Od analityczności do uwięzienia: spektralny model kwarków

Enrique Ruiz Arriola (Grenada) + WB

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- ERA + WB, Spectral quark model and low-energy hadron phenomenology, hep-ph/0301202, Phys. Rev. **D** (in print)

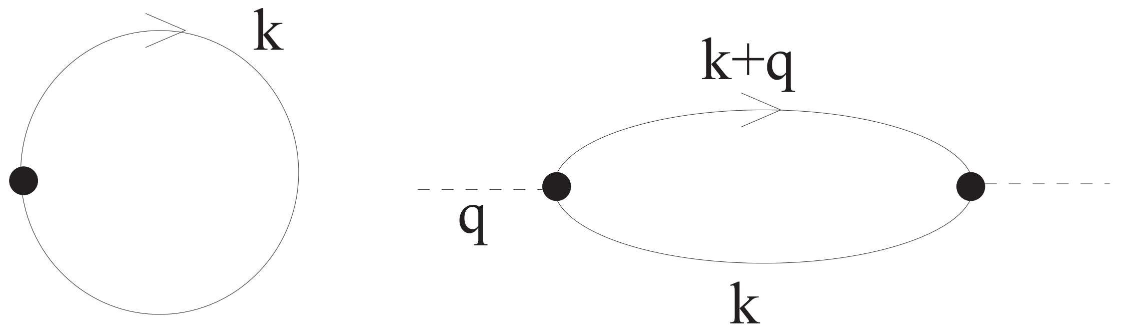
- ERA + WB, Pion light-cone wave function and pion distribution amplitude in the Nambu–Jona-Lasinio model, hep-ph/0207266, Phys. Rev. **D66** (2002) 094016

What is a Chiral Quark Model?

Prototype: Nambu-Jona–Lasinio, UV cut-off, interactions $\rightarrow$ χSB: massive quarks, Goldstone pions

One-loop (leading-N_c)

20 years of vast applications: low-energy hadron phenomenology (pions, nucleons, quark matter, ...)



The momentum running around the loop is cut

$$k < \Lambda$$

Requirements

1. Give finite values for hadronic observables
2. Satisfy the electromagnetic and chiral Ward-Takahashi identities, thus reproducing all necessary symmetry requirements
3. Satisfy the anomaly conditions
4. Comply to the QCD factorization property, in the sense that the expansion of a correlator at a large Q is a pure twist-expansion, involving only the inverse powers of Q^2, without the $\log Q^2$ corrections
5. Have usual dispersion relations

ERA's review

All simultaneously – far from trivial!

Spectral representation

We introduce a novel approach, the spectral regularization of the chiral quark model, based the Lehmann representation for the quark propagator,

$$S(p) = \int_C d\omega \frac{\rho(\omega)}{\not{p} - \omega}$$

where $\rho(\omega)$ is the spectral function and C denotes a contour in the complex ω plane chosen in a suitable way

Examples: free theory has $\rho(\omega) = \delta(\omega - m)$, perturbative QCD yields at LO

Non-perturbative?

$$\rho(\omega) = \delta(\omega - m) + \mathrm{sign}(\omega) \frac{\alpha_S C_F}{4\pi} \frac{1-\xi}{\omega} \theta(\omega^2 - m^2)$$

Quark condensate

$$\langle \bar{q}q \rangle \equiv -iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} S(p) = -4iN_c \int d\omega \rho(\omega) \int \frac{d^4p}{(2\pi)^4} \frac{\omega}{p^2 - \omega^2}$$

The integral over p is quadratically divergent, which requires the use of an auxiliary regularization, *removed* at the end

$$\langle \bar{q}q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \omega \rho(\omega) \left[2\Lambda^2 + \omega^2 \log \left(\frac{\omega^2}{4\Lambda^2} \right) + \omega^2 + \mathcal{O}(1/\Lambda) \right]$$

The finiteness of the result at $\Lambda \to \infty$ requires the conditions

$$\int d\omega \omega \rho(\omega) = 0, \quad \int d\omega \omega^3 \rho(\omega) = 0$$

and thus

$$\langle \bar{q} q \rangle = -\frac{N_c}{4\pi^2} \int d\omega \log(\omega^2) \omega^3 \rho(\omega).$$

The spectral condition allowed for rewriting $\log(\omega^2/\Lambda^2)$ as $\log(\omega^2)$, hence no scale dependence (no "dimensional transmutation") is present in the final expression.

With the accepted value of

$$\langle \bar{q} q \rangle = \simeq -(243 \text{ MeV})^3$$

we infer the value of the third log-moment. The negative sign of the quark condensate shows that

$$\int d\omega \log(\omega^2) \omega^3 \rho(\omega) > 0.$$

Spectral moments

$$\rho_0 \equiv \int d\omega \rho(w) = 1,$$

$$\rho_n \equiv \int d\omega \omega^n \rho(\omega) = 0, \quad \text{for } n = 1, 2, 3, \ldots$$

Observables are given by inverse moments

Such a $\rho(\omega)$ exists!

$$\rho_{-k} \equiv \int d\omega \omega^{-k} \rho(\omega), \quad \text{for } k = 1, 2, 3, \ldots$$

as well as by the “log moments”,

$$\rho'_n \equiv \int d\omega \log(\omega^2) \omega^n \rho(\omega), \quad \text{for } n = 2, 3, 4, \ldots$$

Vacuum energy density

$$\langle\theta^{\mu\nu}\rangle = -iN_cN_f\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\times$$
$$\mathrm{Tr}\frac{1}{\not p-\omega}\left[\frac{1}{2}(\gamma^\mu p^\nu+\gamma^\nu p^\mu)-g^{\mu\nu}(\not p-\omega)\right]=Bg^{\mu\nu}+\langle\theta^{\mu\nu}\rangle_0,$$

where $\langle\theta^{\mu\nu}\rangle_0$ is the energy-momentum tensor for the free theory, evaluated with $\rho(\omega)=\delta(\omega)$, and B (bag constant) is the vacuum energy density:

$$B=-iN_cN_f\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\frac{\omega^2}{p^2-\omega^2},$$

The conditions that have to be fulfilled for B to be finite are

$$\rho_2=0,\quad \rho_4=0$$

Then

$$B = -\frac{N_c N_f}{16\pi^2}\rho_4' \equiv -\frac{3N_c}{16\pi^2}\int d\omega \log(\omega^2)\omega^4\rho(\omega)$$

for $N_f = 3$.

The even conditions (here quadratic and quartic) imply that $\rho(\omega)$ *cannot be positive definite*; otherwise the even moments could not vanish!

According to the most recent QCD sum rules analysis

$$B = -\frac{9}{32}\langle\frac{\alpha}{\pi}G^2\rangle = -(224^{+35}_{-70}\mathrm{MeV})^4$$

The negative sign of B enforces

$$\rho_4' > 0$$

Gauge technique and the vertex functions

In QCD, the vector and axial currents are:

$$J_V^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \frac{\lambda_a}{2} q(x), \quad J_A^{\mu,a}(x) = \bar{q}(x)\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q(x)$$

CVC and PCAC:

$$\partial_\mu J_V^{\mu,a}(x) = 0, \quad \partial_\mu J_A^{\mu,a}(x) = \bar{q}(x)\hat{M}_0 i\gamma_5 \frac{\lambda_a}{2} q(x)$$

This implies Ward-Takahashi identities (WTI), based on

$$\left[J_V^{0,a}(x), q(x')\right]_{x_0=x_0'} = -\frac{\lambda_a}{2} q(x)\delta(\vec{x} - \vec{x}')$$

$$\left[J_A^{0,a}(x), q(x')\right]_{x_0=x_0'} = -\gamma_5 \frac{\lambda_a}{2} q(x)\delta(\vec{x} - \vec{x}')$$

A number of results are then obtained essentially for free:

- Pions arise as Goldstone bosons, with standard current-algebra properties

- At high energies parton-model features, such as scaling or the spin-1/2 nature of hadronic constituents, are recovered

Vertices with one current

The vector and axial unamputated vertex functions are:

$$\Lambda^{\mu,a}_{V,A}(p',p) \;=\; \int d^4x d^4x' \langle 0 | T\left\{ J^{\mu,a}_{V,A}(0) q(x') \bar{q}(x) \right\} |0\rangle e^{ip'\cdot x' - ip\cdot x}$$

$$(p'-p)_\mu \Lambda^{\mu,a}_V(p',p) \;=\; S(p')\frac{\lambda_a}{2} - \frac{\lambda_a}{2}S(p)$$

$$(p'-p)_\mu \Lambda^{\mu,a}_A(p',p) \;=\; S(p')\frac{\lambda_a}{2}\gamma_5 + \gamma_5\frac{\lambda_a}{2}S(p)$$

The gauge technique consists of writing a solution

Delburgo & West

$$\Lambda^{\mu,a}_V(p',p) \;=\; \int d\omega \rho(\omega) \frac{i}{\not p' - \omega}\gamma^\mu \frac{\lambda_a}{2}\frac{i}{\not p - \omega}$$

$$\Lambda^{\mu,a}_A(p',p) \;=\; \int d\omega \rho(\omega) \frac{i}{\not p' - \omega}\left(\gamma^\mu - \frac{2\omega q^\mu}{q^2}\right)\gamma_5 \frac{\lambda_a}{2}\frac{i}{\not p - \omega}$$

"Transverse ambiguity"

The above ansätze fulfill the WTI's. They are determined up to *transverse pieces*.

This ambiguity appears in all effective models. Current conservation fixes only the longitudinal pieces. Example:

$$j_\mu = \bar{\psi} \left(f_1 \gamma_\mu + i f_2 \sigma_{\mu\nu} q^\nu \right) \psi$$

The condition $q^\mu j_\mu = 0$ does not constrain the f_2-term, since $\sigma_{\mu\nu} q^\nu q^\mu = 0$ from antisymmetry.

Pion-quark coupling

Near the pion pole $(q^2 = 0)$ we get

$$\Lambda_A^{\mu,a}(p+q,p) \to -\frac{q^\mu}{q^2}\Lambda_\pi^a(p+q,p),$$

where

$$\Lambda_\pi^a(p+q,p) = \int d\omega \rho(\omega) \frac{i}{\not p + \not q - \omega} \frac{\omega}{f_\pi} \gamma_5 \lambda_a \frac{i}{\not p - \omega}$$

We recognize in our formulation the Goldberger-Treiman relation for quarks:

$$g_\pi(\omega) = \frac{\omega}{f_\pi}$$

Vertices with two currents

The vertices with two currents, axial or vector, are constructed similarly. The vacuum polarization is (dimensional regulariztion):

$$
\begin{aligned}
i\Pi^{\mu a,\nu b}_{VV}(q) &= \delta^{ab}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\bar{\Pi}_{VV}(q) = \int d^4x e^{-iq\cdot x}\langle 0|T\left\{J^{\mu a}_V(x)J^{\nu b}_V(0)\right\}|0\rangle \\
&= -N_c\int d\omega\rho(\omega)\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not p - \not q - \omega}\gamma_\mu\frac{\lambda_a}{2}\frac{i}{\not p - \omega}\gamma_\nu\frac{\lambda_b}{2}\right]
\end{aligned}
$$

transverse

$$
\begin{aligned}
\bar{\Pi}_{VV}(q) &= \ldots \\
I(q^2,\omega) &= -\frac{1}{(4\pi)^2}\int_0^1 dx\log\left[\omega^2 + x(1-x)q^2\right]
\end{aligned}
$$

Dispersion relation

The twice-subtracted dispersion relation holds:

$$\bar{\Pi}_V(q^2) = \frac{q^4}{\pi} \int_0^\infty \frac{dt}{t^2} \frac{\mathrm{Im}\bar{\Pi}_V(t)}{t - q^2 - i0^+}$$

This is in contrast to quark models formulated in the Euclidean space, where the dispersion relation is postulated, but never deduced!

$e^+e^- \to$ **hadrons**

At large s we find

proportional to $\mathrm{Im}\bar{\Pi}_V$

$$\sigma(e^+e^- \to \mathrm{hadrons}) \to \frac{4\pi\alpha_{\mathrm{QED}}^2}{3s}\left(\sum_i e_i^2\right)\int d\omega\rho(\omega),$$

where e_i is the electric charge of the quark of species i. This is the proper asymptotic QCD result, provided

$$\int d\omega\rho(\omega) = 1$$

Pion weak decay

The pion weak-decay constant, defined as

$$\langle 0 \,|J_A^{\mu a}(x)|\, \pi_b(q)\rangle = i f_\pi q_\mu \delta_{a,b} e^{iq\cdot x},$$

can be computed from the axial-axial correlation function.
The result is

$$f_\pi^2 \;=\; 4N_c \int d\omega \rho(\omega)\omega^2 I(0,\omega)$$

A finite value for f_π requires the condition $\rho_2 = 0$. Then

$$f_\pi^2 = -\frac{N_c}{4\pi^2}\int d\omega \log(\omega^2)\omega^2\rho(\omega) \equiv -\frac{N_c}{4\pi^2}\rho_2'$$

The value $f_\pi = 93\text{MeV}$ determines ρ_2'. The sign is

$$\rho_2' < 0$$

Pion electromagnetic form factor

The electromagnetic form factor for a positively charged pion, $\pi^+ = u\bar{d}$, is defined as

$$\langle \pi^+(p')|J_\mu^{\rm em}(0)|\pi^+(p)\rangle \;=\; (p^\mu + p'^\mu) e F_\pi^{\rm em}(q^2)$$

For on-shell massless pions the electromagnetic form factor reads

$$F_\pi^{em}(q^2) = \frac{4N_c}{f_\pi^2}\int dw\rho(\omega)\omega^2 I(q^2,\omega)$$

The low-momentum expansion is $F_\pi^{em}(0) = 1$

$$F_\pi^{em}(q^2) = 1 + \frac{1}{4\pi^2 f_\pi^2}\left(\frac{q^2\rho_0}{6} + \frac{q^4\rho_{-2}}{60} + \frac{q^6\rho_{-4}}{240} + \dots\right)$$

The mean squared radius reads

$$\langle r_\pi^2 \rangle = 6 \frac{dF}{dq^2}|_{q^2=0} = \frac{N_c}{4\pi^2 f_\pi^2} \int d\omega \rho(\omega) = \frac{N_c}{4\pi^2 f_\pi^2},$$

which coincides with the unregularized-quark-loop result. The numerical value is

$$\langle r^2 \rangle_\pi^{\rm em} \Big|_{\rm th} = 0.34 {\rm fm}^2, \qquad \langle r^2 \rangle_\pi^{\rm em} \Big|_{\rm exp} = 0.44 {\rm fm}^2,$$

which is a reasonable agreement (χPT corrections).

The knowledge of the pion electromagnetic form factor allows to determine the even negative moments of $\rho(\omega)$.

Twist expansion and spectral conditions

In the limit of large momentum

$$F_\pi^{em}(q^2) \sim \frac{N_c}{4\pi^2 f_\pi^2}\int d\omega\rho(\omega)\omega^2\{2-\frac{1}{\epsilon}-\log(q^2)+ \frac{2\omega^2}{q^2}\left[\log(-q^2/\omega^2)+1\right]+\frac{2\omega^4}{q^4}\left[\log(-q^2/\omega^2)-\frac{1}{2}\right]\ldots\}$$

With help of the spectral conditions for $n=2,4,6,...$ we get

$$F_\pi^{em}(q^2) \sim -\frac{N_c}{4\pi^2 f_\pi^2}\left[\frac{2\rho_4'}{q^2}+\frac{2\rho_6'}{q^4}+\frac{4\rho_8'}{q^6}+...\right]$$

All spectral conditions needed!

The imposition of the spectral conditions removed all the logs from the expansion, leaving a pure expansion in inverse powers of q^2!

Anomalous form factor in $\pi^0 \to \gamma\gamma$

$$\Gamma^{\mu\nu}_{\pi^0\gamma\gamma}(q_1, q_2) = \epsilon_{\mu\nu\alpha\beta} q_1^\alpha p^\beta F_{\pi\gamma\gamma}(p, q_1, q_2) = -N_c \int d\omega \rho(w) \int \frac{d^4k}{(2\pi)^4} \times$$

$$\mathrm{Tr}\left[-\frac{\omega}{f_\pi} \gamma_5 \tau_0 \frac{i}{\not k - \not q_2 - \omega} i\hat{Q}\gamma^\mu \frac{i}{\not k - \omega} i\hat{Q}\gamma^\nu \frac{i}{\not k - \not q_1 - \omega} \right] + \text{crossed}$$

where $\hat{Q} = 1/2N_c + \tau_3/2$. We find

$$\begin{aligned} F_{\pi\gamma\gamma}(0,0,0) &= \frac{8}{f_\pi} \int d\omega \rho(\omega) \omega^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2 - \omega^2)^3} \\ &= \frac{1}{4\pi^2 f_\pi} \int d\omega \rho(\omega) = \frac{1}{4\pi^2 f_\pi} \end{aligned}$$

which coincides with the standard QCD result!

Blin, Hiller & Schaden

Pion-photon transition form factor

For two off-shell photons with momenta q_1 and q_2 one defines the asymmetry, A, and the total virtuality, Q^2:

$$
\begin{aligned}
A &= \frac{q_1^2 - q_2^2}{q_1^2 + q_2^2}, \qquad -1 \le A \le 1 \\
Q^2 &= -(q_1^2 + q_2^2)
\end{aligned}
$$

At the soft pion point we find the expansion,

$$F_{\pi\gamma\gamma}(Q^2, A) = -\frac{1}{2\pi^2 f_\pi} \int_0^1 dx \left[\frac{2\rho_2'}{Q^2(1 - A^2(2x-1)^2)} + \dots \right]$$

We can confront this with the standard twist decomposition of the pion transition form factor ,

$$F_{\gamma\gamma\pi}(Q^2, A) = J^{(2)}(A)\frac{1}{Q^2} + J^{(4)}(A)\frac{1}{Q^4} + \dots ,$$

Brodsky-Lepage, Praszałowicz-Rostworowski, Dorokhov which yields

$$J^{(2)}(A) = \frac{4f_\pi}{N_c}\int_0^1 dx \frac{\varphi(x;Q_0)}{1-(2x-1)^2A^2}$$

with the leading-twist pion distribution amplitude (PDA) given by

$$\varphi(x;Q_0) = 1$$

Pion light-cone wave function

An analogous calculation to the one presented in ERA+WB produces the following light-cone pion wave function in the present model:

$$\Psi(x,k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}\int d\omega \rho(\omega)\frac{\omega^2}{k_\perp^2+\omega^2}\theta(x)\theta(1-x)$$

(at the low-energy scale of the model, Q_0). It has correct normalization, since $\int d^2k_\perp \Psi(x,k_\perp) = \varphi(x) = 1$. At $k_\perp = 0$ it satisfies the condition:

$$\Psi(x,0) = \frac{N_c}{\pi f_\pi} F_{\pi\gamma\gamma}(0,0,0) = \frac{N_c}{4\pi^3 f_\pi^2}$$

In QCD one has a similar relation holding for quantities integrated over x.

QCD evolution of PDA

All results of the effective, low-energy model, refer to a soft energy scale, Q_0. In order to compare to experimental results, obtained at large scales, Q, the QCD evolution must be performed. Initial condition:

$$\varphi(x; Q_0) = \theta(x)\theta(1-x).$$

The evolved distribution amplitude reads

$$\begin{aligned} \varphi(x; Q) &= 6x(1-x) \sum_{n=0}^{\infty} C_n^{3/2}(2x-1) a_n(Q) \\ a_n(Q) &= \frac{2}{3} \frac{2n+3}{(n+1)(n+2)} \left(\frac{\alpha(Q^2)}{\alpha(Q_0^2)} \right)^{\gamma_n^{(0)}/(2\beta_0)} \end{aligned}$$

where $C_n^{3/2}$ are the Gegenbauer polynomials, $\gamma_n^{(0)}$ are appropriate anomalous dimensions, and $\beta_0 = 9$.

Results extracted from the experimental data of CLEO provide $a_2(2.4\mathrm{GeV}) = 0.12 \pm 0.03$, which we use to fix

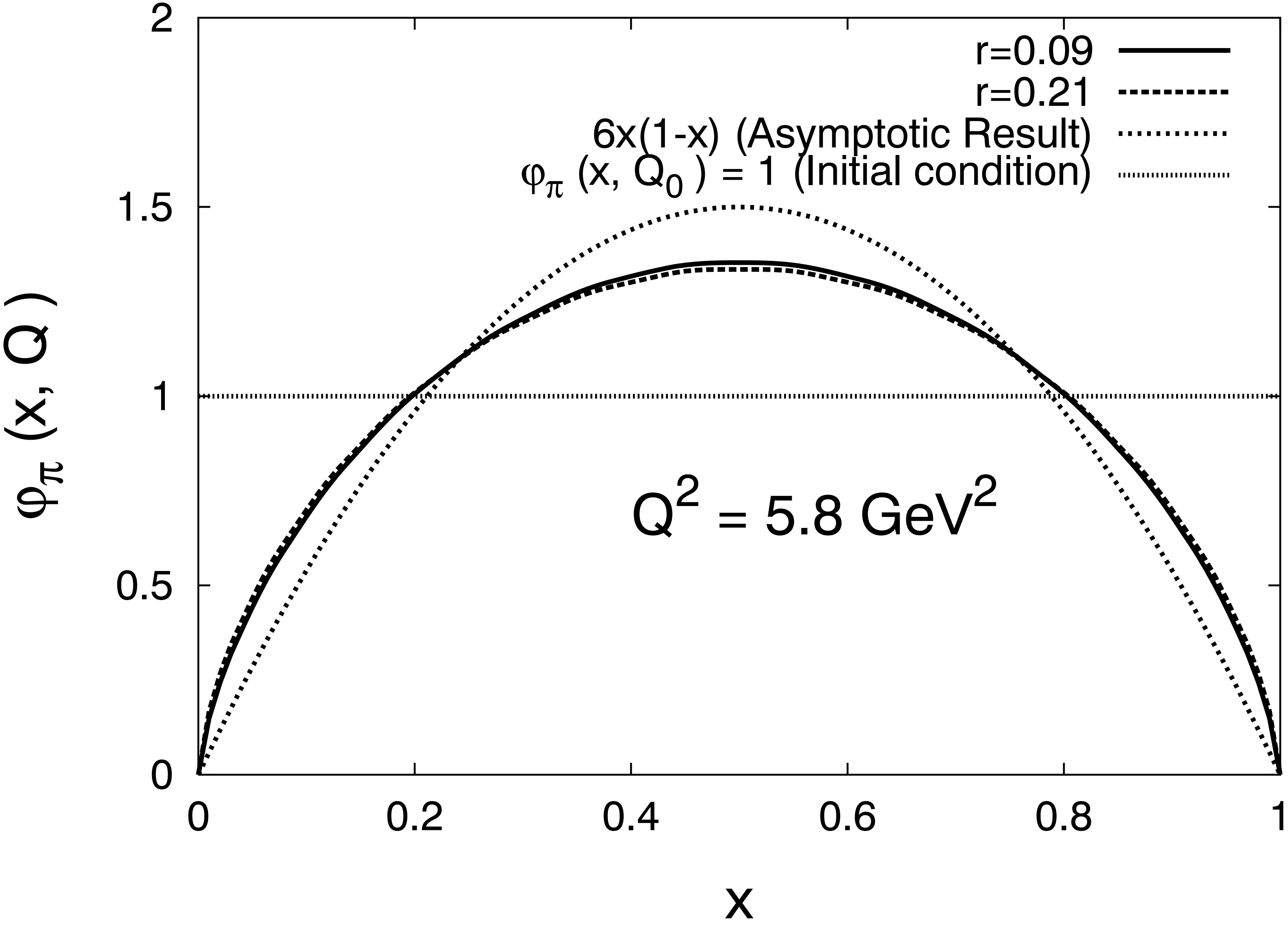
$$\alpha(Q = 2.4\mathrm{GeV})/\alpha(Q_0) = 0.15 \pm 0.06$$

At LO this correspondsto $Q_0 = 322 \pm 45$ MeV

Now we can predict

$$\begin{aligned} a_4(2.4\mathrm{GeV}) &= 0.06 \pm 0.02 \ \ (\mathrm{exp} : -0.14 \pm 0.03 \mp 0.09) \\ a_6(2.4\mathrm{GeV}) &= 0.02 \pm 0.01 \end{aligned}$$

Encouraging, with leading-twist and LO QCD evolution!



r=0.09
r=0.21
6x(1-x) (Asymptotic Result)
φπ (x, Q0) = 1 (Initial condition)
Q2 = 5.8 GeV2
φπ (x, Q)
x
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

The $\gamma \to \pi^+\pi^0\pi^-$ decay

Here we consider an example of a low-energy process involving a quark box diagram, which similarly to the neutral pion decay is related to the QCD anomaly in the soft pion limit. The amplitude for $\gamma(q,e) \to \pi^+(p_1)\pi^0(p_2)\pi^-(p_3)$ is

$$T_{\gamma(q,\varepsilon)\to\pi^+(p_1)\pi^0(p_2)\pi^-(p_3)} \equiv F(p_1,p_2,p_3)\varepsilon_{\alpha\beta\sigma\tau}e^\alpha p_1^\beta p_2^\sigma p_3^\tau.$$

In the limit of all momenta going to zero we get, with the spectral normalization condition,

$$F(0,0,0) = \frac{1}{4\pi^2 f_\pi^3}\int d\omega\rho(\omega) = \frac{1}{4\pi^2 f_\pi^3}$$

which is the correct result.

Pion structure function

We take π^+ for definiteness and get

$$u_\pi(x) = \bar{d}_\pi(1-x) = \theta(x)\theta(1-x),$$

independent of the spectral function $\rho(\omega)$. Thus one recovers scaling in the Bjorken limit, the Callan-Gross relation, the proper support, and the correct normalization.

Davidson & Arriola in NJL

The $k_\perp$-unintegrated parton distribution can be shown to be equal to

$$q(x, k_\perp) = \frac{N_c}{4\pi^3 f_\pi^2} \int d\omega \rho(\omega) \frac{\omega^2}{k_\perp^2 + \omega^2} \theta(x)\theta(1-x),$$

hence at Q_0 one has an interesting relation

$$q(x, k_\perp) = \bar{q}(1-x, k_\perp) = \Psi(x, k_\perp).$$

At $k_\perp = 0$ we have

$$q(x, 0_\perp) = \frac{N_c}{4\pi^3 f_\pi^2}.$$

Finally, via integrating with respect to $k_\perp$ the following identity between the PDF and the PDA is obtained at the scale Q_0:

$$q(x) = \varphi(x)$$

The first moment of the PDF is responsible for the momentum sum rule. We also find

$$\int_0^1 dx\, x q(x) = \int_0^1 dx\, x \bar{q}(x) = \frac{1}{2}.$$

quarks carry all momentum

QCD evolution of PDF

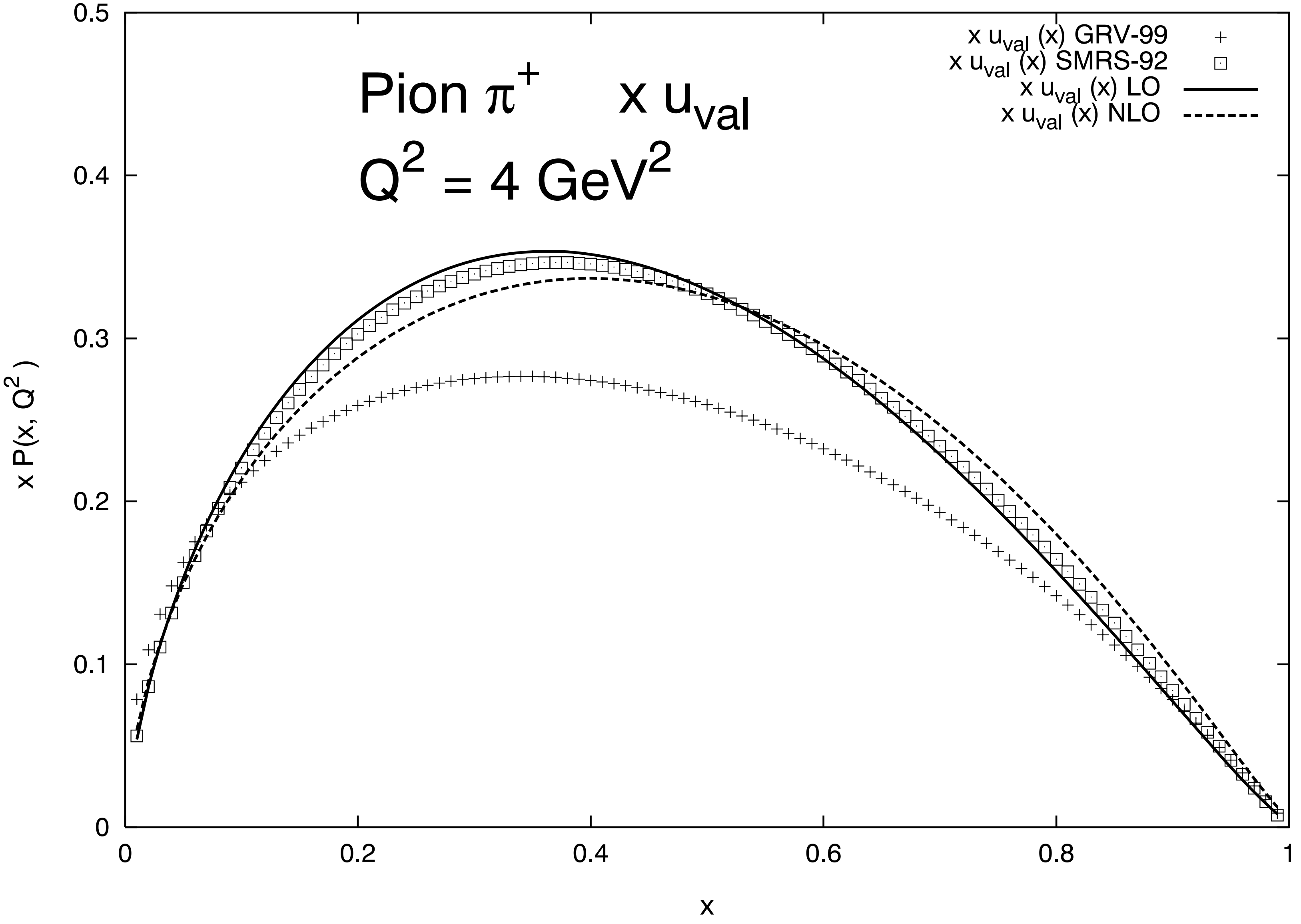
The QCD evolution of the constant PDF has been treated in detail by Davidson & ERA at LO and NLO. In particular, the non-singlet contribution to the energy-momentum tensor evolves as

$$\frac{\int dx\, xq(x,Q)}{\int dx\, xq(x,Q_0)} = \left(\frac{\alpha(Q)}{\alpha(Q_0)}\right)^{\gamma_1^{(0)}/(2\beta_0)} ,$$

In has been found that at $Q^2 = 4\mathrm{GeV}^2$ the valence quarks carry $47 \pm 0.02\%$ of the total momentum fraction in the pion. Downward LO evolution yields that at the scale

$$Q_0 = 313^{+20}_{-10}\mathrm{MeV}$$

the quarks carry 100% of the momentum. The agreement of the evolved PDF with the SMRS data analysis is impressive


Pion π^+ x u_{val}
$Q^2 = 4\ GeV^2$
x u_{val} (x) GRV-99
x u_{val} (x) SMRS-92
x u_{val} (x) LO
x u_{val} (x) NLO
x P(x, Q^2)
x
0
0.1
0.2
0.3
0.4
0.5
0.2
0.4
0.6
0.8
1

Résumé

Spectral condition	Physical significance
normalization	
$\rho_0 = 1$	proper normalization of the quark propagator
	preservation of anomalies
	proper normalization of the pion distribution amplitude
	proper normalization of the pion structure function
	reproduction of the large-N_c quark-model values of the Gasser-Leutwyler coefficients
	relation $M_V^2 = 24\pi^2 f_\pi^2 / N_c$ in the VMD model
positive moments	
$\rho_1 = 0$	finiteness of the quark condensate, $\langle \bar{q} q \rangle$
	vanishing quark mass at asymptotic Euclidean momenta,
$\rho_2 = 0$	finiteness of the vacuum energy density, B
	finiteness of the pion decay constant, f_π
$\rho_3 = 0$	finiteness of the quark condensate, $\langle \bar{q} q \rangle$
$\rho_4 = 0$	finiteness of the vacuum energy density, B
$\rho_n = 0,\ n = 2, 4 \ldots$	absence of logs in the twist expansion of vector amplitudes
$\rho_n = 0,\ n = 5, 7 \ldots$	finiteness of nonlocal quark condensates, $\langle \bar{q} (\partial^2)^{(n-3)/2} q \rangle$
	absence of logs the twist expansion of the scalar pion form factor

Spectral condition	Physical significance
negative moments	
$\rho_{-2} > 0$	positive quark wave-function normalization at vanishing momentum
$\rho_{-1}/\rho_{-2} > 0$	positive value of the quark mass at vanishing momentum, $M(0) > 0$
ρ_{-n}	low-momentum expansion of correlators
log-moments	
$\rho'_2 < 0$	$f_\pi^2 = -N_c/(4\pi^2)\rho'_2$
$\rho'_3 > 0$	negative value of the quark condensate, $\langle \bar{q}q \rangle = -N_c/(4\pi^2)\rho'_3$
$\rho'_4 > 0$	negative value of the vacuum energy density, $B = -N_c/(4\pi^2)\rho'_4$
$\rho'_5 < 0$	positive value of the squared vacuum virtuality of the quark, $\lambda_q^2 = -\rho'_5/\rho'_3$
ρ'_n	high-momentum (twist) expansion of correlators

Vector-meson dominance

Now we construct explicitly an example of $\rho(\omega)$. Vector-meson dominance (VMD) of the pion form factor is assumed:

$$F_V(t) = \frac{M_V^2}{M_V^2 + t}.$$

with $M_V = m_\rho$. In our approach

$$F_V(t) = \frac{N_c}{4\pi^2 f_\pi^2} \sum_{n=1}^{\infty} \rho_{2-2n} \frac{2^{-2n-1}\sqrt{\pi}\Gamma(n+1)}{n\Gamma(n+3/2)} (-t)^n .$$

Comparison yields

$$\rho_{2-2k} = \frac{2^{2k+3}\pi^{3/2} f_\pi^2}{N_c M_V^{2k}} \frac{k\,\Gamma(k+3/2)}{\Gamma(k+1)}, k = 1, 2, 3, \ldots$$

In particular, the normalization condition, $\rho_0 = 1$, yields

$$M_V^2 = \frac{24\pi^2 f_\pi^2}{N_c}$$

This relation is usually obtained when matching chiral quark models to VMD, yielding $M_V = 826\ \mathrm{MeV}$ with $f_\pi = 93$ MeV, and $M_V = 764\ \mathrm{MeV}$ with $f_\pi = 86$ MeV (in the chiral limit).

Even though we have determined the negative even moments of the spectral function, the positive even moments, obtained by analytic continuation in the index n fulfill the spectral conditions for the positive moments due to the fact that $\Gamma(n)$ has single poles at non-positive integers, $n = 0, -1, -2, \ldots$ Miracle!

$$\rho_{2n} = 0, \qquad n = 1, 2, 3 \ldots$$

For the log-moments we have

$$\rho'_{2n} = \left(-\frac{M_V^2}{4}\right)^n \frac{\Gamma(n)\,\Gamma\left(\frac{5}{2}-n\right)}{\Gamma(\frac{5}{2})}, \qquad n = 1, 2, 3\ldots$$

The first few values are

$$\rho'_2 = -\frac{4f^2\pi^2}{N_c}, \quad \rho'_4 = \frac{2f^2 M_V^2 \pi^2}{N_c}$$

We may write the following interesting relation coming out from VMD and the spectral approach:

$$B = -\frac{9\pi^2 f_\pi^4}{N_c} = -\frac{N_c M_V^4}{64\pi^2} = -(202 - 217 \text{ MeV})^4$$

which agrees within errors with the QCD SR estimate.

Inverse problem

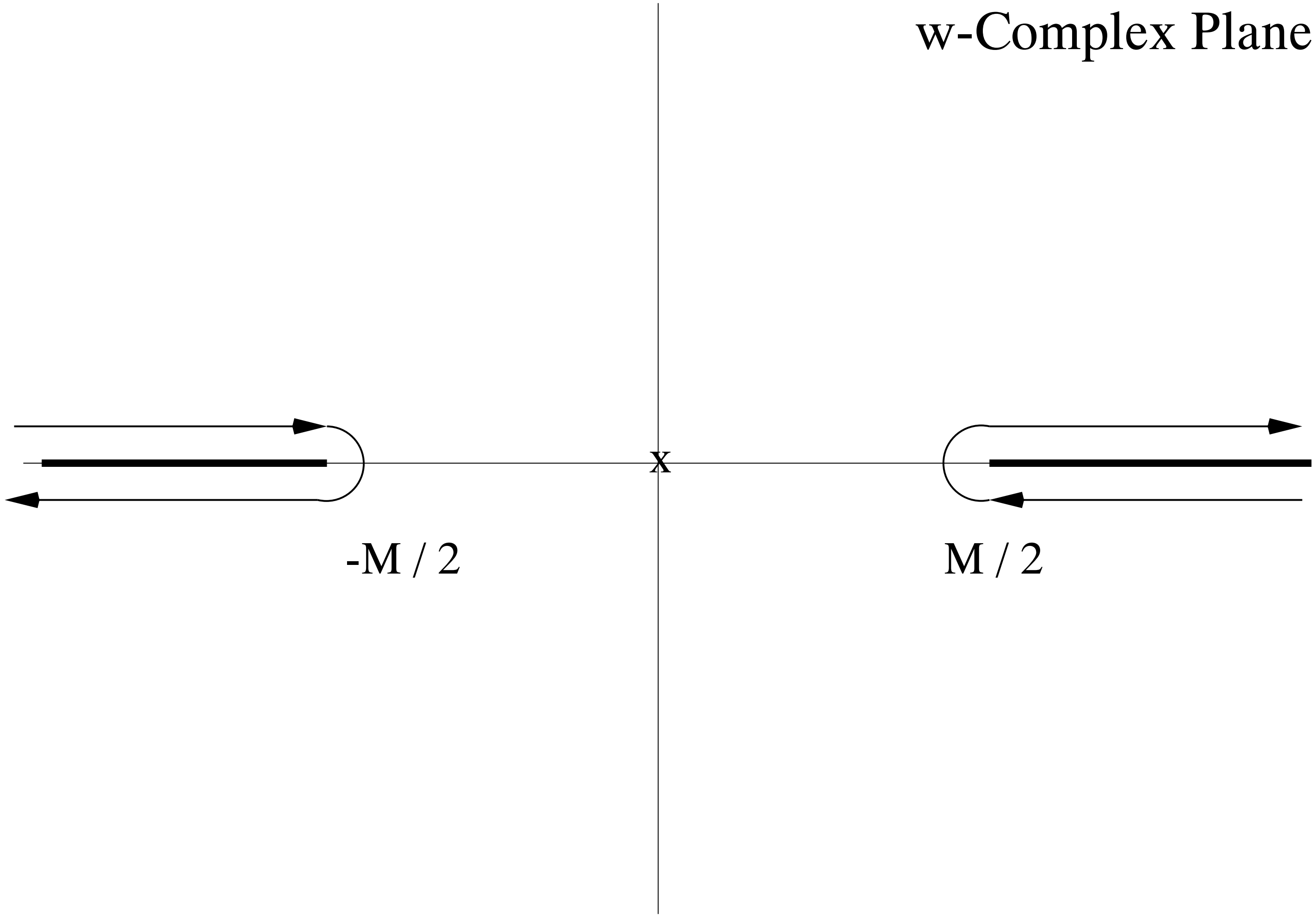
The mathematical problem is now to invert the formula

$$\rho_{2n} = \int_C d\omega \omega^{2n} \rho_V(\omega)$$

The solution is given by the following surprisingly simple function

$$\rho_V(\omega) = \frac{1}{2\pi i} \frac{3\pi^2 M_V^3 f_\pi^2}{4N_c} \frac{1}{\omega} \frac{1}{(M_V^2/4 - \omega^2)^{5/2}}.$$

The function $\rho_V(\omega)$ has a single pole at the origin and branch cuts starting at $\pm$ half the meson mass, $\omega = \pm M_V/2$.



contour C

Scalar spectral function

For the case of the scalar spectral function (controling odd moments) we proceed heuristically, by proposing its form in an analogy to the form of ρ_V $(\rho = \rho_V + \rho_S)$

$$\rho_S(\omega) = \frac{1}{2\pi i} \frac{16(d_S - 1)(d_S - 2)\rho_3'}{M_S^4(1 - 4\omega^2/M_S^2)^{d_S}}$$

where the normalization is chosen in such a way that $\rho_3' = -4\pi^2\langle\bar{q}q\rangle/N_c$. The admissible values of d_S are half-integer, since only then the integration around the half-circles at the branch points vanishes. The preferred value turns out to be $d_S = 5/2$.

The analytic structure of $\rho_S(\omega)$ is similar to the case of $\rho_V(\omega)$, except for the absence of the pole at $\omega = 0$.

Quark propagator

$$S(p) = A(p)\not{p} + B(p) = Z(p)\frac{\not{p} + M(p)}{p^2 - M^2(p)}$$

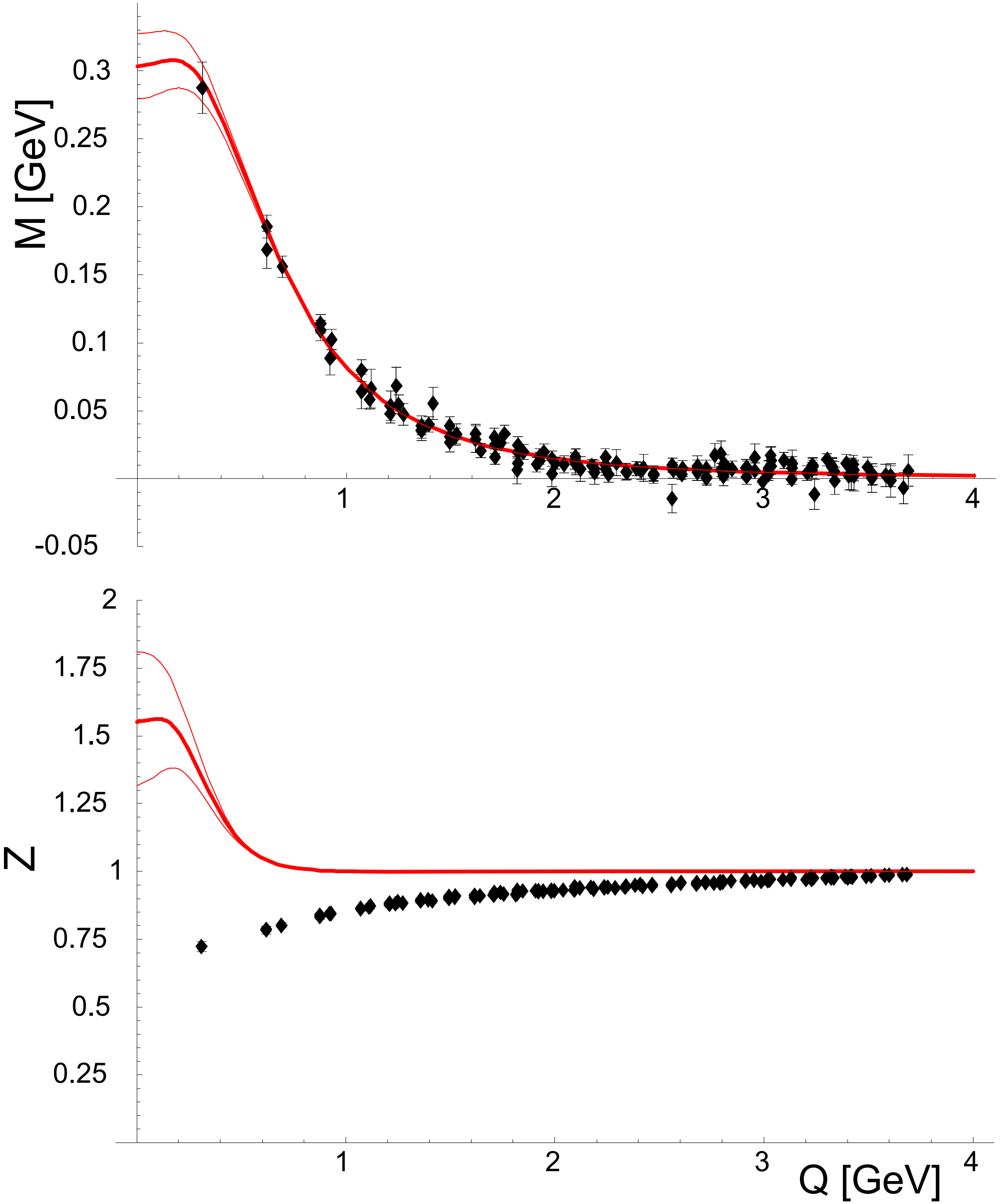
$$A(p^2) \equiv \int_C d\omega \frac{\rho_V(\omega)}{p^2-\omega^2} = \frac{1}{p^2}\left[1 - \frac{1}{(1-4p^2/M_V^2)^{d_V}}\right]$$

$$B(p^2) \equiv \int_C d\omega \frac{\omega\rho_S(\omega)}{p^2-\omega^2} = \frac{64(d_S-2)(d_S-1)\pi^2\langle\bar{q}q\rangle}{M_S^4 N_c(1-4p^2/M_S^2)^{d_S}}$$

No poles in the whole complex plane! Only branch cuts starting at $p^2 = 4M^2$, where M is the relevant mass.

The absence of poles is sometimes called "analytic confinement"

Poles would lead to cuts in f.f.



M [GeV]
0.3
0.25
0.2
0.15
0.1
0.05
-0.05
1
2
3
4
Z
2
1.75
1.5
1.25
1
0.75
0.5
0.25
1
2
3
4
Q [GeV]

$M(Q^2)$ decreases as $1/Q^3$ at large Euclidean momenta, which is favored by the recent lattice calculations. The χ^2 fit results in the following optimum values,

$$\begin{aligned} M_0 &= 303 \pm 24 \text{ MeV}, \\ M_S &= 970 \pm 21 \text{ MeV}, \end{aligned}$$

with the optimum value of χ^2 per degree of freedom equal to 0.72. The corresponding value of the quark condensate is

$$\langle \bar{q}q \rangle = -(243.0^{+0.1}_{-0.8} \text{ MeV})^3.$$

Other predictions

Pion light-cone wave function:

$$\Psi(x, k_\perp) = \frac{3M_V^3}{16\pi(k_\perp^2 + M_V^2/4)^{5/2}} \theta(x)\theta(1-x)$$

Passing to the impact-parameter space yields

$$\begin{aligned} \Psi(x, b) &\equiv 2\pi \int_0^\infty k_\perp dk_\perp \Psi(x, k_\perp) J_0(k_\perp b) \\ &= \left(1 + \frac{bM_V}{2}\right) \exp\left(-\frac{M_V b}{2}\right) \theta(x)\theta(1-x) \end{aligned}$$

The average transverse momentum squared is equal to

$$\langle k_\perp^2 \rangle \equiv \int d^2k_\perp \, k_\perp^2 \Psi(x, k_\perp) = \frac{M_V^2}{2}$$

which numerically gives $\langle k_\perp^2 \rangle = (544 \text{ MeV})^2$ (at Q_0).

AG & JK $\to$ spreading

Unintegrated PDF:

$$q(x, k_\perp) = \overline{q}(1-x, k_\perp) = \frac{3M_V^3}{16\pi(k_\perp^2 + M_V^2/4)^{5/2}} \theta(x)\theta(1-x)$$

Quark propagator in the coordinate representation:

$$A(x) = \frac{48 + 24M_V\sqrt{-x^2} - 6M_V^2 x^2 + M_V^3(-x^2)^{3/2}}{96\pi^2 x^4} \exp(-M_V\sqrt{-x^2}/2)$$

$$B(x) = \langle \overline{q}q \rangle / (4N_c) \exp(-M_S\sqrt{-x^2}/2)$$

Final remarks

1. What has been done? Spectral representation, one loop, gauge technique. All works! Symmetries, anomalies, normalizations guaratied. Dynamics encoded in moments

2. What does not work? Weinberg sum rule II (modify transverse pieces?)

3. Need for QCD evolution of all quantities in order to pass from Q_0 to Q.

4. The method is very simple and predictive

5. Interesting particular realization (VMD)

6. Improve $Z(Q^2)$

Lots of applications