\right|_{y=0} \quad \text{(equality for } m = 0\text{)}$$

Curvature at mid-pseudorapidity - minimum possible

Let $j(y) \equiv dy/d\eta$. Then from the chain rule

$$\frac{d^2}{d\eta^2}\frac{d^3N}{d^2p_T d\eta} = j(y)\frac{d}{dy}j(y)\frac{d}{dy}j(y)\frac{d^3N}{d^2p_T dy}$$
$$= (j(y)j'(y)^2 + j(y)^2 j''(y))\frac{d^3N}{d^2p_T dy} + 3j(y)^2 j'(y)\frac{d^4N}{d^2p_T d^2y} + j(y)^3\frac{d^5N}{d^2p_T d^3y}$$

Since $j(y)$ is symmetric, at midrapidity $j'(0) = 0$. Furthermore, $j(0) = p_T/m_T$ and $j''(0) = p_T/m_T + m_T/p_T$. Combining these,

$$\left.\frac{d^2}{d\eta^2}\frac{d^3N}{d^2p_T d\eta}\right|_{\eta=0} = \left.\frac{m^2 p_T}{m_T^3}\frac{d^3N}{d^2p_T dy}\right|_{y=0} + \left.\frac{p_T^3}{m_T^3}\frac{d^2}{dy^2}\frac{d^3N}{d^2p_T dy}\right|_{y=0}$$

and, upon integration over d^2p_T,

$$\left.\frac{d^2}{d\eta^2}\frac{dN}{d\eta}\right|_{\eta=0} = \int d^2p_T \left.\frac{m^2 p_T}{m_T^3}\frac{d^3N}{d^2p_T dy}\right|_{y=0} + \int d^2p_T \left.\frac{p_T^3}{m_T^3}\frac{d^2}{dy^2}\frac{d^3N}{d^2p_T dy}\right|_{y=0}$$

positive typically negative (if $d^3N/d^2p_T dy$ max. at $y = 0$)

Mid-pseudorapidity depletion in a schematic model

Homework

Check the formulas from the previous slide and discuss the final result, in particular its dependence on m.

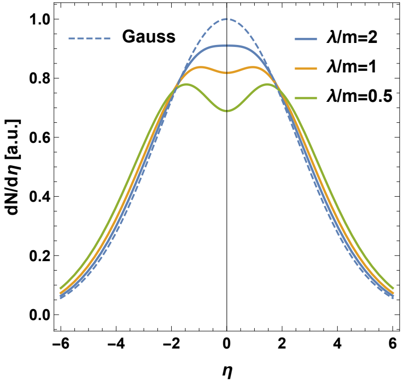
Now consider any value of η. For the sake of example, we take an ansatz factorized in $y - p_T$ (in reality need not be so)

$$\frac{d^3N}{d^2p_T dy} = \frac{d^3N}{d^2m_T dy}(m_T, y) = f(m_T)\frac{dN}{dy}(y).$$

with $dN/(dy\, m_T\, dm_T) = A\exp(-y^2/2\sigma^2)\exp(-m_T/\lambda), \quad \sigma = 2.5$
Typically, $f(m_T) = Ae^{-m_T/\lambda}$, with λ a parameter. Then

$$\frac{dN}{d\eta} = A\int d^2m_T e^{-m_T/\lambda}\frac{dy}{d\eta}(\eta, m_T, m)\frac{dN}{dy}(\eta, m_T, m)$$

Mid-pseudorapidity depletion in a schematic model 2



Depletion, possible 2 maxima, and broadening!

Effect stronger as (averaged) p_T/m_T (or λ/m) decreases, or m increases

Of course, normalization is preserved, $\int d\eta \frac{dN}{d\eta} = \int dy \frac{dN}{dy}$

Homework – carry out numerical integration (Mathematica, python)

Do an example with some other dependence of the spectrum on m_T and y.

Space-time rapidity η_{PS} and proper time τ

Exactly the same construction as for (4-momentum) rapidity but with the 4-coordinates (t, x, y, z)

$$t = \tau \cosh \eta_{PS}, \quad z = \tau \sinh \eta_{PS}$$

or

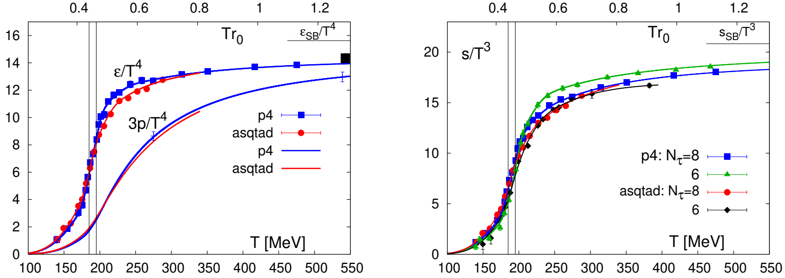
$$\eta_{PS} = \tfrac{1}{2} \log \frac{t+z}{t-z}, \quad \tau = \sqrt{t^2 - z^2}$$

Important in covariant description of dynamics

Hints for Quark-Gluon Plasma

QGP from lattice QCD

At high temperatures the thermal motion is so high, that also the momenta transferred are large and hadrons are melted into quarks and gluons. An early expectation was that weakly-interacting quark-gluon plasma (QGP) should be formed. As seen from the lattice QCD, this is not really the case at accessible temperatures!



[Bazavov et al., PRD 80(2009)014504, arXiv:0903.4379]

QGP → sQGP – strongly interacting QGP. But things happen!

Reminder: the Stefan-Boltzmann law ($\epsilon, P \sim T^4$)

With the grand-canonical ensemble the pressure is

$$P = -\Omega(T,V,\mu)/V = \pm\gamma T \int \frac{d^3k}{(2\pi)^3} \log\left(1 \pm e^{-(E(k)-\mu)/T}\right)$$

$+$ fermions, $-$ bosons, $E(k) = \sqrt{m^2+k^2}$, V - volume, T - temperature, μ - chemical potential, γ - degeneracy factor.
For $m = 0$ and $\mu = 0$: $P = \gamma\frac{\pi^2}{90}T^4$ for bosons and $P = \gamma\frac{7}{8}\frac{\pi^2}{90}T^4$ for fermions, whereas energy density $\epsilon \equiv E/V = 3P$ and entropy density $s = (\epsilon + P)/T = 4P/T$.

QGP (gluons and quarks+antiquarks)

$\gamma = 8 \times 2(\text{color} \times \text{spin}) + 7/8 \times 2 \times 3 \times 2 \times N_f([q + \bar{q}] \times (\text{color} \times \text{spin} \times \text{flavor})$
$N_f = 2$ and 3: $P/T^4 = \simeq 4.06$ and $\simeq 5.21$, respectively (+ bag constant in some models)

$s \simeq 14/\text{fm}^3$ for $T = 175$ MeV

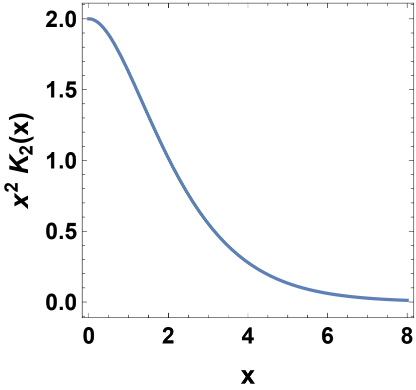
Noninteracting massive pions

$P/T^4 = \gamma_\pi \sum_{k=1}^{\infty} \frac{m^2 K_2\left(\frac{km}{T}\right)}{2\pi^2 k T^2}$, with $\gamma_\pi = 3$

A dramatic growth of the number of degrees of freedom, as seen on the lattice!

Modified Bessel function of rank 2

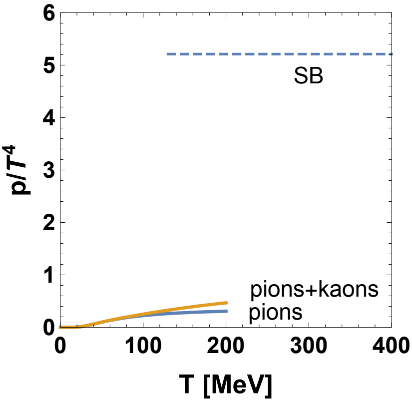
It will appear frequently in thermal analyses . . .



Gas of pions and kaons vs QGP

Homework

Derive P for the noninteracting pion gas

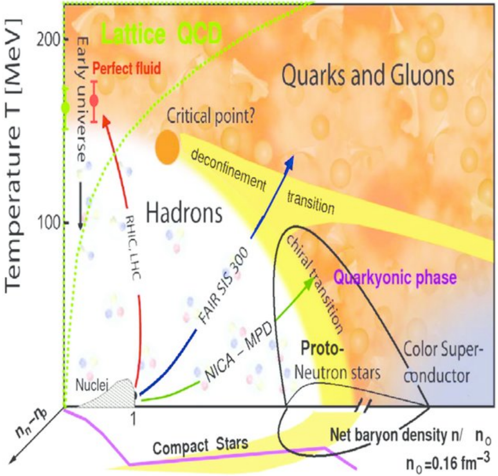


We see a huge jump in p/T^4, but more hadrons contribute (see later on)

Experimental signatures of QGP

- Harmonic flow, radial flow (collectivity)
- Jet quenching
- Strangeness enhancement
- J/ψ production
- ...

Phase diagram of QCD



[from D. P. Menezes]

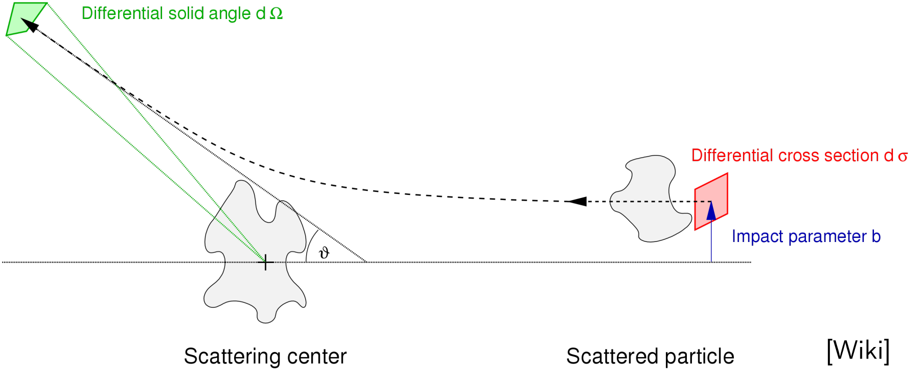
Lecture 2

Basics of scattering

Classical scattering

[recall your QM class!]

Differential cross section $\frac{d\sigma}{d\Omega}$



[Wiki]

Total cross section: $\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega}$

(QM: $d\sigma/d\Omega = |f(\Omega)|^2$)

Experimental determination

$$d\sigma(\Omega) = \frac{dN(\Omega)}{j}$$

$dN(\Omega)$ – number of particles scattered per unit time into $[\Omega, \Omega + d\Omega]$
j – number of incident particles passing per unit time through per unit area perpendicular to the beam (incident flux)
Another interpretation following from the above definition:

$$\sigma = \frac{\text{probability of transition}}{\text{density of scatterers per transverse area}}$$

For a single scatterer it is thus the characteristic area attributed to it
Yet another way:

$$\sigma = \frac{\text{probability of transition per unit of time}}{\text{density of scatterers per volume} \times \text{velocity}}$$

[standard stuff]

Quantum field theoretic definition – full truth :)

Homework

Digest the formulas from the previous slide

For the process $a + b \to 1 + 2 + \dots n$:

$$d\sigma(a + b \to 1 + 2 + \dots n) = \frac{(2\pi)^4 \delta^4(p_a + p_b - p_1 - p_2 - \dots - p_n)}{v_{ab}\, 2E_a\, 2E_b}$$

$$\times \left|\langle p_a p_b \mid T \mid p_1 p_2 \dots p_n \rangle\right|^2 \prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

momentum conservation / flux $\times$ transition rate $\times$ phase space of the final state

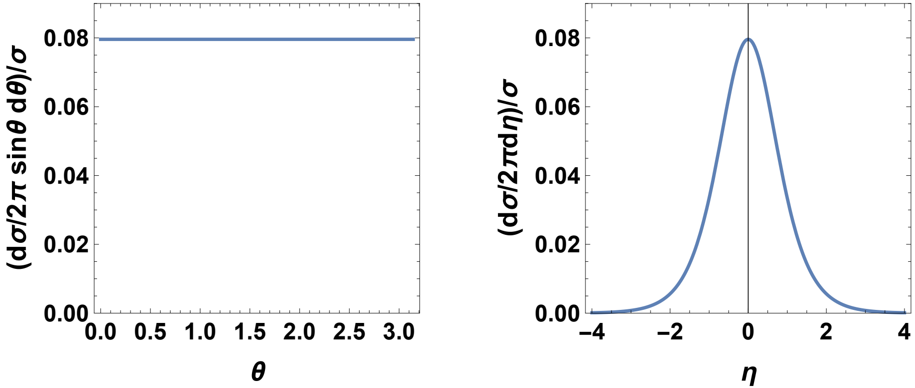
Invariantly, $v_{ab} E_a E_b = \sqrt{(p_a p_b)^2 - (m_a m_b)^2}$

The total cross section is $\sigma = \frac{1}{S} \int d\sigma$, where S is the symmetry factor accounting for identical particles in the final state

Examples

- Classical elastic scattering on a hard sphere: $\frac{d\sigma}{d\Omega} = R^2/4$, $\sigma = \pi R^2$
- QM elastic sc. on a hard sphere, low energy limit: $\frac{d\sigma}{d\Omega} = R^2$, $\sigma = 4\pi R^2$
- – high energy limit: $\sigma = 2\pi R^2$ (wave nature, diffraction)
- Ratherford scattering (Lab): $\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 \alpha_{\rm QED}}{4E_{\rm kin} \sin^2 \frac{\theta}{2}} \right)^2$, $\sigma = \infty$

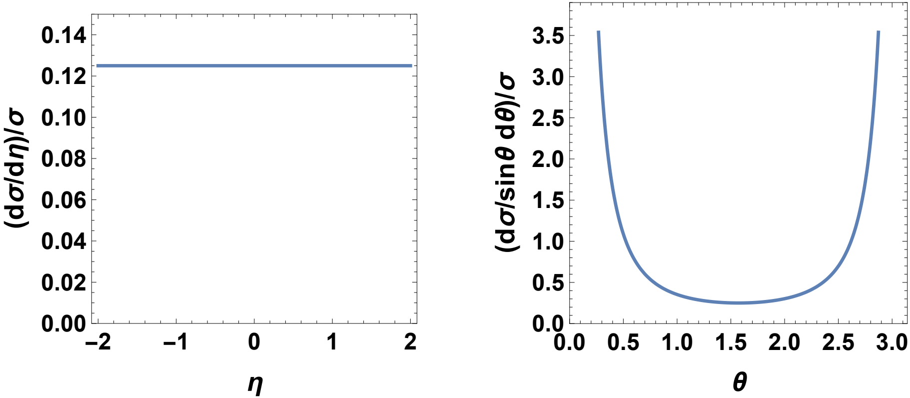
$d\sigma/d\Omega$ uniform in Ω plotted vs pseudorapidity



Homework

Recreate the plots from this and the next slide

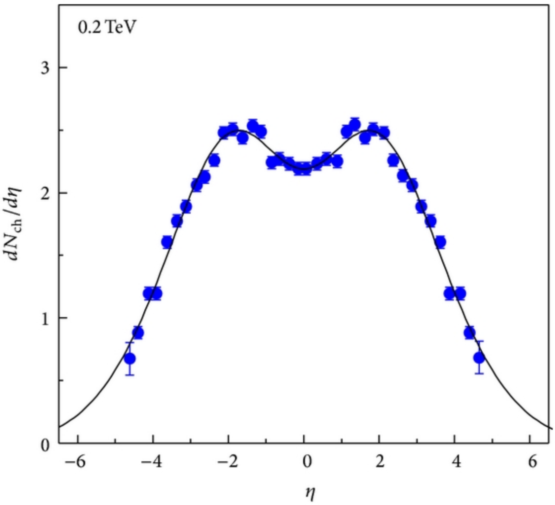
$d\sigma/d\eta$ uniform in $\eta \in [-2, 2]$ plotted vs θ



Message:

Cross section flat in η corresponds to very forward/backward physics!

Spectra in pseudorapidity, pp collisions, UA5 Collaboration



Units

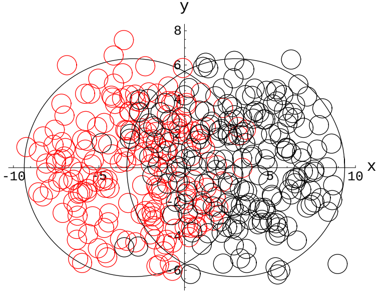
1 fm = (femtometer, fermi) $= 10^{-15}$ m
1 mb $= 10^{-31}$ m^2 $= 10^{-1}$ fm^2

Natural units:

$\hbar$=1, $c = 1$, $k_B = 1$
fm GeV=$\simeq \frac{1}{0.1973}$

Participants and spectators

Au+Au collision at RHIC
(view along the beam)



all nucleons

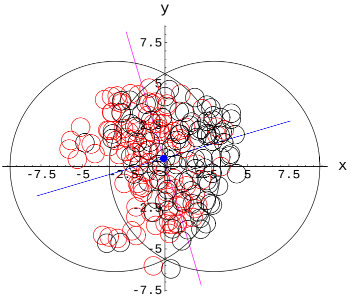
- Participants – nucleons that collide
- Spectators– nucleons that fly by without colliding

Participants may collide elastically and inelastically (i.e., producing particles). In the latter case they are referred to as wounded nucleons

Roughly, the min. transverse distance (impact parameter) d between the colliding nucleons must satisfy the geometric condition $\pi d^2 \lesssim \sigma_{NN}^{\rm inel}$ (inelastic cross section)

Participants and spectators

Au+Au collision at RHIC
(view along the beam)



participants

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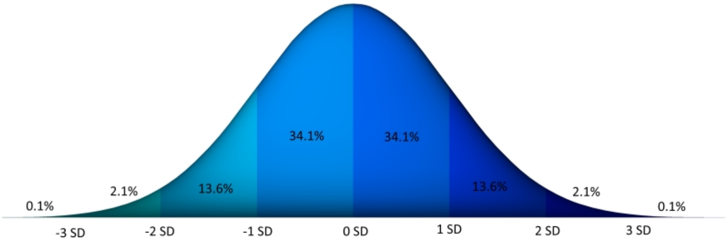
Quantiles

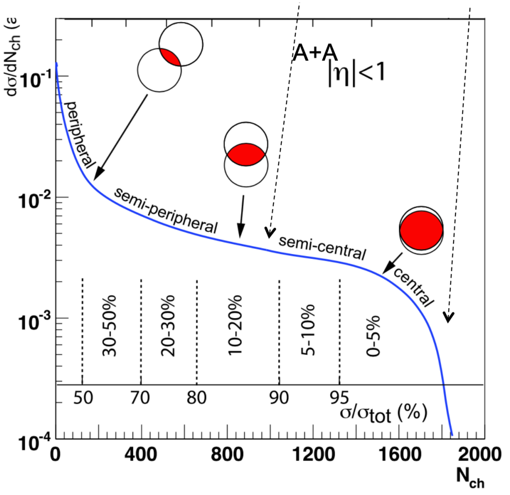
A statistical sample with some characteristics (e.g., people with height), can be divided into quantiles:



In collision experiment the same methodology: events have some characteristics: multiplicity of produced hadrons, response of various detectors, . . . One can cut the sample of events into quantiles according to. a chosen feature (e.g., multiplicity), here called centrality classes.

Centrality as quantiles of multiplicity

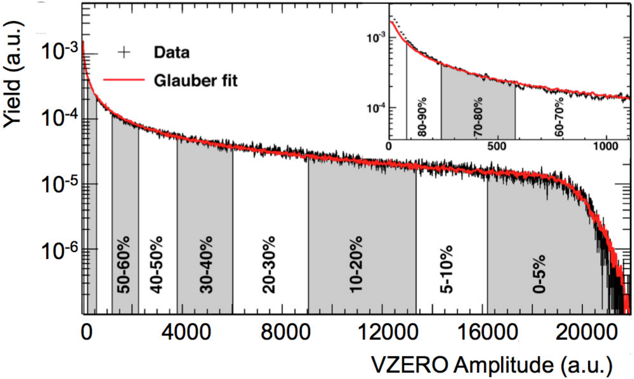
geometric intuition [fig. from J. Jia]



in pp – activity

Centrality as quantiles of a forward detector

Not one definition! Depends on the feature chosen (which detector used). Analogy: for people, one an arrange with height or with weight. Result not the same (they would be exactly the same if weight were strictly proportional to hight, with no fluctuations)



[ALICE, arXiv:1306.3130]

Centrality vs impact parameter

From geometry, $N_{\rm ch}$ decreases with b, as the overlap region is smaller. If this is strictly monotonic (i.e, no fluctuations), centrality classes for both features would be the same. Then (see below)

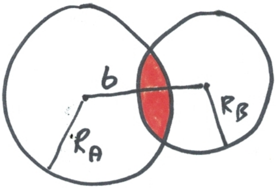
$$c \simeq \frac{\pi b^2}{\sigma_{\rm inel}^{AB}} \simeq \frac{b^2}{(R_A+R_B)^2},$$

where R_A and R_B are the radii of the nuclei

[WB, Florkowski, PRC 65(2002)024905]

To see this consider an archery competition: probability $\sim 2\pi b db \to$ cumulative distribution function:

$$c(b) = \int_0^b P(b')db' = \frac{b^2}{b^2_{\rm max}} = \frac{b^2}{(R_A+R_B)^2}$$

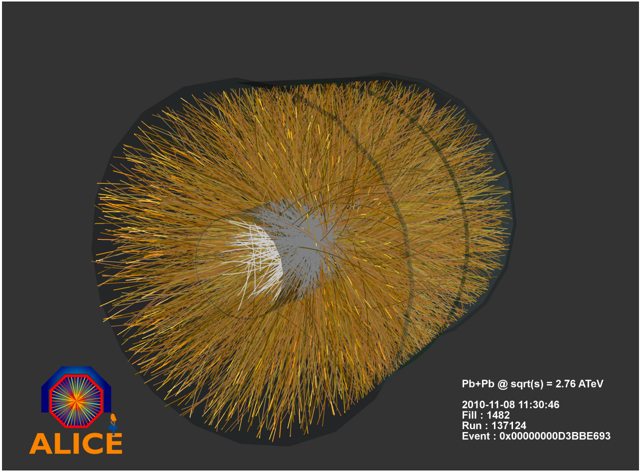


Homework:

Take the human data from [link] and generate histograms for distributions and cumulative distributions according to 1) heigh 2) weight. Determine "centrality classes" for the two cases and compare the results.

Fireball

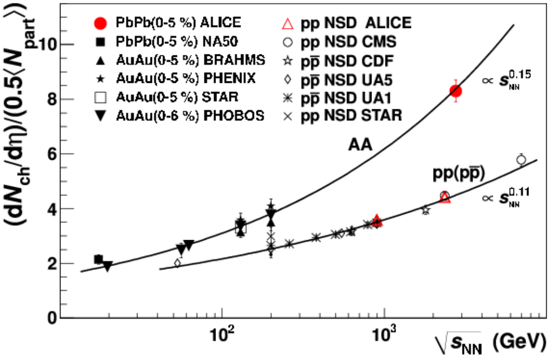
An ALICE event



In a single relativistic heavy-ion collision thousands of particles are formed, observed (and identified)

Multiplicities

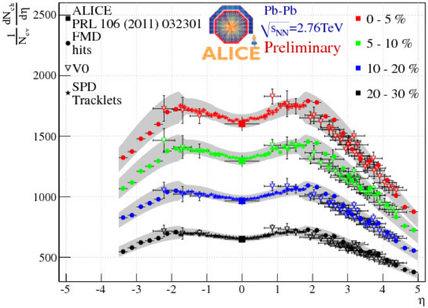
Growth with $\sqrt{s_{NN}}$, but not superposition of p+p

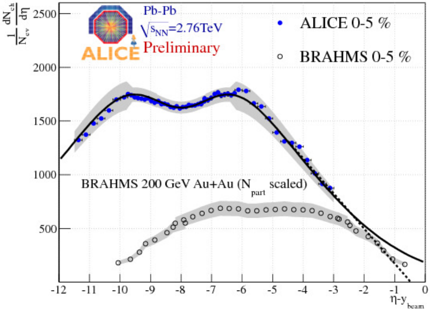


[Aamodt et al. (ALICE) PRL 105(2010)252301]

$N_{\rm part}$ – number of participating nucleons

Spectra in pseudorapidity





Kinematic range in rapidity is $\sim y_{\text{beam}} = \text{arccosh}[\sqrt{s_{NN}}/(2m_N)]$
(≃ 8 at 2.76 TeV, ≃5.4 at 200 GeV)

As already said, for identified particles y is typically used. Recall that η distributions are wider and lower in the center.

1600 of charged hadrons per unit of η! (~ 2400 for all hadrons)

Statistical (thermal) model of hadronization

[Fermi, Pomeranchuk, Hagedorn, Kapusta, Koch, Muller, Rafelski, Sollfrank, Heinz, Becattini, Braun-Munzinger, Stachel, Redlich, Cleymans, Gazdzicki, ...]

Large multiplicities $\to$ Large densities $\to$ statistical description – the higher collision energies, the better!

By counting all the particles we cannot obtain the temperature T, as we do not know the volume V. Idea: look at identified hadron multiplicities and take ratios to divide out V.

For the simplified case of the Boltzmann distribution $(\hbar = k_B = c = 1)$

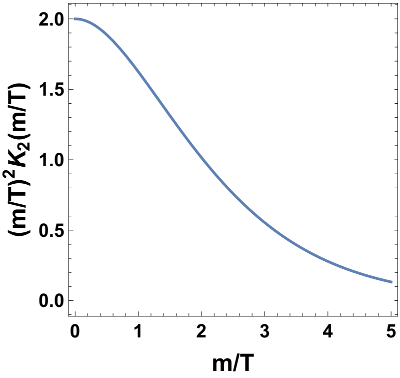
$$N = V \int \frac{d^3p}{(2\pi)^3} e^{-(E-\mu)/T} = V e^{\mu/T} \int \frac{d^3p}{(2\pi)^3} e^{-\sqrt{m^2+p^2}/T} = \frac{VT^3}{2\pi^2} e^{\mu/T} \left(\frac{m}{T}\right)^2 K_2\left(\frac{m}{T}\right)$$

In chemical equilibrium

$$\mu = B\mu_B + S\mu_S + I_3\mu_{I_3}$$

Bessel functions

Modified Bessel function of the second kind



- higher m (at fixed T) $\rightarrow$ lower yield of a species

Chemistry

For boost-invariant systems (approximately satisfied at midrapidity) the ratio of abundances of species i and j is

$$\frac{dN_i/dy}{dN_j/dy} = \frac{N_i}{N_j} \simeq \frac{2J_i+1}{2J_j+1} e^{(\mu_i-\mu_j)/T} \frac{m_i^2 K_2(m_i/T)}{m_j^2 K_2(m_j/T)}$$

where $\mu_i = B_i\mu_B + S_i\mu_S + I_{3,i}\mu_{I_3}$. For instance (hadron symbols here denote their multiplicities)

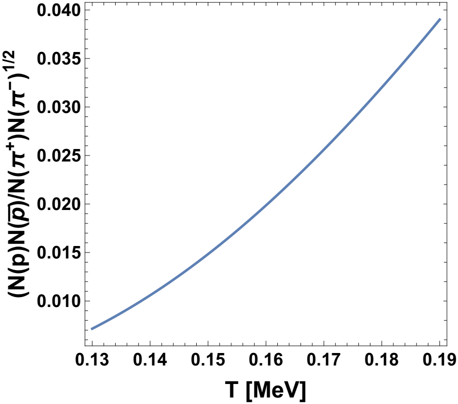
$$\frac{p}{\bar{p}} = e^{2\mu_B/T}, \ \frac{K^+}{K^-} = e^{2\mu_S/T}, \ \frac{p\,\bar{p}}{\pi^+\pi^-} = \left(\frac{1}{2}\frac{m_p^2 K_2(m_p/T)}{m_\pi^2 K_2(m_\pi/T)}\right)^2$$

3 equations allow to find the thermal parameters T, μ_B, μ_S.

In practice μ_S and μ_{I_3} are determined by requiring that the strangeness of the system is zero, and the ratio of the baryon number to the electric charge densities is the same as in the colliding nuclei $\to$ solve overdetermined system for many ratios in the χ^2 sense. Actually, one needs to include resonance decays (see the following) for the analysis to be realistic.

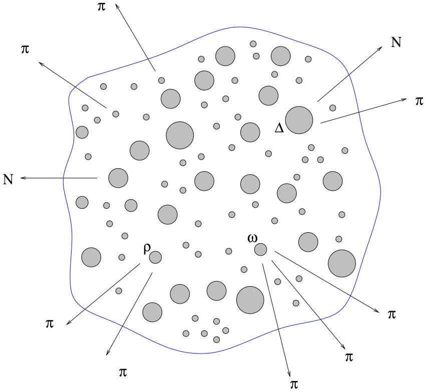
Sensitive thermometer

μ-independent combination



- lower $T \rightarrow$ more difference between species

Resonance decays



Very important: ∼75% of pions come from resonance decays (!) Although resonances are heavier, hence suppressed, they are very numerous. Their role increases with larger temperatures. Cascades possible.

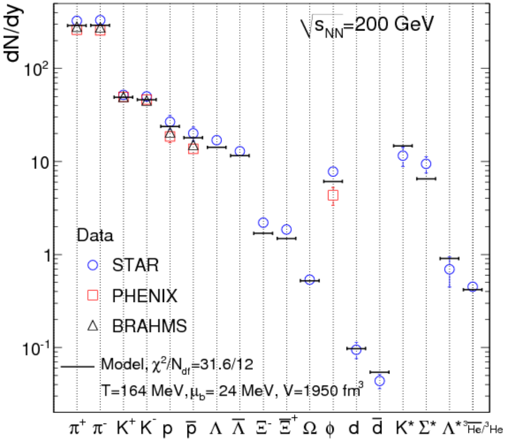
SHARE, THERMUS - publicly available codes carrying out statistical hadronization with decays of all resonances from Particle Data Tables

Example: Au+Au at RHIC, $\sqrt{s_{NN}} = 130$ GeV

Au+Au	model	experiment
T_{chem} [MeV]	**165±7**	
μ^B_{chem} [MeV]	**41±5**	
μ^S_{chem} [MeV]	9	
μ^I_{chem} [MeV]	-1	
χ^2/n	**0.97**	
π^-/π^+	1.02	1.00 ± 0.02 , 0.99 ± 0.02
$\overline{p}/\pi^-$	0.09	0.08 ± 0.01
K^-/K^+	0.92	0.88 ± 0.05, 0.78 ± 0.12 0.91 ± 0.09, 0.92 ± 0.06
K^-/π^-	0.16	0.15 ± 0.02
K^*_0/h^-	0.046	0.060 ± 0.012
$\overline{K^*_0}/h^-$	0.041	0.058 ± 0.012
$\overline{p}/p$	0.65	0.61 ± 0.07, 0.54 ± 0.08 0.60 ± 0.07, 0.61 ± 0.06
$\overline{\Lambda}/\Lambda$	0.69	0.73 ± 0.03
$\overline{\Xi}/\Xi$	0.76	0.82 ± 0.08

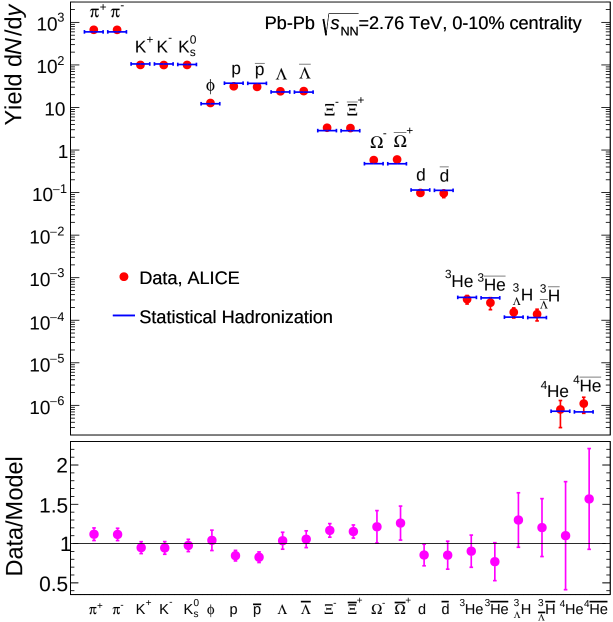
[Florkowski, WB, Michalec, Acta Phys. Polon. B33 (2002) 761]

RHIC success



[Andronic et al., PLB 697(2011)203, arXiv:1010.2995]

LHC



[A. Andronic, P. Braun-Munzinger, K. Redlich, J. Stachel, Nature 561 (2018) 321]

(note, however, a 50% or so discrepancy for p and $\bar{p}$)

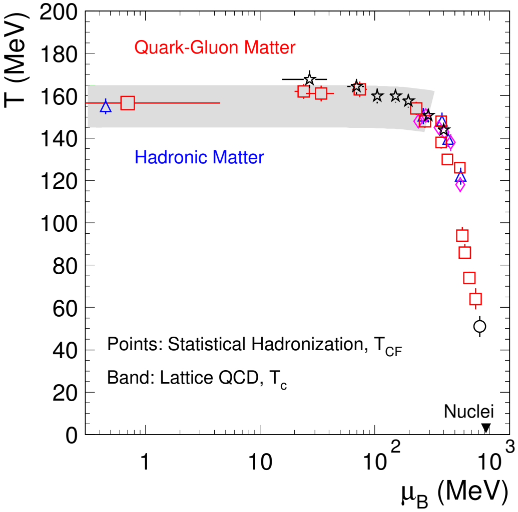
Yields of light nuclei – "Snowflakes in hell"

9 orders of magnitude!
But:

- Fundamentally not possible to understand the production of the light nuclei (albeit described) in the statistical hadronization model. Too weakly bound to achieve thermal equilibrium during the fireball's lifetime. Too large compared to the inter-particle spacing.
- Recent quantitative and detailed discussion: [Y. Cai, T. D. Cohen, B. A. Gelman, Y. Yamauchi, PRC 100 (2019) 2, 024911, arXiv:1905.02753]
- Alternative approach: coalescence, see [S. Bazak, S. Mrówczyński, Mod. Phys. Lett. A33 (2018) 1850142]

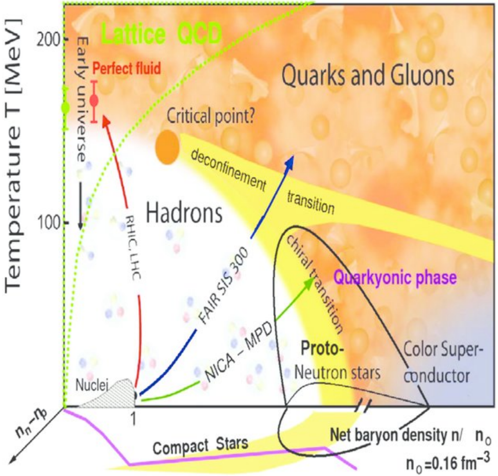
Open problem!

T-μ_B diagram – Cleymans-Redlich curve



[A. Andronic, P. Braun-Munzinger, K. Redlich, J. Stachel, Nature 561 (2018) 321]

Phase diagram of QCD



[from D. P. Menezes]

Expanding and cooling fireball

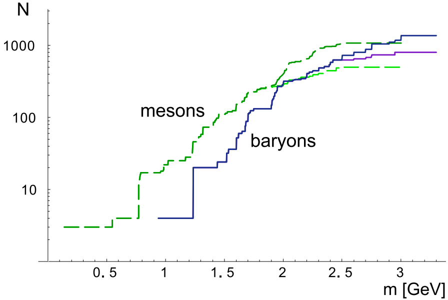
Working scenario:

- Due to huge density, a fireball is formed (QGP), which expands (nothing to hold it) and thus cools down.
- At some temperature T_f a phase transition (more precisely: crossover) occurs to the hadronic phase.
- The hadron "soup" must include all resonances – this has a huge effect at $T_f \sim 160$ MeV.
- After the freeze-out, the hadrons may still rescatter, resonances decay, and finally the stable particles reach detectors.
- The proton puzzle, the "snowflakes in hell" question, ...

Hagedorn spectrum

Exponential growth of the density of states:

$$\rho(m) = f(m) e^{m/T_H}, \quad N(m) = \sum_i \gamma_i \theta(m - m_i)$$



[WB, Florkowski, Glozman, PRD 70(2004)117503]

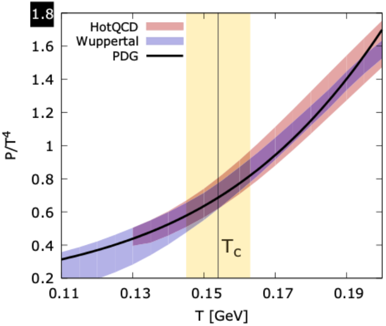
Limiting temperature: the temperature of the system cannot surpass T_H, as then the partition function becomes infinite (pre-QCD considerations)

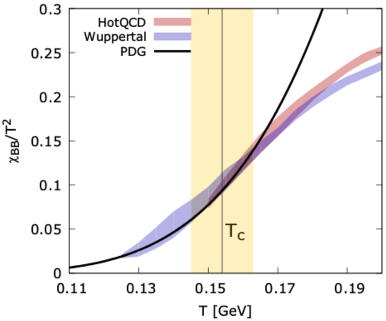
Hadron resonance gas (HRG) vs lattice QCD

For temperatures below the cross-over the thermodynamic functions are expressed via sums over all hadronic resonances. The pressure is

$$P(T,\mu_B,\mu_S,\mu_{I_3}) = \pm T \sum_i (2J_i+1) \int \frac{d^3k}{(2\pi)^3} \log\left(1 \pm e^{-\left[\sqrt{m_i^2+k^2}-\mu_i\right]/T}\right)$$

baryonic susceptibility $\chi_{BB} = \partial^2 P/\partial\mu_B^2$





[Lo, Marczenko, Redlich, Sasaki, Eur.Phys. J. A52 (2016) 235]

Other effects

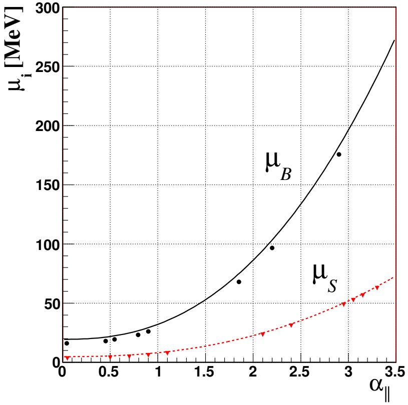
Homework

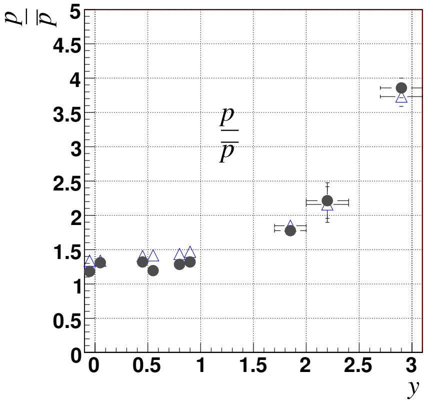
Derive the explicit formulas for the baryon density $n_B = \partial P(T, \mu_B)/\partial \mu_B$ and for $\chi_{BB} = \partial^2 P(T, \mu_B)/\partial \mu_B^2$ in HRG. Plot the contributions of nucleons and antinucleons as a function of T. At the LHC you can take $\mu_B = 0$.

- To satisfy the baryon number and strangeness conservation laws $\rightarrow$ canonical ansatz
- To satisfy the energy conservation $\rightarrow$ microcanonical ansatz - relevant for systems with small numbers of particles
- Short-range repulsion, excluded volume
- Incomplete equilibrium (Rafelski's fugacity factors)
- Strangeness enhancement
- Hierarchy of freeze-outs, based on hierarchy of cross sections

Off mid-rapidity

μ_i depend on the spatial rapidity $\alpha_\parallel = \frac{1}{2}\log\left(\frac{t+z}{t-z}\right)$





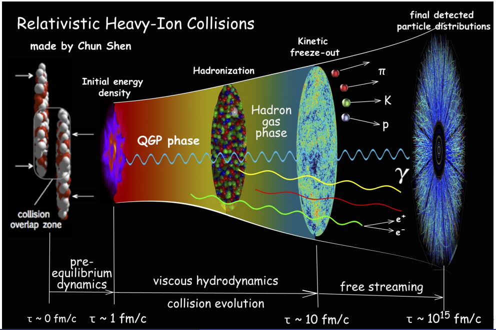
[B. Biedroń, WB, PRC 75(2007)054905]

Pb+Pb E_{cm}=5.5 TeV
t=-19.00 fm/c
H. Weber / UrQMD Frankfurt/M

Summary of thermal approach

- Dense system with numerous collisions
- Estimate: after freeze-out typically one collision per particle (as it should be)
- Thermal and chemical equilibrium (at FO) explain the hadron abundances
- The embarrassing success of light (hyper)nuclei production
- Resonances crucial, HRG
- HRG compares reasonably well to lattice QCD

The system (at least near the end of the evolution) is close to thermal and chemical equilibrium



Lecture 3

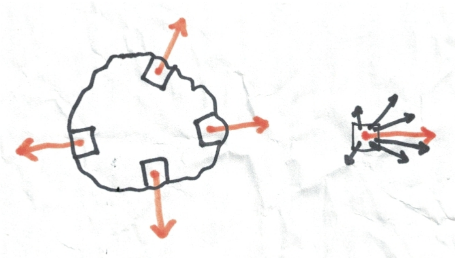
Expansion and flow

The key concept of the approach to collectivity

Flow (and jet quenching) are the two major discoveries of the ultra-relativistic heavy-ion program!

Inevitability of expansion

No container! $\rightarrow$ the fireball expands (and cools down)
Think in terms of gas/fluid - dense medium, high pressure, short mean-free path, multiple rescattering

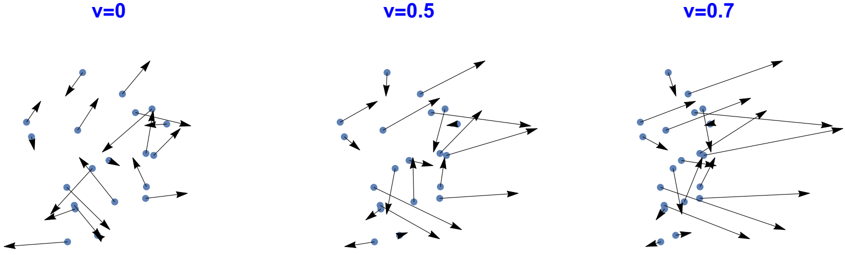


Flow is generic to a system with copious rescattering: hydro, transport, ...

Obviously, the expansion affects the momentum spectra, as the velocity of the fluid element yields the Doppler effect

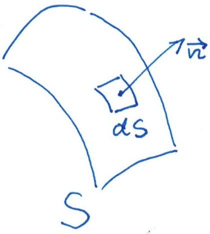
Boosting the distribution

Some space (points) and momentum (arrows) distribution of thermal pions ($T = 160$ MeV) in a fluid element at rest, and moving to the right at a velocity v (in units of c)



We observe the shift of velocities in the direction of the boost (and also the Lorenz contraction of the distribution, not relevant here).

Reminder from differential geometry



Surface integral

$$I = \iint_S d\vec{S} \cdot \vec{F} = \iint_S F_x dy\, dz + F_y dz\, dx + F_z dx\, dy = \iint_S \epsilon^{ijk} F_i dr_j dr_k$$

$\vec{r} = (x, y, z)$, $d\vec{S} = dS\,\vec{n}$, $\vec{n}$ - normal vector to the element S, $\vec{n}^2 = 1$
(recall the concept of the flux through a surface)

Surface integral, cont.

In curvilinear coordinates S can be parametrized with 2 variables (α, β). Then

$$I = \iint d\alpha\, d\beta \left[F_x \frac{\partial(y,z)}{\partial(\alpha,\beta)} + F_y \frac{\partial(z,x)}{\partial(\alpha,\beta)} + F_z \frac{\partial(x,y)}{\partial(\alpha,\beta)} \right]$$

The jacobians are

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{vmatrix}$$

etc.

$$I = \iint d\alpha\, d\beta\, \epsilon^{ijk} F_i \frac{\partial r_j}{\partial\alpha} \frac{\partial r_k}{\partial\beta} = \iint d\alpha\, d\beta \begin{vmatrix} F_x & F_y & F_z \\ \partial x/\partial\alpha & \partial y/\partial\alpha & \partial z/\partial\alpha \\ \partial x/\partial\beta & \partial y/\partial\beta & \partial z/\partial\beta \end{vmatrix}$$

Example: area of the sphere

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta,\ \vec{n} = \vec{r}/r$

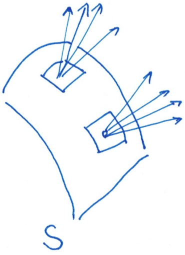
$$I = \iint_S d\vec{S}\cdot\vec{r}/r = \iint d\theta\, d\phi \begin{vmatrix} x/r & y/r & z/r \\ \partial x/\partial\theta & \partial y/\partial\theta & \partial z/\partial\theta \\ \partial x/\partial\phi & \partial y/\partial\phi & \partial z/\partial\phi \end{vmatrix} =$$

$$\cdots = r^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 4\pi r^2$$

Homework

Complete the above derivation.

Emission from a surface



Imagine particles are emitted from a surface in the direction $\vec{k}$ ($\vec{k}^2 = 1$), with some probability distribution $f(\vec{n} \cdot \vec{k})$. Then

$$\frac{d^2N}{dk^2} = \iint_S dS f(\vec{n}(x,y,z) \cdot \vec{k})$$

Note that particles emitted in the direction $\vec{k}$ originate from different positions on S. At a given direction, we collect from various surface elements!

Homework

Assume particles are emitted from a half-sphere with perfect collimation to the normal, i.e, $\vec{n} \parallel \vec{k}$, or $f(\vec{n} \cdot \vec{k}) = \delta(\vec{n} - \vec{k})$. Parametrize k in spherical angles and find the distribution of emission in these angles.

The Frye-Cooper formula

Emission from a fireball is conceptually analogous to the above examples, with the following differences: it occurs from volume elements, it is not static (the fireball expands), and relativity/Lorentz invariance must be taken into account.

If things were static, then one would collect particles (hadrons) produced from various fluid elements at rest with isotropic emission distribution depending on the hadrons energy, $f(E)$:

$$\frac{d^3N}{d^3p} = \int_V dV\, f_i(E)$$

Rewrite invariantly: $u^\mu = \frac{1}{\sqrt{1-v^2}}(1, \vec{v})$ (four-velocity, $u_\mu u^\mu = 1$), at rest $u^\mu = (1,0,0,0)$). Then $E = p^0 \to p \cdot u$, also recall $E/d^3p = 1/d^2p_T\, dy$ – Lorentz invariant $\to$

$$\frac{E\, d^3N}{d^3p} = \frac{d^3N}{d^2p_T dy} = \int_V dV E\, f_i[p \cdot u(x)]$$

It remains to write down invariantly $dV E$.

Frye-Cooper 2

Answer: we should substitute

$$dV\,E \to d^3\Sigma_\mu(x)p^\mu$$

where

$$d^3\Sigma_\mu(x) = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial p}\frac{\partial x^\beta}{\partial q}\frac{\partial x^\gamma}{\partial r}dp\,dq\,dr$$

is a 3-D volume element (from hypersurface) in the 4-D space-time. Indeed, for a "static" fireball $x^0 = t =$const and using $p = x, q = y, r = z$ we find immediately $dV = dxdydz$, hence indeed there is matching to the case where emission occurs everywhere at the same time t.

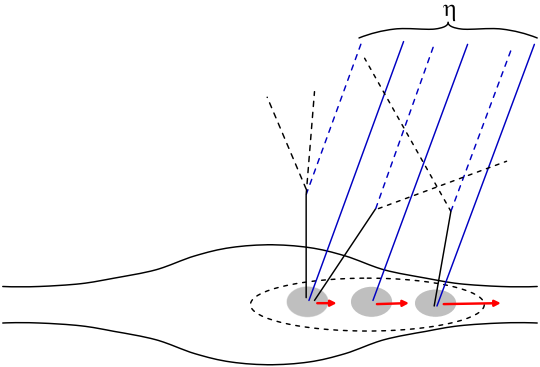
In general, this is of course not the case and one has to use the general formula

Frye-Cooper

$$\frac{Ed^3N_i}{d^3p} = \int_\Sigma d^3\Sigma_\mu(x)p^\mu\, f_i[p\cdot u(x)]$$

Example

Collecting from fluid elements along the longitudinally extending (and expanding – red arrows) fireball:



Hypersurface

$$d^3\Sigma_\mu(x) = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial x^\alpha}{\partial p}\frac{\partial x^\beta}{\partial q}\frac{\partial x^\gamma}{\partial r}dp\,dq\,dr$$

x^α - coordinates in space-time, p, q, r - parameters of a 3-dim. hypersurface
Example:

- Boost-inv. freeze-out [Schnedermann, Sollfrank, Heinz, PRC48 (1993) 2462]
  $x^\mu = (t, x, y, z) = \left(\tau(\zeta)\mathrm{cosh}\alpha_\parallel, \rho(\zeta)\cos\phi, \rho(\zeta)\sin\phi, \tau(\zeta)\mathrm{sinh}\alpha_\parallel\right) \to$
  $d^3\Sigma^\mu = \left(\frac{d\rho}{d\zeta}\mathrm{cosh}\alpha_\parallel, \frac{d\tau}{d\zeta}\cos\phi, \frac{d\tau}{d\zeta}\sin\phi, \frac{d\rho}{d\zeta}\mathrm{sinh}\alpha_\parallel\right)\rho(\zeta)\tau(\zeta)d\zeta d\alpha_\parallel d\phi$

Above $\tau = \sqrt{t^2 - z^2}$ and $\rho = \sqrt{x^2 + y^2}$ are some known functions of ζ (adjusted somehow to "physics"), and $(\zeta, \alpha_\parallel, \phi)$ parametrize the hypersurface.

Homework

Derive $d^3\Sigma^\mu$ in the above model.

With a complementary hypothesis for u^μ one may obtain model results without running lengthy hydrodynamics simulations.

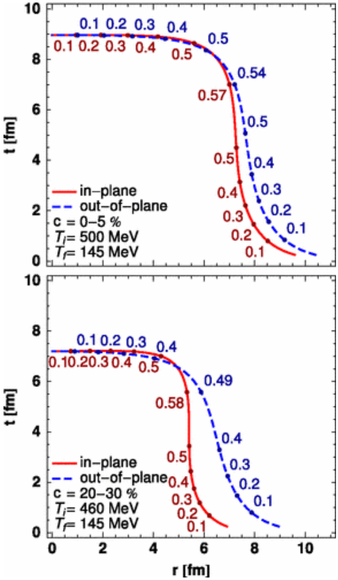
Freeze-out from perfect hydrodynamics

Hydrodynamics provides $d^3\Sigma^\mu$ and u_μ when a freeze-out condition is met (typically, $T = T_f$) as a numerical output

In different locations, the freeze-out occurs at different times (this is so also in non-relativistic explosions)

More elementary discussion of freeze-out parameterizations can be found in [W. Florkowski, WB, Acta Phys.Polon. B35 (2004) 2895]

Freeze-out from perfect hydrodynamics

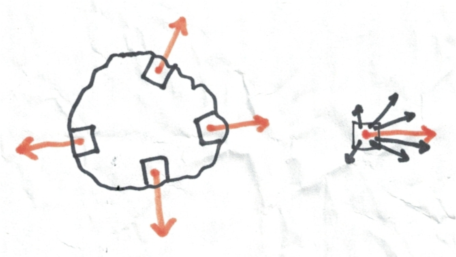


RHIC@200 GeV
r - transverse radius, t - time
labels - transverse flow velocity v/c

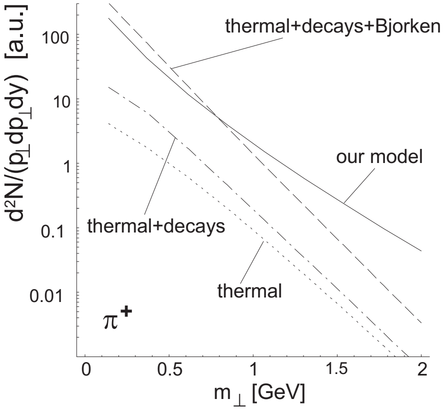
Sections through the hypersurface defined by $T_f = 145$ MeV (3-D object in 4-D space) at $z = 0$ and $y = 0$ (red) or $x = 0$ (blue).

[WB, M. Chojnacki, W. Florkowski, A. Kisiel, PRL 101 (2008) 022301]

Recall the Doppler effect



Effects on the p_T spectra



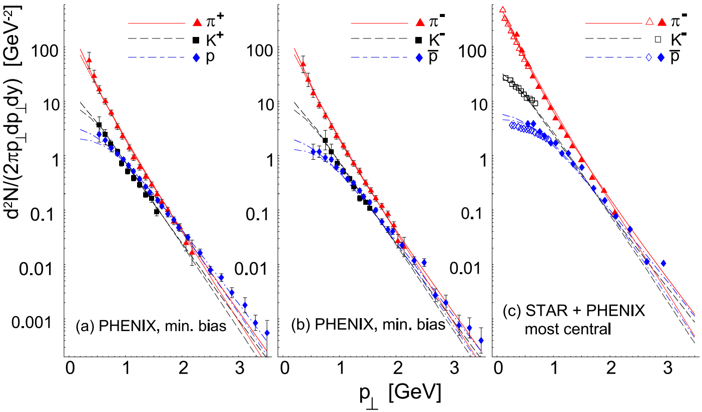
– thermal: pion spectrum from a static fireball
– thermal+decays: initial and secondary pions, which lead to a decrease of the inverse slope
– Bjorken: pure longitudinal expansion $\rightarrow$ redshift, as all fluid elements move away from the observer $\rightarrow$ cooling of the spectrum.

– our model: transverse flow added, hence some fluid elements move towards the observer $\rightarrow$ blueshift

Radial flow $\rightarrow$ blueshift and redshift $\rightarrow$ convex

[WB, W. Florkowski, PRL 87(2001)272302]

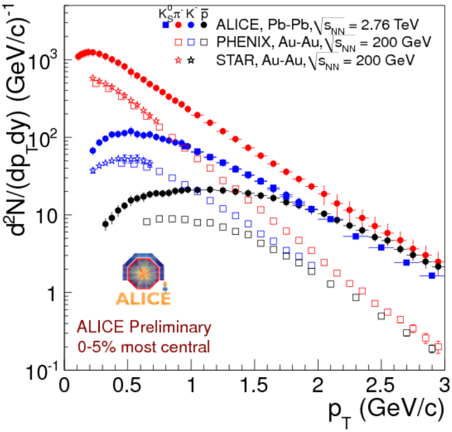
Example p_T spectra @130 GeV



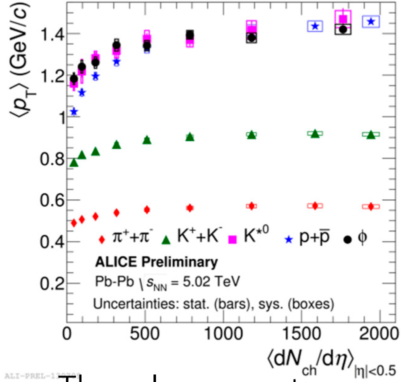
$T_f = 165$ MeV, $\mu_B = 41$ MeV [WB, W. Florkowski, PRL 87(2001)272302]

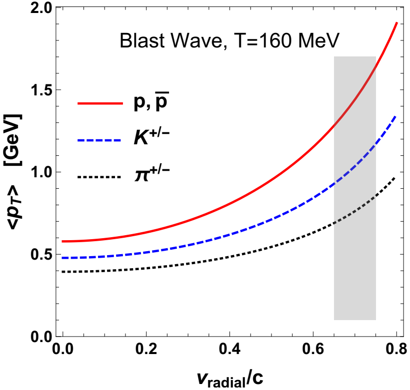
– mass hierarchy (from thermal motion and from transverse flow)

p_T spectra, RHIC vs the LHC



More flow with increasing energy → more Doppler shift!





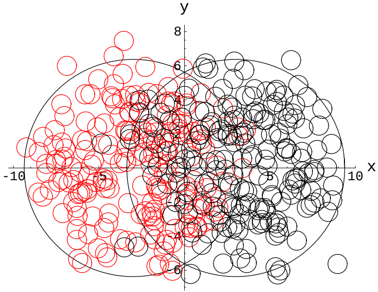
- Thermal component
- Radial flow component

Blast wave model: $\rightarrow$ enhancement of the mass hierarchy

$$\frac{dN}{dy\,d^2p_T} = \text{const} \times m_T I_0\left(\frac{p_T \sinh\alpha}{T}\right) K_1\left(\frac{m_T \cosh\alpha}{T}\right), \quad v_r/c = \tanh\alpha$$

Initial geometry

Au+Au collision at RHIC
(view along the beam)



1. Participants determine the geometry of the overlap region
2. Initial entropy distribution in more microscopic approaches (IP Glasma) also follows the geometry of the overlap region
3. Strong radial flow
4. Initial eccentricity $\rightarrow$ anisotropic flow of hadrons [Ollitrault 1992]

Initial geometry

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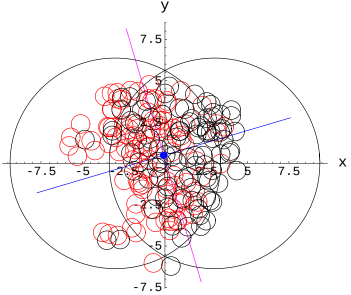

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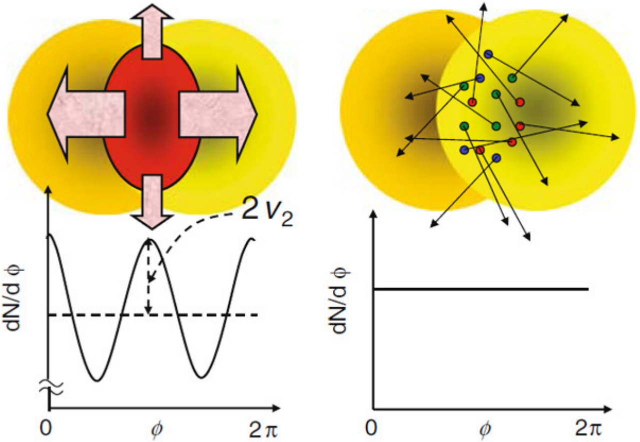
Rescattering/collectivity essential



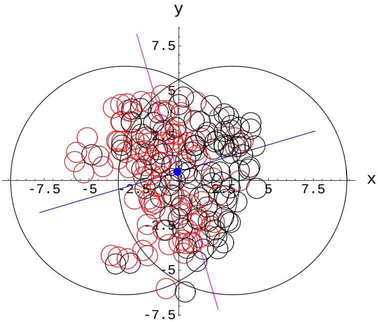
[ALICE]

In each event, define the harmonic flow coefficients and event-plane angles:

$$dN/d\phi \propto 1 + 2\sum_n v_n \cos[n(\phi - \Psi_n)]$$

Fluctuations

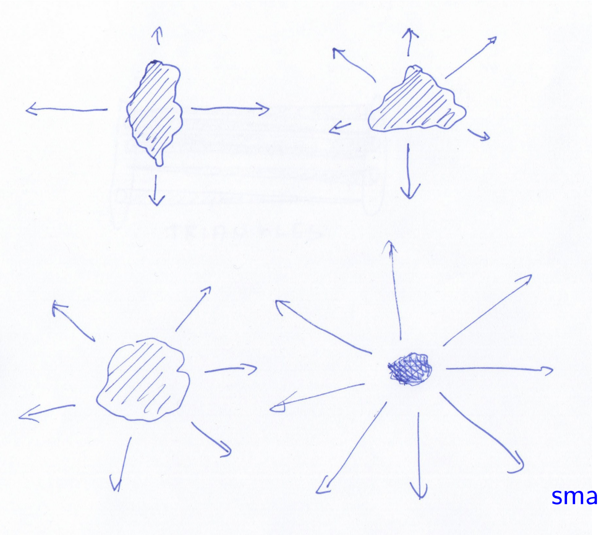
Collapse of the nuclear wave function → each Little Bang different



1. Higher Fourier components appear
2. Odd harmonics also show up, triangular flow
3. Fluctuations dominant for central A+A and for *small systems*, such as p+A (see later on)

New thinking since [Miller and Snellings 2003]

Collectivity: shape/size – flow transmutation

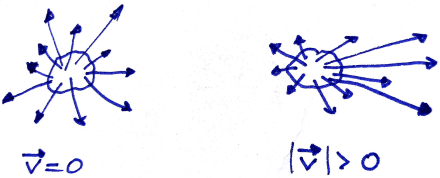


smaller → faster

Any rescattering will do!

Collimation from the Doppler effect

- Emission from a fast moving element of fluid
- Collimation of hadrons (increasing with mass)

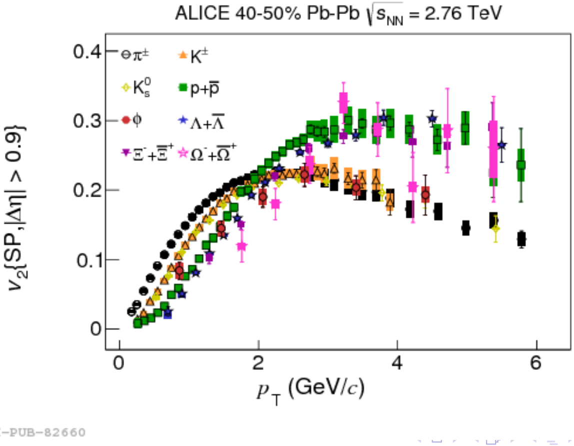


Multi-particle correlations in the azimuth are used in the cumulant or other methods to extract the flow coefficients without the non-flow contamination (from jets, resonance decays, . . .)

[Borghini, Ollitrault 2001]

Features of harmonic flow

1. Mass ordering of harmonic flow coefficients v_n
2. Higher harmonics suppressed
3. Near-side ridge (discussed later on) - considered the “proof” of harmonic flow



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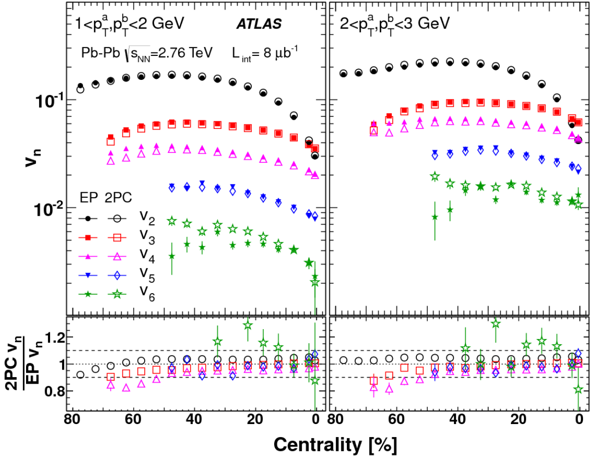
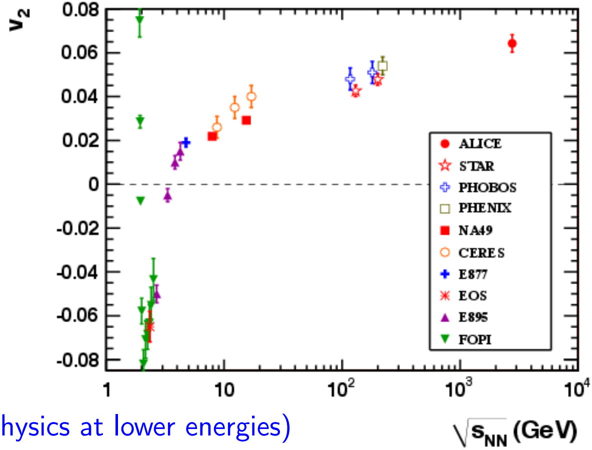
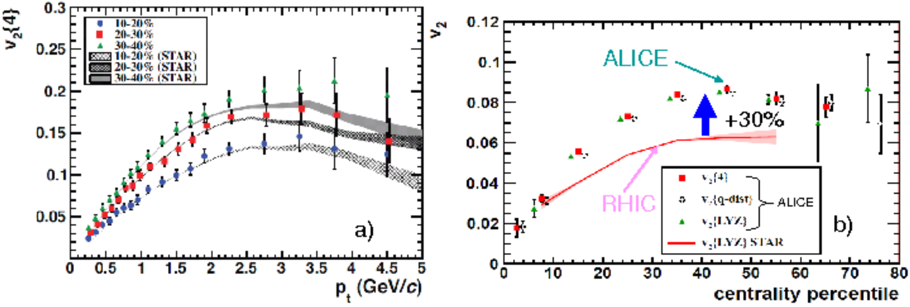


v_2 vs $\sqrt{s_{NN}}$



(different physics at lower energies)



[ALICE, PRL 105(2010)252302]

At the LHC the differential elliptic flow is the same as at RHIC, but "sampling" is at higher p_T

Lecture 4

Flow (radial and harmonic) leads to correct phenomenology of the p_T spectra and v_n, with proper dependence on the geometry (shape-flow transmutation), collision energy, and mass hierarchy

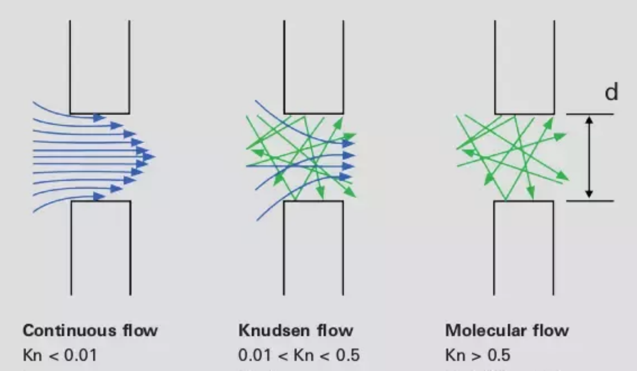
Hydrodynamics

What produces the flow (collectivity)?

Flow (and jet quenching) are the two major discoveries of the ultra-relativistic heavy-ion program!

Basics

- Fluid $\equiv$ substance that cannot resist any shear force (gas, liquid, plasma), continuously deforms
- size of particles $\ll$ fluid element $\ll$ size of the system
- Knudsen number: $\mathrm{Kn} = \lambda/L$, λ mean free path, L - system's size
- $\mathrm{Kn} \ll 1 \rightarrow$ fluid description



[Wikipedia]

Assumes the shape of the container...

Assumes the shape of the container...

Reminder: the energy-momentum tensor $T^{\mu\nu}$

Noether theorem

Any continuous symmetry of dynamics $\to$ conserved current j^μ

i.e., there is a continuity equation $\partial_\mu j^\mu = 0$. Explicitly, $\partial_0 j_0(t,\vec{x}) + \vec{\nabla}\cdot\vec{j}(t,\vec{x}) = 0$.
Integrating over a volume and using Gauss' law yields

$$d/dt\, Q(t) = d/dt \int_V dV\, j_0 = -\int_V dV\, \vec{\nabla}\cdot\vec{j} = -\int_S \vec{n}\cdot\vec{j}$$

If the volume contains the whole system, nothing leaks (the flux through S is 0), then the charge Q is conserved.
Four translations in space-time are continuous symmetries $\to$ four Noether currents. They can be grouped into one Lorentz-covariant object,

Energy-momentum tensor

$T^{\mu\nu}$, satisfying the continuity equations $\partial_\nu T^{\mu\nu} = 0$

In our case $T^{\mu\nu} = T^{\nu\mu}$ (some issues here in field theory, in general relativity it is symmetric)

$T^{\mu\nu}$ in kinetic theory

In general, both particles and fields contribute to $T^{\mu\nu}$. In kinetic theory one introduces the space-time – momentum distribution $f(x,p)$ of particles. Then one can show that

$T^{\mu\nu}$ in relativistic kinetic theory

$$T^{\mu\nu}(x) = \int \frac{d^3p}{p_0} p^\mu p^\nu f(x,p)$$

The interpretation of the $\mu = 0$ terms is obvious:

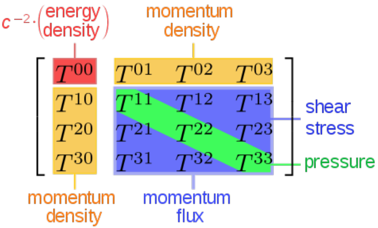
$$T^{00} = \int d^3p\, p^0 f(x,p) - \text{energy}, \quad T^{0i} = \int d^3p\, p^i f(x,p) - \text{momentum}$$

The ij terms may be viewed as "correlations" $T^{ij} = \int \frac{d^3p}{p_0} p^i p^j f(x,p)$. Note that the trace

$$T^\mu_\mu = T^{00} - \sum_i T^{ii} = \int \frac{d^3p}{p_0} (p_0^2 - \vec{p}^2) f(x,p) = m^2 \int \frac{d^3p}{p_0} f(x,p) \propto m^2$$

(in QCD the trace anomaly (quantum effect) makes the trace non-zero despite (nearly) massless quarks and gluons! $T^\mu_\mu \propto G^{\mu\nu}G_{\mu\nu}$)

$T^{\mu\nu}$ in a fluid



In general all independent 10 components may arise. Imagine, however, $f = f(x, p \cdot u)$, where u is a Lorentz vector (it becomes the four-velocity from the next slide). Then by Lorentz covariance

$$T^{\mu\nu}(x) = \int \frac{d^3p}{p_0} p^\mu p^\nu f(x, p \cdot u) = A u^\mu u^\nu + B g^{\mu\nu}$$

(the only symmetric tensors to our disposal are $u^\mu u^\nu$ and $g^{\mu\nu}$)

Homework

Evaluate the current $\int \frac{d^3p}{p_0} p^\mu f(x, p \cdot u)$, where $f \propto e^{-p \cdot u/T}$, with $u \cdot u = 1$ for the case of massless particles, $p_0 = |\vec{p}|$.

Perfect (a.k.a. ideal) hydrodynamics (no viscosity)

Local thermal equilibrium at point x: $T^{\mu\nu}(x) = \int \frac{d^3p}{p_0} p^\mu p^\nu f_{\rm eq}(x, u\cdot p; T, \mu)$

Landau's definition of the four-velocity of the fluid

$$T^{\mu\nu}(x) u_\mu(x) = \lambda(x) u^\nu(x)$$

$$u_\mu u^\mu = 1, \quad u^\mu = \gamma(1, v_x, v_y, v_z) = \frac{1}{\sqrt{1-v^2}}(1, v_x, v_y, v_z)$$

The perfect hydro form follows (u^μ and $g^{\mu\nu}$ for disposal):

$$T^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu} \quad (\lambda = \varepsilon)$$

In the fluid element's rest frame $u^\mu = (1,0,0,0)$
and

$$T^{\mu\nu}(x) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}, \quad \varepsilon = \varepsilon(T(x), \mu(x)),\ P = P(T(x), \mu(x))$$

The perfect hydro equations

Energy-momentum conservation → Euler eqns.

$$\partial_\mu T^{\mu\nu}(x) = 0,$$

4 equations for 5 unknown functions: $\vec{v}$, ε, P – need the equation of state, which is specific for a given system, linking ε and P to close the system

- Example: massless particles → $\varepsilon = 3P$

Homework

1. (trivial) Obtain the equation of state for massless particles.
2. Write down the equation of state for dust (occurs in the Universe): very cold tiny grains of matter.

Conservation of entropy

(here checked for the case of 0 chemical potential)

$$\partial_\mu T^{\mu\nu} = \partial_\mu \left[\varepsilon(T(x)) + P(T(x))u^\mu(x)u^\nu(x) - P(T)g^{\mu\nu}\right] = 0$$
$$0 = \partial_\mu 1 = \partial_\mu u \cdot u = 2u_\nu \partial_\mu u^\nu \to u_\nu \partial_\mu u^\nu = 0$$
$$\partial_\mu \varepsilon = \partial_T \varepsilon \, \partial_\mu T, \quad \partial_\mu P = \partial_T P \, \partial_\mu T$$

Then

$$0 = u_\nu \partial_\mu T^{\mu\nu} = \partial_T \varepsilon \, u^\mu \partial_\mu T + (\varepsilon + P)\partial_\mu u^\mu$$

The first law of thermodynamics

$dE = TdS - PdV \to dE/dV = TdS/dV - P \to \varepsilon = Ts - P \to s = (\varepsilon + P)/T$

In addition, there are thermodynamic (Maxwell) relations, stemming from differentiability of thermodynamic potentials. The relevant one here (for the case of 0 chemical potential where there is only one thermodynamic variable T) follows from the Gibbs free energy $F = E - TS$, where $dF = -SdT - PdV$

$$\partial_T P|_V = \partial_V S|_T \to \partial_T P|_V = s \to \frac{dP}{dT} = s = (\varepsilon + P)/T$$

Conservation of entropy cont.

With the formulas from the previous slide it is straightforward to check that the entropy current is su^μ is indeed conserved:

Entropy current conservation

$$\partial_\mu (s u^\mu) = 0$$

Homework (tedious)

Carry out the calculations yielding the entropy conservation.

Similar result for non-vanishing chemical potentials for conserved charge currents: $\partial_\mu (n u^\mu) = 0$

Detailed reference: W. Florkowski's book

In essence, perfect hydro has no friction forces (viscosity), so no entropy is generated in the evolution. Also, no net baryon number, net strangeness,..., is produced.

Sound velocity

Consider perturbation on a static background

$$\varepsilon(x) = \varepsilon_0 + \delta\varepsilon(x), \;\; P(x) = P_0 + \delta P(x)$$

and a small velocity $u^\mu = (1, \delta v_x, \delta v_y, \delta v_z)$ (recall the 5 variables). To first order

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon_0 + \delta\varepsilon & (\varepsilon_0 + P_0)\delta v_x & (\varepsilon_0 + P_0)\delta v_y & (\varepsilon_0 + P_0)\delta v_z \\ (\varepsilon_0 + P_0)\delta v_x & P_0 + \delta P & 0 & 0 \\ (\varepsilon_0 + P_0)\delta v_y & 0 & P_0 + \delta P & 0 \\ (\varepsilon_0 + P_0)\delta v_z & 0 & 0 & P_0 + \delta P \end{pmatrix}$$

$$\partial_0 T^{00} + \partial_i T^{i0} \to \partial_t \delta\varepsilon + (\varepsilon_0 + P_0)\vec{\nabla} \cdot \delta\vec{v}$$

$$\partial_0 T^{0j} + \partial_i T^{ij} \to (\varepsilon_0 + P_0)\delta\partial_t v^j + \nabla^j \delta P$$

Combining,

$$\partial_t^2 \delta\varepsilon - \nabla^2 \delta P = 0$$

For zero chemical potentials there is only one thermodynamic parameter T. Then $\delta P = \frac{dP}{d\varepsilon}\delta\varepsilon = c_s^2(T)\delta\varepsilon$. We thus arrive at the wave equation

$$\partial_t^2 \delta\varepsilon - c_s^2 \nabla^2 \delta\varepsilon = 0$$

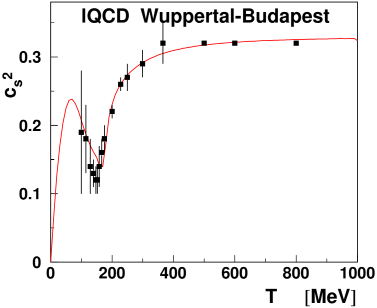
with c_s interpreted as the sound velocity, dependent on T (or ϵ)

A simple form hydro for $\mu = 0$

In the case of vanishing chemical potentials one may rewrite the perfect hydro equations + eq. of state in the elegant and instructive form

$$s\, u^\mu \partial_\mu d u^\nu = c_s^2(s)(g^{\mu\nu} - u^\mu u^\nu)\partial_\mu s, \quad \partial_\mu(s u^\mu) = 0$$

The sound velocity inputs a property of the medium, with $c_s^2(T) = \frac{dP}{d\varepsilon} = \frac{s}{T}\frac{dT}{ds}$ (the latter from thermodynamic equalities)

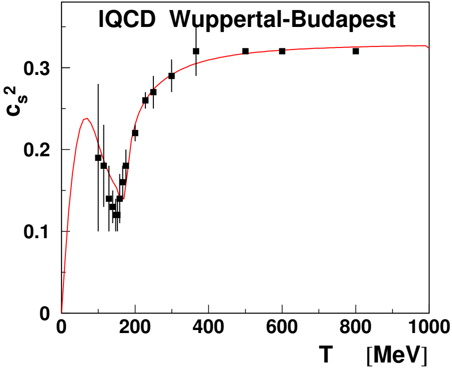


- no first order phase transition (there $\frac{dP}{d\varepsilon}$ would vanish!)
- no shock or rarefaction waves (!)
- laminar flow, no turbulence $\rightarrow$ "easy"

[M. Chojnacki, W. Florkowski 2007]

Digression on hadronization

As the system cools down, quarks and gluons are gradually replaced with hadrons



- Hadronization is conveniently carried over "behind the back", hidden in the eq. of state
- Fluid changed into particles via the Frye-Cooper mechanism, no need for elusive fragmentation models

Bjorken flow

Purely longitudinal expansion $u^\mu = \frac{1}{\tau}(t,0,0,z)$, assumed boost invariance involves dependence on the proper time $\tau = \sqrt{t^2+z^2}$ only
$\partial_\mu u^\mu = \frac{1}{\tau}$, $\partial_\mu \tau = u_\mu$

$$0 = \partial_\mu (s u^\mu) = \frac{ds(\tau)}{d\tau} + \frac{s(\tau)}{\tau} \ \rightarrow \ s(\tau) = s(\tau_0)\frac{\tau_0}{\tau}$$

Thermodynamic relations for $\mu = 0$: $\varepsilon + P = Ts$, $d\varepsilon = T\,ds$, $dP = s\,dT$, from where (for ultra-relativistic particles, where $P = c_s^2 \varepsilon$)

$$\varepsilon(\tau) = \varepsilon(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{1+c_s^2}, \ \ T(\tau) = T(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{c_s^2}$$

$\rightarrow$ estimates based on entropy conservation. From known experimental hadronic yields per unit of rapidity one infers $\varepsilon_{\rm QGP}(\tau_0) \simeq 4$ GeV/fm^3 (for comparison, the saturation density of nuclear matter is only $\simeq 0.16$ GeV/fm^3)

Homework

Recompute the Bjorken model

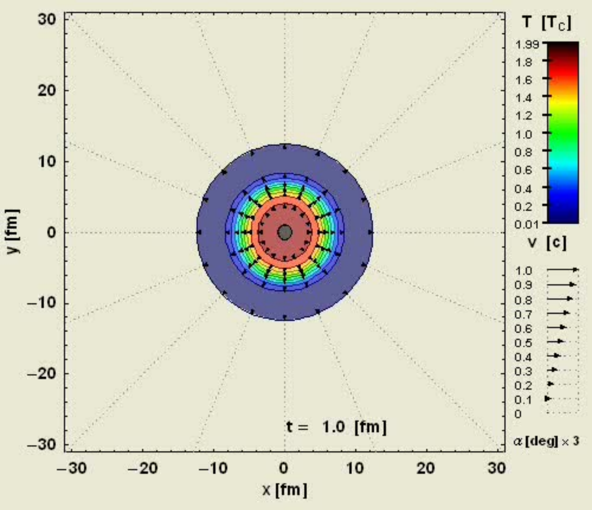
Resume

The general concept is very clear: energy-momentum conservation, baryon number conservation, ... What we need:

- Low mean-free path compared to a characteristic size, $\lambda \ll L$
- A fluid element must contain many particles. So, obviously, in total one needs very many particles
- Thermal equilibrium (or not too far) must occur in the fluid element
- Equation of state as the property of the medium. Can be obtained from lattice QCD/hadronic gas.
- Very important: Initial conditions must be provided on a (space-like) hyper-surface. Hydro is an initial-value problem and reflects the properties of the medium as well as the initial condition. This is typically provided at a constant proper time $\tau = \sqrt{t^2 - z^2}$ =const., whereas in the transverse plane it assumes some specific shape (see the previous lecture on the shape-flow transmutation).
- The initial conditions may be "single shot" - some average, or varying event-by-event (e-by-e).

Relativistic 2+1D perfect hydro (boost invariant)

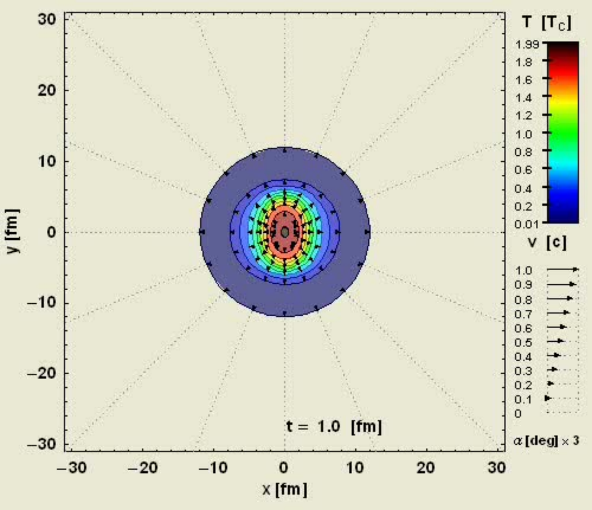
central (0-20%)



[M. Chojnacki, W. Florkowski]

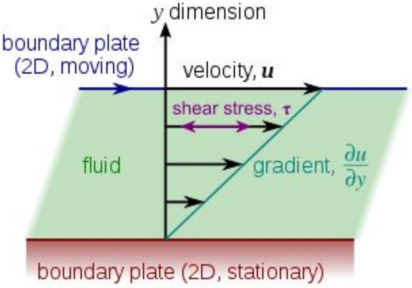
Relativistic 2+1D perfect hydro (boost invariant)

non-central (40-60%)



[M. Chojnacki, W. Florkowski]

Kinetic arguments for viscosity



[Wikipedia]

$$F/A = \eta \partial_y v_x$$

$$\mathrm{Re} = \frac{\rho v L}{\eta}$$

estimates for QGP:
$\mathrm{Re} \sim 10$ - very small!
(thousands needed for turbulence)

Navier-Stokes equations:

$$\rho \left(\partial_t v_i + \vec{v} \cdot \vec{\nabla} v_i \right) = -\nabla_i P + \eta \nabla^2 v_i$$

one of Millennium Problems!

Various materials

material	η [Pa s]	η/s [$\hbar/k_B$]
water	3×10^{-4}	8
honey	1000	5×10^{7}
superfluid ^{4}He	10^{-6}	2
ultra-cold ^{6}Li	$< 10^{-15}$	< 0.3
QGP	$< 2 \times 10^{11}$	< 0.4
pitch	2×10^{11}	10^{16}

U. of Queensland, 8 drops since 1927, Ig Nobel prize

Bounds on shear viscosity

dilute gas: $\eta = \frac{1}{3} npl$ (density $\times$ momentum $\times$ mean free path)

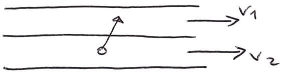
Quantum limit

Heisenberg uncertainty principle: $pl \geq \hbar$ and $s \sim k_B n \quad \rightarrow \quad \eta/s \geq\sim \hbar/k_B$

[P. Danielewicz and M. Gyulassy, PRD 31 (1985) 53]

KSS bound based on AdS/CFT: $\eta/s \geq \frac{1}{4\pi}\hbar/k_B$

[P. Kovtun, D. T. Son, and A. O. Starinets, PRL 94 (2015) 111601]



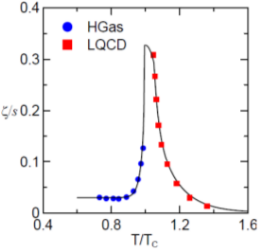
- $l = \frac{1}{n\sigma_{\rm el}} \rightarrow \eta = \frac{p}{3\sigma_{\rm el}}$ – counterintuitive!

Shear and bulk

shear viscosity η – resistance to deformation

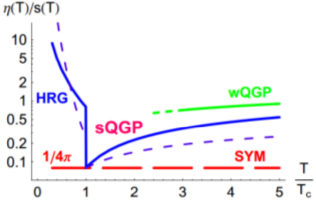
bulk viscosity ζ – resistance to expansion (volume change)

Bulk viscosity



Karsh&Kharzeev&Tuchin
Noronha&Noronha&Greiner

Shear viscosity



Hirano&Gyulassy

Friction makes things smoother!

[from G. Denicol]

Adding viscosities into relativistic hydro

Recent review: [P. Romatschke, U. Romatschke, arXiv:1712.05816]

Israel-Stewart second-order hydro: perfect fluid

$$T_0^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu}$$

+ stress corrections from shear π (traceless) and bulk Π viscosities

$$T^{\mu\nu} = T_0^{\mu\nu} + \pi^{\mu\nu} + \Pi \Delta^{\mu\nu}$$

$$\partial_\mu T^{\mu\nu} = 0$$

The viscous corrections are solutions of 6 additional equations:

$$\Delta^{\mu\alpha}\Delta^{\nu\beta} u^\gamma \partial_\gamma \pi_{\alpha\beta} = \frac{2\eta\sigma^{\mu\nu} - \pi^{\mu\nu}}{\tau_\pi} - \frac{4}{3}\pi^{\mu\nu}\partial_\alpha u^\alpha$$

$$u^\gamma \partial_\gamma \Pi = \frac{-\zeta \partial_\gamma u^\gamma - \Pi}{\tau_\Pi} - \frac{4}{3}\Pi \partial_\alpha u^\alpha$$

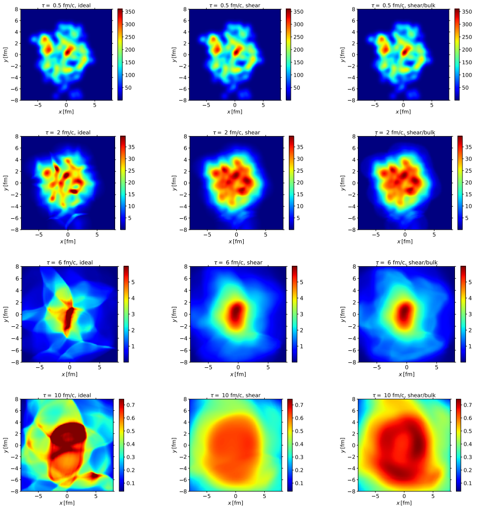
$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \; \nabla^\mu = \Delta^{\mu\nu}\partial_\nu$$

$$\sigma_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3}\Delta_{\mu\nu}\partial_\alpha u^\alpha\right)$$

The relaxation time is taken as $\tau_\pi = \tau_\Pi = \frac{3\eta}{Ts}$

Quenching of flow

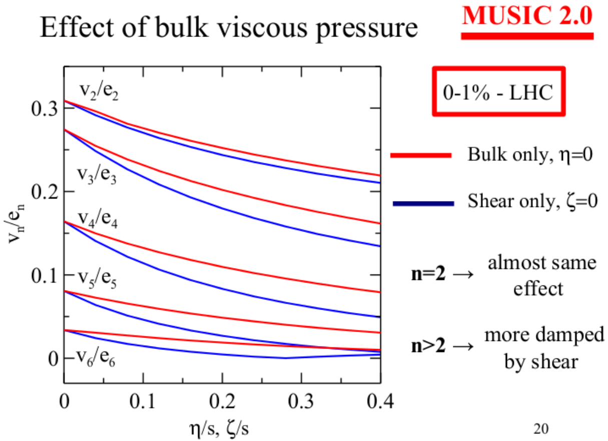
perfect shear shear+bulk



- Quenching of flow with viscosity
- Increasing with the Fourier rank
- Sets limits on viscosity, which is close to the KSS bound $\eta/s = 1/4\pi$
- ... but many other model parameters

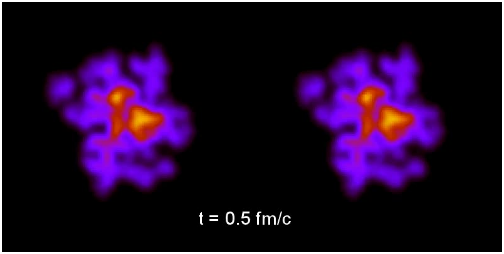
Figure:
[Bazow, Heinz, Strickland 2016]

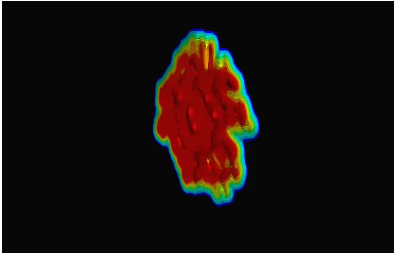
Damping of flow



[from G. Denicol]

3D numerics





[B. Schenke https://quark.phy.bnl.gov/∼bschenke]

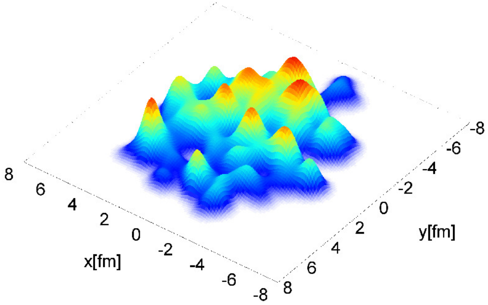
[other codes]

Initial conditions

- Initial value problem for partial differential equations $\rightarrow$ need to choose initial conditions for the functions on a time-like hypersurface, e.g, with constant $\tau = \sqrt{t^2 - z^2}$
- These conditions fluctuate event-by-event ...
- ... and (in hydro) are carried over to the freeze-out deterministically
- The starting time τ must be very short (a fraction of fm) for sufficient flow to develop (phenomenological statement)

However, on the general grounds of the fluctuation-dissipation theorem, hydro must also bring in some fluctuations

[J. I. Kapusta, B. Mueller, M. Stephanov, Phys.Rev. C85 (2012) 054906 – Bjorken flow
L. Yan, H. Grönqvist, JHEP 1603 (2016) 121 – Gubser flow:
"...the effect of hydrodynamical noise on flow harmonics is found to be negligible, especially in the ultra-central Pb-Pb collisions ..."]



[Schenke, Tribedy, Venugopalan, PRL 108(2012)252301, arXiv:1202.6646]

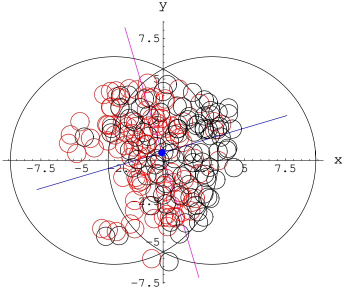
[Białas, Błeszyński, Czyż, NPB 111 (1976) 461]

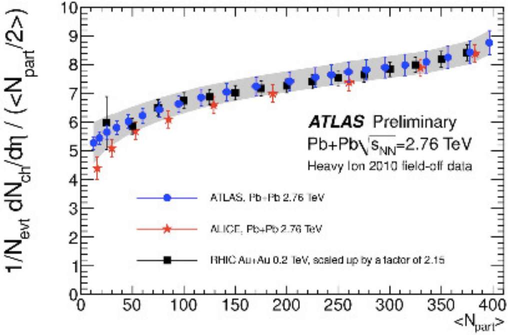
wounded + binary: $N \sim (1-\alpha)N_w/2 + \alpha N_{\rm bin}$, $\alpha \sim 0.14$

[D. Kharzeev, M. Nardi, PLB 507 (2001) 121]

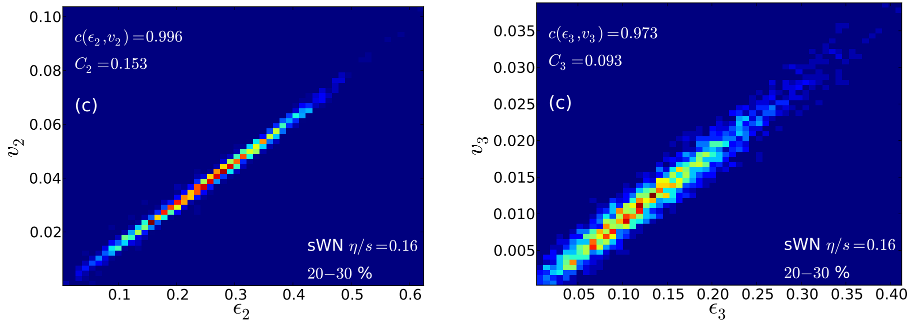
soft – wounded (a nucleon gets wounded only once)

hard – binary





Proportionality of harmonic flow to the initial eccentricity



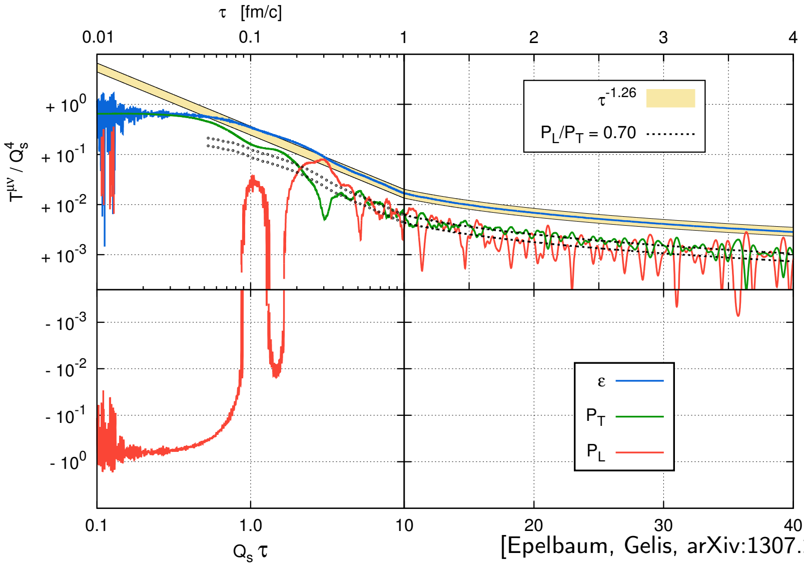
[Niemi, Denicol, Holopainen, Huovinen 2012]

"Hydro without hydro" – linearity of the shape-flow transmutation

$$v_n = \kappa_n \epsilon_n, \quad (n = 2, 3)$$

- κ_n depend on the collision energy, multiplicity, viscosity ...
- Approximate linearity allows us to build scale-less combinations independent of the response coefficient κ_n (see later)

Isotropization in Color Glass Condensate (with $SU_c(2)$)

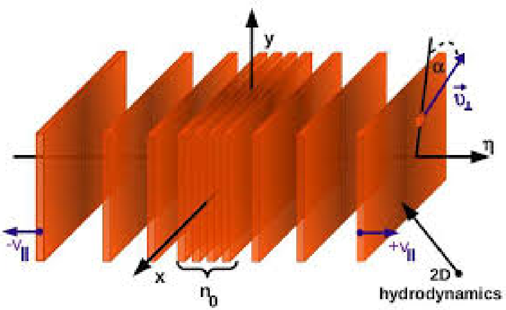


[Epelbaum, Gelis, arXiv:1307.2214]

[Review: Berges, Blaizot, Gelis, J. Phys. G 39(2012)085115]

Longitudinal-transverse anisotropy

[Florkowski, Ryblewski, 2008]

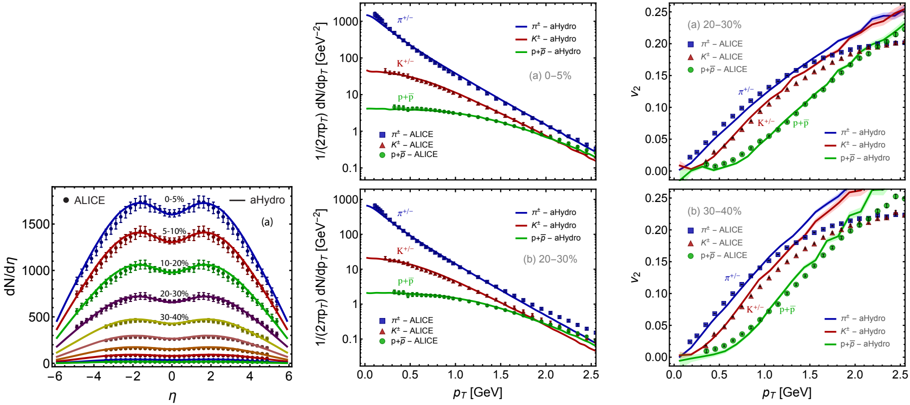


Thermal equilibrium only in the transverse direction

Anisotropic hydro

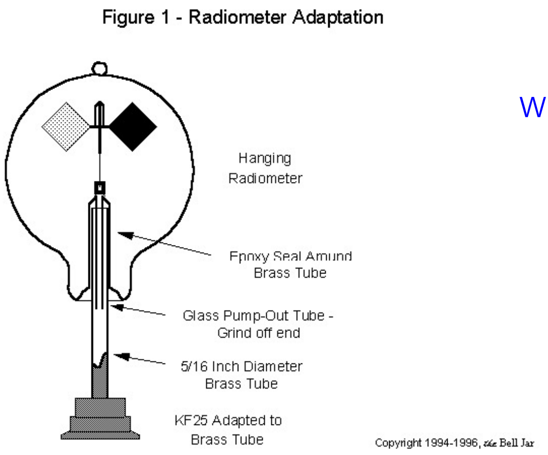
One can obtain satisfactory phenomenology in approaches without isotropization, where $P_T \geq P_L$

[Alqahtani, Nopoush, Ryblewski, Strickland, PRL 119 (2017) 042301]



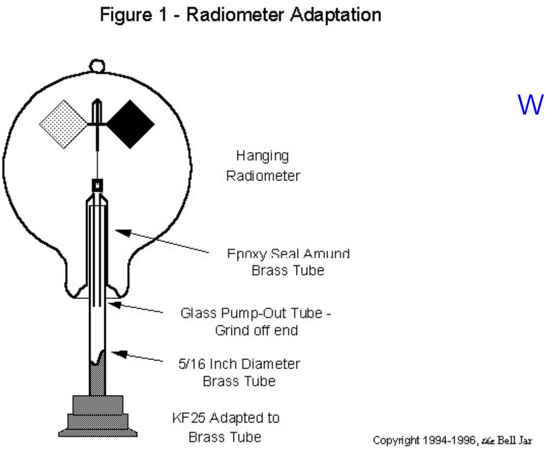
That things are nontrivial even classically...

The Crooks radiometer (1873)



Which way will it turn?

The Crooks radiometer (1873)



Which way will it turn?

- Not the light pressure!
- Not Navier-Stokes
- The Kortweg equations (capillarity) do it (arxiv:1702.00831)

https://www.quantamagazine.org/famous-fluid-equations-are-incomplete-20150721/

Summary of hydrodynamics

- Thermal equilibrium at freeze-out $\to$ species ratios
- Radial flow $\to$ $\langle p_T \rangle$, mass hierarchy, shape of p_T spectra
- Initial anisotropy + shape-flow transmutation from copious rescattering (hydrodynamics) $\to$ harmonic flow (any rescattering would do!)
- Viscosity $\to$ smoothing effect
- Viscous hydrodynamics difficult!
- Early thermalization $\to$ early hydrodynamization (no need for strict thermal equilibrium or isotropy)
- Successful phenomenology achieved, but numerous parameters (starting time, initial temperature, viscosities, freeze-out condition, regularization of hydrodynamics, model of the initial condition, ...). So we have a global picture rather than a very detailed picture.